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The Number of Rational Points on Singular Curves $y^2 = x(x-a)^2$ over Finite Fields $\mathbf{F}_p$

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***Abstract*—**Let $p \geq 5$ be a prime number and let $\mathbf{F}_p$ be a finite field. In this work, we determine the number of rational points on singular curves $E_a : y^2 = x(x-a)^2$ over $\mathbf{F}_p$ for some specific values of a.

***Keywords*—**Singular curve, elliptic curve, rational points.

I. Introduction

Mordell began his famous paper [9] with the words Mathematicians have been familiar with very few questions for so long a period with so little accomplished in the way of general results, as that of finding the rational points on elliptic curves. The history of elliptic curves is a long one, and exciting applications for elliptic curves continue to be discovered. Recently, important and useful applications of elliptic curves have been found for cryptography [4], [7], [8], for factoring large integers [6] and for primality proving [1], [3]. The mathematical theory of elliptic curves was also crucial in the proof of Fermat's Last Theorem [17].

Let q be a positive integer, $\mathbf{F}_q$ be a finite field and let $\overline{\mathbf{F}}_q$ denote the algebraic closure of $\mathbf{F}_q$ with char($\overline{\mathbf{F}}_q$) $\neq 2, 3$. An elliptic curve E over $\mathbf{F}_q$ is defined by an equation

$$E : y^2 = x^3 + ax^2 + bx,$$

where $a, b \in \mathbf{F}_q$ and $b^2(a^2 - 4b) \neq 0$. The discriminant of E is

$$\Delta = 16b^2(a^2 - 4b).$$

If $\Delta = 0$, then E is not an elliptic curve is a singular curve. We can view an elliptic curve E as a curve in projective plane $\mathbf{P}^2$, with a homogeneous equation

$$y^2z = x^3 + ax^2z^2 + bxz^3,$$

and one point at infinity, namely $(0, 1, 0)$. This point ∞ is the point where all vertical lines meet. We denote this point by O. Let

$$E(\mathbf{F}_q) = \{(x, y) \in \mathbf{F}_q \times \mathbf{F}_q : y^2 = x^3 + ax^2 + bx\} \cup \{O\}$$

denote the set of rational points (x, y) on E. Then it is a subgroup of E. The order of $E(\mathbf{F}_q)$, denoted by $N_q = \#E(\mathbf{F}_q)$, is defined as the number of the rational points on E and is given by

$$\#E(\mathbf{F}_q) = q + 1 + \sum_{x \in \mathbf{F}_q} \left(\frac{x^3 + ax^2 + bx}{\mathbf{F}_q} \right),$$

where $(\frac{\cdot}{\mathbf{F}_q})$ denotes the Legendre symbol (for further details on elliptic curves see [10], [11], [16]).

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II. The Number of Rational Points on Singular Curves $y^2 = x(x-a)^2$ Over $\mathbf{F}_p$.

In [2], [12], [14], we considered some specific elliptic curves (including the number of rational points on them) over finite fields. In this section we will determine the number of rational points on singular curves

$$E_a : y^2 = x(x-a)^2 \tag{1}$$

over finite fields $\mathbf{F}_p$ for primes $p \geq 5$. Let

$$E_a(\mathbf{F}_p) = \{(x, y) \in \mathbf{F}_p \times \mathbf{F}_p : y^2 = x(x-a)^2\} \cup O.$$

Before we consider our problem we give some notations which we need them later.

Lemma 2.1: [5] Let p be an odd prime and let $f(x) \in \mathbf{Z}[x]$ be a polynomial of degree ≥ 1. Then the number $N_p(f)$ of solutions $(x, y) \in \mathbf{F}_p \times \mathbf{F}_p$ of the congruence $y^2 \equiv f(x)(mod\, p)$ is

$$N_p(f) = p + S_p(f), \tag{2}$$

where

$$S_p(f) = \sum_{x=0}^{p-1} (\frac{f(x)}{p}). \tag{3}$$

Also it is showed in [16] that for the polynomial $f(x) = (x-r)^2(x-s)$ of degree 3 for some $r, s \in \mathbf{F}_p$,

$$\sum_{x=0}^{p-1} (\frac{f(x)}{\mathbf{F}_p}) = -(\frac{r-s}{\mathbf{F}_p}). \tag{4}$$

Note that the $f(x) = x(x-a)^2$ is a polynomial of degree 3. So by considering the point 0, we can rewrite the formula (2) as

$$\begin{aligned} \#E_a(\mathbf{F}_p) &= p + 1 + \sum_{x=0}^{p-1} (\frac{x(x-a)^2}{p}) \\ &= p + 1 - (\frac{a}{p}) \end{aligned} \tag{5}$$

by (3) and (4). Therefore if $(\frac{a}{p}) = 1$, then $\#E_a(\mathbf{F}_p) = p$ and if $(\frac{a}{p}) = -1$, then $\#E_a(\mathbf{F}_p) = p + 2$. Therefore the order of E_a over $\mathbf{F}_p$ is depends on whether a is a quadratic residue or not.

Now we can give the following two theorems which I proved them in [13] and [15], respectively.

Theorem 2.1: Let $\mathbf{F}_p$ be a finite field. Then

$$\left(\frac{1}{p}\right) = 1 for\ every\ primes\ p \geq 5$$

$$\left(\frac{2}{p}\right) = \begin{cases} 1 & if\ p \equiv 1,7(8) \\ -1 & if\ p \equiv 3,5(8) \end{cases}$$

$$\left(\frac{3}{p}\right) = \begin{cases} 1 & if\ p \equiv 1,11(12) \\ -1 & if\ p \equiv 5,7(12) \end{cases}$$

$$\left(\frac{4}{p}\right) = 1\ for\ every\ primes\ p \geq 5$$

$$\left(\frac{5}{p}\right) = \begin{cases} 1 & if\ p \equiv 1,9(10) \\ -1 & if\ p \equiv 3,7(10) \end{cases}$$

$$\left(\frac{6}{p}\right) = \begin{cases} 1 & if\ p \equiv 1,5,19,23(24) \\ -1 & if\ p \equiv 7,11,13,17(24) \end{cases}$$

$$\left(\frac{7}{p}\right) = \begin{cases} 1 & if\ p \equiv 1,3,9,19,25,27(28) \\ -1 & if\ p \equiv 5,11,13,15,17,23(28) \end{cases}$$

$$\left(\frac{8}{p}\right) = \begin{cases} 1 & if\ p \equiv 1,7,17,23(24) \\ -1 & if\ p \equiv 5,11,13,19(24) \end{cases}$$

$$\left(\frac{9}{p}\right) = 1\ for\ every\ primes\ p \geq 11$$

$$\left(\frac{10}{p}\right) = \begin{cases} 1 & if\ p \equiv 1,3,9,13,27,31,37,39(40) \\ -1 & if\ p \equiv 7,11,17,19,21,23,29,33,37(40). \end{cases}$$

Theorem 2.2: Let $\mathbf{F}_p$ be a finite field. Then

$$\left(\frac{-1}{p}\right) = \begin{cases} 1 & if\ p \equiv 1(4) \\ -1 & if\ p \equiv 3(4) \end{cases}$$

$$\left(\frac{-2}{p}\right) = \begin{cases} 1 & if\ p \equiv 1,3(8) \\ -1 & if\ p \equiv 5,7(8) \end{cases}$$

$$\left(\frac{-3}{p}\right) = \begin{cases} 1 & if\ p \equiv 1,7(12) \\ -1 & if\ p \equiv 5,11(12) \end{cases}$$

$$\left(\frac{-4}{p}\right) = \begin{cases} 1 & if\ p \equiv 1,5(12) \\ -1 & if\ p \equiv 7,11(12) \end{cases}$$

$$\left(\frac{-5}{p}\right) = \begin{cases} 1 & if\ p \equiv 1,3,7,9(20) \\ -1 & if\ p \equiv 11,13,17,19(20) \end{cases}$$

$$\left(\frac{-6}{p}\right) = \begin{cases} 1 & if\ p \equiv 1,5,7,11,25,29,31,35(48) \\ -1 & if\ p \equiv 13,17,19,23,37,41,43,47(48) \end{cases}$$

$$\left(\frac{-7}{p}\right) = \begin{cases} 1 & if\ p \equiv 1,9,11,15,23,25(28) \\ -1 & if\ p \equiv 3,5,13,17,19,27(28) \end{cases}$$

$$\left(\frac{-8}{p}\right) = \begin{cases} 1 & if\ p \equiv 1,11,17,19,25,35,41,43(48) \\ -1 & if\ p \equiv 5,7,13,23,29,31,37,47(48) \end{cases}$$

$$\left(\frac{-9}{p}\right) = \begin{cases} 1 & if\ p \equiv 1,5,13,17(24) \\ -1 & if\ p \equiv 7,11,19,23(24) \end{cases}$$

$$\left(\frac{-10}{p}\right) = \begin{cases} 1 & if\ p \equiv 1,7,9,11,13,19,23,37(40) \\ -1 & if\ p \equiv 3,17,21,27,29,31,33,39(40). \end{cases}$$

Now we can consider our main problem.

Theorem 2.3: Let E_a be the singular curve defined in (1). Then

$$\#E_1(\mathbf{F}_p) = p\ for\ every\ primes\ p \geq 5$$

$$\#E_2(\mathbf{F}_p) = \begin{cases} p & if\ p \equiv 1,7(8) \\ p+2 & if\ p \equiv 3,5(8) \end{cases}$$

$$\#E_3(\mathbf{F}_p) = \begin{cases} p & if\ p \equiv 1,11(12) \\ p+2 & if\ p \equiv 5,7(12) \end{cases}$$

$$\#E_4(\mathbf{F}_p) = p\ for\ every\ primes\ p \geq 5$$

$$\#E_5(\mathbf{F}_p) = \begin{cases} p & if\ p \equiv 1,9(10) \\ p+2 & if\ p \equiv 3,7(10) \end{cases}$$

$$\#E_6(\mathbf{F}_p) = \begin{cases} p & if\ p \equiv 1,5,19,23(24) \\ p+2 & if\ p \equiv 7,11,13,17(24) \end{cases}$$

$$\#E_7(\mathbf{F}_p) = \begin{cases} p & if\ p \equiv 1,3,9,19,25,27(28) \\ p+2 & if\ p \equiv 5,11,13,15,17,23(28) \end{cases}$$

$$\#E_8(\mathbf{F}_p) = \begin{cases} p & if\ p \equiv 1,7,17,23(24) \\ p+2 & if\ p \equiv 5,11,13,19(24) \end{cases}$$

$$\#E_9(\mathbf{F}_p) = p\ for\ every\ primes\ p \geq 11$$

$$\#E_{10}(\mathbf{F}_p) = \begin{cases} p & if\ p \equiv 1,3,9,13,27,31,37,39(40) \\ p+2 & if\ p \equiv 7,11,17,19,21,23,29,33,37(40) \end{cases}$$

$$\#E_{-1}(\mathbf{F}_p) = \begin{cases} p & if\ p \equiv 1(4) \\ p+2 & if\ p \equiv 3(4) \end{cases}$$

$$\#E_{-2}(\mathbf{F}_p) = \begin{cases} p & if\ p \equiv 1,3(8) \\ p+2 & if\ p \equiv 5,7(8) \end{cases}$$

$$\#E_{-3}(\mathbf{F}_p) = \begin{cases} p & if\ p \equiv 1,7(12) \\ p+2 & if\ p \equiv 5,11(12) \end{cases}$$

$$\#E_{-4}(\mathbf{F}_p) = \begin{cases} p & if\ p \equiv 1,5(12) \\ p+2 & if\ p \equiv 7,11(12) \end{cases}$$

$$\#E_{-5}(\mathbf{F}_p) = \begin{cases} p & if\ p \equiv 1,3,7,9(20) \\ p+2 & if\ p \equiv 11,13,17,19(20) \end{cases}$$

$$\#E_{-6}(\mathbf{F}_p) = \begin{cases} p & if\ p \equiv 1,5,7,11,25,29,31,35(48) \\ p+2 & if\ p \equiv 13,17,19,23,37,41,43,47(48) \end{cases}$$

$$\#E_{-7}(\mathbf{F}_p) = \begin{cases} p & if\ p \equiv 1,9,11,15,23,25(28) \\ p+2 & if\ p \equiv 3,5,13,17,19,27(28) \end{cases}$$

$$\#E_{-8}(\mathbf{F}_p) = \begin{cases} p & if\ p \equiv 1,11,17,19,25,35,41,43(48) \\ p+2 & if\ p \equiv 5,7,13,23,29,31,37,47(48) \end{cases}$$

$$\#E_{-9}(\mathbf{F}_p) = \begin{cases} p & if\ p \equiv 1,5,13,17(24) \\ p+2 & if\ p \equiv 7,11,19,23(24) \end{cases}$$

$$\#E_{-10}(\mathbf{F}_p) = \begin{cases} p & if\ p \equiv 1,7,9,11,13,19,23,37(40) \\ p+2 & if\ p \equiv 3,17,21,27,29,31,33,39(40). \end{cases}$$

Proof: Applying Theorems 2.1 and 2.2 the result is clear. ■

Now we consider the sum of $x-$ and $y-$coordinates of all rational points (x,y) on E_a over F_p. Let $[x]$ and $[y]$ denote the $x-$and $y-$coordinates of the points (x,y) on E_a, respectively. Then we have the following the results.

Theorem 2.4: The sum of $[x]$ on E_a is

$$\sum_{[x]} E_a(\mathbf{F}_p) = \begin{cases} \frac{p^3-p-12a}{12} & if\ (\frac{a}{p})=1 \\ \frac{p^3-p+12a}{12} & if\ (\frac{a}{p})=-1. \end{cases}$$

Proof: Let $U_p=\{1,2,\cdots,p-1\}$ be the set of units in $\mathbf{F}_p$. Then then taking squares of elements in U_p, we would obtain the set of quadratic residues $Q_p=\{1^2,2^2,\cdots,(\frac{p-1}{2})^2\}$. Then the sum of all elements in Q_p hence

$$\sum_{x\in Q_p} x = \frac{p^3-p}{24}.$$

Now let $(\frac{a}{p})=1$. Then a is a quadratic residue. But for this values of a, there is one rational point $(a,0)$ on E_a. Let $H=Q_p-\{a\}$. Then

$$\begin{aligned} \sum_{x\in H} x &= \left(\sum_{x\in Q_p} x\right) - a \\ &= \frac{p^3-p}{24} - a \\ &= \frac{p^3-p-24a}{24}. \end{aligned}$$

We know that every element x of H makes $x(x-a)^2$ is a square. Let $x(x-a)^2\equiv t^2(mod\ p)$. Then $y^2\equiv t^2(mod\ p)$. So there are two rational points (x,t) and $(x,p-t)$ on E_a. The sum of $x-$coordinates of these two points is $2x$, that is, for every $x\in H$, the sum of $x-$coordinates of (x,t) and $(x,p-t)$ is $2x$. So the sum of $x-$coordinates of all points on E_a is

$$2\sum_{x\in H} x.$$

Further we said above that the point $(a,0)$ is also on E_a. Consequently

$$\sum_{[x]} E_a(\mathbf{F}_p) = 2\left(\sum_{x\in H} x\right) + a = \frac{p^3-p-12a}{12}.$$

Let $(\frac{a}{p})=-1$. Then a is not a quadratic residue. But every element x of Q_p makes $x(x-a)^2$ a square. So there are two rational points on E_a and hence the sum of $x-$coordinates of these two points is $2x$. Further $(a,0)$ is also a rational point on E_a. Consequently

$$\sum_{[x]} E_a(\mathbf{F}_p) = 2\left(\sum_{x\in Q_p} x\right) + a = \frac{p^3-p+12a}{12}.$$

■

Theorem 2.5: The sum of $[y]$ on E_a is

$$\sum_{[y]} E_a(\mathbf{F}_p) = \begin{cases} \frac{p^2-3p}{2} & if\ (\frac{a}{p})=1 \\ \frac{p^2-p}{2} & if\ (\frac{a}{p})=-1. \end{cases}$$

Proof: Let $(\frac{a}{p})=1$. Then a is a quadratic residue but again for this value of a, there is one rational point $(a,0)$ on E_a. Also every element x of Q_p makes $x(x-a)^2$ a square. Let $x(x-a)^2\equiv t^2(mod\ p)$. Then

$$y^2\equiv t^2(mod\ p) \Leftrightarrow y\equiv \pm t(mod\, p).$$

So there are two points (x,t) and $(x,p-t)$ on E_a. The sum of $y-$coordinates of these two points is p. We know that there are $\frac{p-1}{2}-1=\frac{p-3}{2}$ points x such that $x(x-a)^2$ is a square. So the sum of $y-$coordinates of all points (x,y) on E_a is

$$p(\frac{p-3}{2}) = \frac{p^2-3p}{2}.$$

Now let $(\frac{a}{p})=-1$. Then a is not a quadratic residue. But every element x of Q_p makes $x(x-a)^2$ a square. Let $x(x-a)^2\equiv t^2(mod\ p)$. Then

$$y^2\equiv t^2(mod\ p) \Leftrightarrow y\equiv \pm t(mod\, p).$$

So there are two points (x,t) and $(x,p-t)$ on E_a. The sum of $y-$coordinates of these two points is p. We know that there are $\frac{p-1}{2}$ points x in Q_p such that $x(x-a)^2$ is a square. So the sum of $y-$coordinates of all points (x,y) on E_a is

$$p(\frac{p-1}{2}) = \frac{p^2-p}{2}.$$

■

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