

# Octonionic Reformulation of Vector Analysis

Bhupendra C. S. Chauhan, P. S. Bisht and O. P. S. Negi

*Abstract*—According to celebrated Hurwitz theorem, there exists four division algebras consisting of  $\mathbb{R}$  (real numbers),  $\mathbb{C}$  (complex numbers),  $\mathbb{H}$  (quaternions) and  $\mathbb{O}$  (octonions). Keeping in view the utility of octonion variable we have tried to extend the three dimensional vector analysis to seven dimensional one. Starting with the scalar and vector product in seven dimensions, we have redefined the gradient, divergence and curl in seven dimension. It is shown that the identity  $n(n-1)(n-3)(n-7) = 0$  is satisfied only for 0, 1, 3 and 7 dimensional vectors. We have tried to write all the vector inequalities and formulas in terms of seven dimensions and it is shown that same formulas loose their meaning in seven dimensions due to non-associativity of octonions. The vector formulas are retained only if we put certain restrictions on octonions and split octonions.

*Keywords*—Octonions, Vector Space and seven dimensions

## I. INTRODUCTION

Octonions were first introduced in Physics by Jordan, Von Neuman and Wigner [1], who investigated a new finite Hilbert space, on replacing the complex numbers by Octonions [2]. According to celebrated Hurwitz theorem [3] there exists four-division algebra consisting of  $\mathbb{R}$  (real numbers),  $\mathbb{C}$  (complex numbers),  $\mathbb{H}$  (quaternions) and  $\mathbb{O}$  (octonions). All four algebras are alternative with antisymmetric associators. Real numbers and complex numbers are limited only up to two dimensions, quaternions are extended to four dimensions (one real and three imaginaries) while octonions represent eight dimensions (one scalar and seven vectors namely one real and seven imaginaries). In 1961, Pais [4] pointed out a striking similarity between the algebra of interactions and the split octonion algebra. Accordingly, some attention has been drawn to octonions in theoretical physics with the hope of extending the 3+1 space-time framework of the theory to eight dimensions in order to accommodate the ever increasing quantum numbers

Bhupendra C. S. Chauhan is in the Department of Physics, Kumaun University, S. S. J. Campus, Almora – 263601 Uttarakhand, India (email: bupendra.123@gmail.com).

P. S. Bisht is in the Department of Physics, Kumaun University, S. S. J. Campus, Almora – 263601 Uttarakhand, India (email: ps\_bisht123@rediffmail.com).

O. P. S. Negi permanent address: Department of Physics, Kumaun University, S. S. J. Campus, Almora – 263601 Uttarakhand, India (email: ops\_negi@yahoo.co.in); present address: from November 08- December 22, 2010: **Universität Konstanz, Fachbereich Physik, Postfach-M 677, D-78457 Konstanz, Germany**

and internal symmetries assigned to elementary particles and gauge fields. A lot of literature is available [5], [6], [7], [8], [9], [10], [11], [12] on the applications of octonions to interpret wave equation, Dirac equation, and the extension of octonion non - associativity to physical theories. Keeping in view the utility of octonion variable, in the present paper, we have tried to extend the three dimensional vector analysis to seven dimensional one. Starting with the scalar and vector product, we have redefined the gradient, divergence and curl in seven dimension with the definitions of octonion variables. It is shown that the identity  $n(n-1)(n-3)(n-7) = 0$  is satisfied only for 0, 1, 3 and 7 dimensional vectors. We have tried to write all the vector inequalities and formulas in terms of seven dimensions and it is shown that same formulas loose their meaning in seven dimensions due to non - associativity of octonions. In this context we have tried to reformulate the vector analysis and it is shown that some vector identities are losing their original form of three dimensional space in seven dimension vector space and looking for more generalized representation in seven dimension space.

## II. OCTONION DEFINITION

An octonion  $x$  is expressed as a set of eight real numbers

$$x = e_0 x_0 + \sum_{j=1}^7 e_j x_j \quad (1)$$

where  $e_j (\forall j = 1, 2, \dots, 7)$  are imaginary octonion units and  $e_0$  is the multiplicative unit element. Set of octets  $(e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7)$  are known as the octonion basis elements and satisfy the following multiplication rules

$$e_0 = 1; e_0 e_j = e_j e_0 = e_j; \\ e_j e_k = -e_k e_j + f_{jkl} e_l \quad (\forall j, k, l = 1, 2, \dots, 7). \quad (2)$$

The structure constants  $f_{jkl}$  is completely antisymmetric and takes the value 1 for following combinations,

$$f_{jkl} = +1$$

$$\forall(j, k, l) = (123), (471), (257), (165), (624), (543), (736) \quad (3)$$

It is to be noted that the summation convention is used for repeated indices. Here the octonion algebra  $\mathcal{O}$  is described over the algebra of real numbers having the vector space of dimension 8. Octonion conjugate is defined as

$$\bar{x} = e_0 x_0 - \sum_{j=1}^7 e_j x_j \quad (4)$$

where we have used the conjugates of basis elements as  $\bar{e}_0 = e_0$  and  $\bar{e}_A = -e_A$ . Hence an octonion can be decomposed in terms of its scalar ( $Sc(x)$ ) and vector ( $Vec(x)$ ) parts as

$$\begin{aligned} Sc(x) &= \frac{1}{2}(x + \bar{x}) \\ Vec(x) &= \frac{1}{2}(x - \bar{x}) = \sum_{j=1}^7 e_j x_j. \end{aligned} \quad (5)$$

Conjugates of product of two octonions is described as  $\overline{(xy)} = \bar{y} \bar{x}$  while the own conjugate of an octonion is written as  $\overline{(\bar{x})} = x$ . The scalar product of two octonions is defined as

$$\langle x, y \rangle = \sum_{\alpha=0}^7 x_\alpha y_\alpha. \quad (6)$$

The norm  $N(x)$  and inverse  $x^{-1}$  (for a nonzero  $x$ ) of an octonion are respectively defined as

$$\begin{aligned} N(x) = x\bar{x} = \bar{x}x &= \sum_{\alpha=0}^7 x_\alpha^2 \cdot e_0; \\ x^{-1} &= \frac{\bar{x}}{N(x)} \implies xx^{-1} = x^{-1}x = 1(7) \end{aligned}$$

The norm  $N(x)$  of an octonion  $x$  is zero if  $x = 0$ , and is always positive otherwise. It also satisfies the following property of normed algebra

$$N(xy) = N(x)N(y) = N(y)N(x). \quad (8)$$

Equation (2) directly leads to the conclusion that octonions are not associative in nature and thus do not form the group in their usual form. Non-associativity of octonion algebra  $\mathcal{O}$  is described by the associator  $(x, y, z) = (xy)z - x(yz) \forall x, y, z \in \mathcal{O}$  defined for any 3 octonions. If the associator is totally

antisymmetric for exchanges of any 2 variables, i.e.  $(x, y, z) = -(z, y, x) = -(y, x, z) = -(x, z, y)$ , the algebra is called alternative. Hence, the octonion algebra is neither commutative nor associative but, is alternative.

### III. MULTI-DIMENSIONAL VECTOR ANALYSIS

Following Silagadze [11], let us consider  $n$ -dimensional vector space  $\mathbb{R}^n$  over the field of real numbers with standard Euclidean scalar product. So, it is natural to describe the generalization of usual 3-dimensional vector space to  $n$ -dimensional vector space in order to represent the following three dimensional vector products of two vectors in  $n$ -dimensional vector space i.e.

$$\vec{A} \times \vec{A} = 0, \quad (9)$$

$$(\vec{A} \times \vec{B}) \cdot \vec{A} = (\vec{A} \times \vec{B}) \cdot \vec{B} = 0, \quad (10)$$

$$|\vec{A} \times \vec{B}| = |\vec{A}| |\vec{B}|, \text{ if } (\vec{A} \cdot \vec{B}) = 0, \quad (11)$$

$$(\vec{A} \times \vec{B}) \cdot (\vec{A} \times \vec{B}) = (\vec{A} \cdot \vec{A})(\vec{B} \cdot \vec{B}) - (\vec{A} \cdot \vec{B})^2, \quad (12)$$

$$\vec{A} \times (\vec{B} \times \vec{A}) = (\vec{A} \cdot \vec{A}) \vec{B} - (\vec{A} \cdot \vec{B}) \vec{A}, \quad (13)$$

where  $\vec{A}$  and  $\vec{B}$  are the vectors in  $n$ -dimensional vector space. However the familiar identity

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B} (\vec{A} \cdot \vec{C}) - \vec{C} (\vec{A} \cdot \vec{B}) \quad (14)$$

is not satisfied in general for all values of  $n$ -dimensional vector space. This identity (14) is verified only for the space dimension  $n$  satisfying [11] the equation

$$n(n-1)(n-3)(n-7) = 0. \quad (15)$$

As such, the space dimension must be equal to magic number seven for the unique generalization of ordinary three dimensional vector products. It shows that the identity (15) is satisfied only for  $n = 0, 1, 3$  and  $7$  dimensional vectors. The identity (15) is thus the direct consequence of celebrated Hurwitz theorem [3] which shows that there exists four-division algebra consisting of  $\mathbb{R}$  (real numbers) ( $n = 0$ ),  $\mathbb{C}$  (complex numbers) ( $n = 1$ ),  $\mathbb{H}$  (quaternions) ( $n = 3$ ) and  $\mathcal{O}$  (octonions) ( $n = 7$ ).

### IV. OCTONION ANALYSIS OF VECTOR SPACE

Using the octonion multiplication rules (2), we may define the seven dimensional vector product as

$$\vec{e}_j \times \vec{e}_k = \sum_{l=1}^{k=7} f_{jkl} \vec{e}_l, \quad \forall j, k, l = 1, 2, \dots, 6, 7 \quad (16)$$

where  $f_{jkl}$  is described by equation (3). It is a totally  $G_2$ -invariant anti-symmetric tensor. As such we have

$$f_{jkl} f_{lmn} = g_{jkmn} + \delta_{jm} \delta_{kn} - \delta_{jn} \delta_{km} = -f_{lmn} f_{jkl} \quad (17)$$

where  $g_{jkmn} = \vec{e}_j \cdot \{\vec{e}_k, \vec{e}_m, \vec{e}_n\}$  is a totally  $G_2$ -invariant anti-symmetric tensor [11] and  $g_{jkmn} = -g_{mkjn}$ . The only independent components are  $g_{1254} = g_{1267} = g_{1364} = g_{1375} = g_{2347} = g_{2365} = g_{4576} = 1$ . Thus, we may write the left hand side of equation (14) as

$$\begin{aligned} \vec{A} \times (\vec{B} \times \vec{C}) &= - \sum_{jk} \sum_{pq} g_{pqjk} A_j B_p C_q \hat{e}_k \\ &\quad - \sum_{jk} A_j B_j C_k \hat{e}_k + \sum_{jk} A_j C_j B_k \hat{e}_k \\ &= \text{irreducible term} + \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B}). \end{aligned} \quad (18)$$

Hence, the identity of Triple Product in seven dimensional vector space is not satisfied unless we apply the definition of octonion to define the irreducible term as a ternary product such that

$$\text{irreducible term} = - \sum_{jk} \sum_{pq} g_{pqjk} A_j B_p C_q \hat{e}_k = \{ \vec{A}, \vec{B}, \vec{C} \}, \quad (19)$$

which gives rise to

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B}) + \{ \vec{A}, \vec{B}, \vec{C} \}. \quad (20)$$

The  $\{ \vec{A}, \vec{B}, \vec{C} \}$  is called as the associator for the case of octonions. Applying the condition of alternativity to octonions the equation (20) reduces to the well known vector identity (14) for the various permutation values of structure constant  $f_{jkl}$  for which the associator is going to be vanished.

Let us use the vector calculus for seven dimensional vector space from the definition of octonion variables. Defining the seven dimensional differential operator *Nabla* as  $\vec{\nabla} = \sum_i \hat{e}_i \frac{\partial}{\partial x_i}$ , we may now define the gradient, curl and divergence of scalar and vector quantities as

$$\text{grad } u = \vec{\nabla} u = \sum_i \hat{e}_i \frac{\partial u}{\partial x_i}, \quad (21)$$

$$\text{curl } \vec{A} = \vec{\nabla} \times \vec{A} = \sum_{ijk} f_{ijk} \frac{\partial A_j}{\partial x_i} \hat{e}_k, \quad (22)$$

$$\text{div } \vec{A} = \vec{\nabla} \cdot \vec{A} = \sum_i \frac{\partial A_i}{\partial x_i}. \quad (23)$$

The Divergent of a curl is zero in usual three dimensional vector-space. This is also applicable to seven dimensional vector-space if we adopt the octonion multiplication rules (2). We may also prove it in the following manner as

$$\begin{aligned} \vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) &= \sum_l \left[ \hat{e}_l \frac{\partial}{\partial x_l} \right] \cdot \left[ \sum_{ijk} f_{ijk} \frac{\partial A_j}{\partial x_i} \hat{e}_k \right] \\ &= \sum_l \sum_{ijk} f_{ijk} \frac{\partial^2 A_j}{\partial x_l \partial x_i} \delta_{lk} \\ &= \sum_{ijk} f_{ijk} \frac{\partial^2 A_j}{\partial x_k \partial x_i} \\ &= \frac{1}{2} \sum_{ijk} [f_{ijk} + f_{kji}] \frac{\partial^2 A_j}{\partial x_i \partial x_k} = 0 \end{aligned} \quad (24)$$

Hence curl of a vector is solenoidal in seven dimensional vector space. We know that the curl of gradient of a vector is zero in 3-dimensional vector-space. thus we see that it also happens for the case of 7- dimensional vector- space by adopting the octonion multiplication rules (2 ) in the following way as

$$\begin{aligned} \vec{\nabla} \times (\vec{\nabla} u) &= \sum_{ijk} f_{ijk} \frac{\partial}{\partial x_i} \left( \sum_j \frac{\partial u}{\partial x_j} \right) \hat{e}_k \\ &= \sum_{ijk} f_{ijk} \frac{\partial^2 u}{\partial x_i \partial x_j} \hat{e}_k = \frac{1}{2} \sum_{ijk} [f_{ijk} + f_{jik}] \frac{\partial^2 u}{\partial x_j \partial x_i} \hat{e}_k = 0. \end{aligned} \quad (25)$$

Thus, the gradient of a curl is also irrotational in seven dimensional vector space with the definition of octonions. Let us see what happens to the vector identities

$$\vec{\nabla} \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\vec{\nabla} \times \vec{A}) - \vec{A} \cdot (\vec{\nabla} \times \vec{B}) \quad (26)$$

in seven dimensional vector-space. The left hand side of equation (26) reduces to

$$\begin{aligned}\vec{\nabla} \cdot (\vec{A} \times \vec{B}) &= \sum_l^7 \hat{e}_l \frac{\partial}{\partial x_l} \cdot \left[ \sum_{ijk}^7 f_{ijk} A_i B_j \hat{e}_k \right] \\ &= \sum_{ijk}^7 \sum_l^7 f_{ijk} \frac{\partial}{\partial x_l} [A_i B_j] \delta_{ij} \\ &= \sum_{ijk}^7 f_{ijk} \left[ A_i \frac{\partial B_j}{\partial x_k} + B_j \frac{\partial A_i}{\partial x_k} \right] \quad (27)\end{aligned}$$

while the right hand side of equation (26) changes to

$$\begin{aligned}\vec{B} \cdot (\vec{\nabla} \times \vec{A}) - \vec{A} \cdot (\vec{\nabla} \times \vec{B}) &= \sum_l^7 \hat{e}_l B_l \cdot \left[ \sum_{mnt}^7 f_{mnt} \frac{\partial A_n}{\partial x_m} \hat{e}_t \right] \\ &\quad - \sum_s^7 \hat{e}_s A_s \cdot \left[ \sum_{pqr}^7 f_{pqr} \frac{\partial B_q}{\partial x_p} \hat{e}_r \right] \\ &= \sum_{ijk}^7 f_{ijk} \left[ A_i \frac{\partial B_j}{\partial x_k} + B_j \frac{\partial A_i}{\partial x_k} \right], \quad (28)\end{aligned}$$

which is equal to the left hand side equation (27). Hence the identity (26) has been satisfied for seven dimensional space. Similarly, on using the octonion multiplication rules (2) we may prove the following identities in seven dimensions i.e.

$$\vec{\nabla} \cdot (u\vec{A}) = u(\vec{\nabla} \cdot \vec{A}) + \vec{A} \cdot (\vec{\nabla} u), \quad (29)$$

$$\vec{\nabla} \times (u\vec{A}) = u(\vec{\nabla} \times \vec{A}) + (\vec{\nabla} u) \times \vec{A}, \quad (30)$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A}. \quad (31)$$

However the vector identity

$$\begin{aligned}\vec{\nabla} \times (\vec{A} \times \vec{B}) &= (\vec{B} \cdot \vec{\nabla})\vec{A} - (\vec{A} \cdot \vec{\nabla})\vec{B} \\ &\quad - \vec{B}(\vec{\nabla} \cdot \vec{A}) + \vec{A}(\vec{\nabla} \cdot \vec{B})\end{aligned} \quad (32)$$

is not satisfied and its left hand side reduces to

$$\begin{aligned}\vec{\nabla} \times (\vec{A} \times \vec{B}) &= \sum_{wkt}^7 f_{wkt} \frac{\partial}{\partial x_w} \left[ \sum_{ijk}^7 f_{ijk} A_i B_j \right] \hat{e}_t \\ &= - \sum_{wkt}^7 \sum_{ijk}^7 f_{ijk} f_{kwt} \frac{\partial}{\partial x_w} [A_i B_j] \hat{e}_t \quad (33)\end{aligned}$$

whereas the right hand expression of equation (32) becomes

$$\begin{aligned}(\vec{B} \cdot \vec{\nabla})\vec{A} - (\vec{A} \cdot \vec{\nabla})\vec{B} - \vec{B}(\vec{\nabla} \cdot \vec{A}) + \vec{A}(\vec{\nabla} \cdot \vec{B}) \\ = \sum_{ij}^7 \frac{\partial A_i}{\partial x_j} B_j \hat{e}_i - \sum_{lm}^7 A_l \frac{\partial B_m}{\partial x_l} \hat{e}_m \\ - \sum_{ij}^7 \frac{\partial A_i}{\partial x_i} B_j \hat{e}_j + \sum_{ij}^7 A_i \frac{\partial B_j}{\partial x_j} \hat{e}_i. \quad (34)\end{aligned}$$

Comparing equations (33) and (34), we get

$$\begin{aligned}\vec{\nabla} \times (\vec{A} \times \vec{B}) &= (\vec{B} \cdot \vec{\nabla})\vec{A} - (\vec{A} \cdot \vec{\nabla})\vec{B} \\ &\quad - \vec{B}(\vec{\nabla} \cdot \vec{A}) + \vec{A}(\vec{\nabla} \cdot \vec{B}) \\ &\quad + \text{irreducible term} \quad (35)\end{aligned}$$

where the irreducible term takes the expression

$$\begin{aligned}\text{irreducible term} &= - \sum_{wt}^7 \sum_{ij}^7 g_{ijwt} \frac{\partial}{\partial x_w} [A_i B_j] \hat{e}_t \\ &= - \sum_{wt}^7 \sum_{ij}^7 g_{ijwt} \frac{\partial}{\partial x_w} [A_i B_j] \hat{e}_t \\ &= \sum_t^7 \hat{e}_t \left[ \vec{\nabla} \cdot \{ \hat{e}_t, \vec{A}, \vec{B} \} \right]. \quad (36)\end{aligned}$$

Hence we get

$$\begin{aligned}\vec{\nabla} \times (\vec{A} \times \vec{B}) + \sum_t^7 \hat{e}_t \left[ \vec{\nabla} \cdot \{ \hat{e}_t, \vec{A}, \vec{B} \} \right] \\ = (\vec{B} \cdot \vec{\nabla})\vec{A} - (\vec{A} \cdot \vec{\nabla})\vec{B} - \vec{B}(\vec{\nabla} \cdot \vec{A}) + \vec{A}(\vec{\nabla} \cdot \vec{B}). \quad (37)\end{aligned}$$

Thus we see that the identity (32) is satisfied only when we have the reducible term vanishing and that can be obtained for different values of permutations of structure constant given by equation (3).

Similarly the other vector identity

$$\begin{aligned}\vec{A} \times (\vec{\nabla} \times \vec{B}) + \vec{B} \times (\vec{\nabla} \times \vec{A}) \\ + (\vec{A} \cdot \vec{\nabla})\vec{B} + \vec{B}(\vec{\nabla} \cdot \vec{A}) \\ = \vec{\nabla}(\vec{A} \cdot \vec{B}) \quad (38)\end{aligned}$$

may also be verified after doing certain modifications also. For this, let us take the first and third terms of left hand side of equation (38) as

REFERENCES

$$\vec{A} \times (\vec{\nabla} \times \vec{B}) + (\vec{A} \cdot \vec{\nabla}) \vec{B} = \sum_{ikst} g_{stik} A_i \frac{\partial B_t}{\partial x_s} \hat{e}_k + \sum_{ik} A_i \frac{\partial B_i}{\partial x_k} \hat{e}_k \quad (39)$$

and the other two terms of left hand side of equation (38) are written as

$$\vec{B} \times (\vec{\nabla} \times \vec{A}) + \vec{B} (\vec{\nabla} \cdot \vec{A}) = \sum_{ikst} g_{stik} B_i \frac{\partial A_t}{\partial x_s} \hat{e}_k + \sum_{ik} B_i \frac{\partial A_i}{\partial x_k} \hat{e}_k \quad (40)$$

However, the right hand side of (38) reduces to

$$\vec{\nabla} (\vec{A} \cdot \vec{B}) = \sum_{ik} A_i \frac{\partial B_i}{\partial x_k} \hat{e}_k + \sum_{ik} B_i \frac{\partial A_i}{\partial x_k} \hat{e}_k. \quad (41)$$

As such, it is concluded that the general form of equation (38) of three dimensional vector space is not satisfied for octonions with the fact that this equation becomes

$$\vec{A} \times (\vec{\nabla} \times \vec{B}) + (\vec{A} \cdot \vec{\nabla}) \vec{B} + \vec{B} \times (\vec{\nabla} \times \vec{A}) + (\vec{B} \cdot \vec{\nabla}) \vec{A} = \vec{\nabla} (\vec{A} \cdot \vec{B}) + \text{irreducible term}. \quad (42)$$

Applying the alternativity relations for octonion basis elements, we see that the the irreducible term is reduced as

$$\begin{aligned} \text{irreducible term} &= - \left[ \sum_{ikst} g_{stik} A_i \frac{\partial B_t}{\partial x_s} \hat{e}_k + \sum_{ikst} g_{stik} B_i \frac{\partial A_t}{\partial x_s} \hat{e}_k \right] \\ &= - \left[ \sum_k \hat{e}_k [\vec{A} \cdot \{\hat{e}_k, \vec{\nabla}, \vec{B}\}] + \sum_k \hat{e}_k [\vec{B} \cdot \{\hat{e}_k, \vec{\nabla}, \vec{A}\}] \right] \\ &= 0 \end{aligned} \quad (43)$$

Hence the identity (38) is verified for seven dimensional vector space in terms of octonion basis elements..

ACKNOWLEDGMENT

One of us OPSN is thankful to Professor H. Dehnen, Universität Konstanz, Fachbereich Physik, Postfach-M 677, D-78457 Konstanz, Germany for his kind hospitality at Universität Konstanz. He is also grateful to German Academic Exchange Service (Deutscher Akademischer Austausch Dienst), Bonn for their financial support under DAAD re-invitation programme.

[1] P. Jordan, J Neuman, Von and E. P. Wigner, Ann. Math., **35**, (1934), 29.  
 [2] A. Cayley, Philos. Mag. **26**, (1845), 208; R. P. Graves, "Life of Sir William Rowan Hamilton", 3 volumes, Arno Press, New York, 1975; J. C. Baez, Bull. Amer. Math. Soc., **39**, (2001), 145; J. H. Conway, "On Quaternions and Octonions: Their Geometry, Arithmetic, and Symmetry," A K Peters Ltd, Massachusetts, 2003.  
 [3] L. E. Dickson, Ann. Math., **20**, (1919), 155; W. R. Hamilton, "Elements of quaternions", Chelsea Publications Co., NY, (1969); P.G.Tait, "An elementary Treatise on Quaternions", Oxford Univ. Press (1875).  
 [4] A. Pais, Phys. Rev. Lett., **7**, (1961), 291.  
 [5] R. Penny, Am. J. Phys., **36**, (1968), 871; Nuovo Cimento, **B3**, (1971), 95.  
 [6] K. Imaeda, H. Tachibana, M. Imaeda and S.Ohta, IL Nuovo Cimento, **B100**, (1987), 53; **B105**, (1990) 1203; Bull. Okayama Univ. Sci., **A24**, (1989) 181; K. Imaeda and H.Tachibana, Il Nuovo Cimento, ; **B104** (1989) 91; Bull. Okayama Univ. Sci., **A19**, (1984), 93; H. Tachibana; Ph.D.Thesis, Okayama Univ. Science (1990).  
 [7] G. C. Joshi, Lett. Nuovo Cimento, **44**, (1985), 449; A. J. Davis and G. C. Joshi, J. Math. Phys., **27**, (1986), 3036; R. Foot and G. C. Joshi, Int. J. Theor. Phys., **28**, (1989), 1449; Phys. Lett., **B199**, (1987), 203.  
 [8] A. Gamba, J. Math. Phys., **8**, (1967), 775; Il Nuovo Cimento, **A111**, (1998) 293; M. Gogberashvili, J. Phys. A: Math.Gen., **39**, (2006) 7099.  
 [9] G. M. Dixon, "Division Algebras: Octonions, Quaternions, Complex Numbers, and the Algebraic Design of Physics", (Kluwer, 1994).  
 [10] S. Okubo, "Introduction to Octonion and Other Non-Associative Algebras in Physics", Montreal Memorial Lecture Series in Mathematical Physics, 2, Cambridge University Press, 1995; C.H. Barton and A. Sudbery, "MAGIC SQUARES OF LIE ALGEBRAS" arXiv:math.RA/0001083.  
 [11] Z. K. Silagadze, J. Phys. A: Math. Gen., **35**, (2002) 4949; M. Rost, Documenta Mathematica, **1**, (1996) 209; J. A. Nieto and L. N. Alejo-Armenta, Int. J. Mod. Phys., **A16**, (2001) 4207.  
 [12] P. S. Bisht and O. P. S. Negi, Ind. J. P. Appl. Phys., **31**, 292 (1993); **32**, 183 (1994); Int. J. Theor. Phys., **47**, 1497 (2008); P. S. Bisht, Shalini Dangwal and O. P. S. Negi, Int. J. Theor. Phys., **47**, 2297 (2008); P. S. Bisht, B. Pandey and O. P. S. Negi, FIZIKA (Zagreb), **B17**, 405 (2008); P. S. Bisht and O.P.S.Negi, Pramana (India), **73**, 105 (2009); B. C. Chanyal, P. S. Bisht and O. P. S. Negi, Int. J. Theor. Phys., **49**, 1333 (2010).