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On modified numerical schemes in vortex element method for 2D flow simulation around airfoils

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Abstract—The problem of incompressible steady flow simulation around an airfoil is discussed. For some simplest airfoils (circular, elliptical, Zhukovsky airfoils) the exact solution is known from complex analysis. It allows to compute the intensity of vortex layer which simulates the airfoil. Some modifications of the vortex element method are proposed and test computations are carried out. It's shown that the these approaches are much more effective in comparison with the classical numerical scheme.

Keywords—Vortex element method, vortex layer, integral equation, ill-conditioned matrix.

I. Introduction

FLOW simulation around an airfoil is a very important problem for number of engineering applications. Different numerical methods have been developed for its solving, most of them presuppose mesh generation in flow region. But there are also the so-called 'meshfree', or 'lagrangian' numerical methods which don't need mesh in flow region at all. Vortex element method [1]–[5] is one of these methods and it is especially effective for flow simulating in aeroelastical problems when the airfoil can be not rigid or it can be elastically fixed. When using vortex element method the airfoil is simulated with vortex layer on airfoil's surface. Its intensity depends on time, so it should be computed at every time step. Accuracy of vortex layer intensity computation defines the accuracy of boundary condition satisfaction on the airfoil's surface and consequently the accuracy of vortex wake simulation near the airfoil. However, the existing well-known numerical schemes, normally being used in vortex element method, sometimes lead to significant errors, especially when simulating flow around airfoils with angular points or sharp edges (wing airfoils). The aim of this paper is to develop some numerical schemes for vortex element method which allow to compute vortex layer intensity on airfoil surface more accurately in comparison with 'classical' schemes.

II. Governing equations

Viscous incompressible media movement is described by the equation of continuity

$$\nabla \cdot \boldsymbol{V} = 0$$

and Navier-Stokes equations

$$\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V} \cdot \nabla)\boldsymbol{V} = \nu \Delta \boldsymbol{V} - \nabla \left(\frac{p}{\rho} \right),$$

where $\boldsymbol{V}(\boldsymbol{r}, t)$ — flow velocity, $p(\boldsymbol{r}, t)$ — pressure, $\rho = \text{const}$ — density of the media, ν — kinematic viscosity coefficient. No-slip boundary condition on the airfoil surface

$$\boldsymbol{V}(\boldsymbol{r}, t) = 0, \quad \boldsymbol{r} \in K,$$

and boundary conditions of perturbation decay on infinity

$$\boldsymbol{V}(\boldsymbol{r}, t) \to \boldsymbol{V}_\infty, \quad p(\boldsymbol{r}, t) \to p_\infty, \quad |\boldsymbol{r}| \to \infty,$$

should be satisfied.

Navier-Stokes equations could be written down in Helmholtz form using vorticity vector $\boldsymbol{\Omega}(\boldsymbol{r}, t) = \nabla \times \boldsymbol{V}(\boldsymbol{r}, t)$:

$$\frac{\partial \boldsymbol{\Omega}}{\partial t} + \nabla \times (\boldsymbol{\Omega} \times \boldsymbol{U}) = 0. \qquad (1)$$

Here $\boldsymbol{U}(\boldsymbol{r}, t) = \boldsymbol{V}(\boldsymbol{r}, t) + \boldsymbol{W}(\boldsymbol{r}, t)$, $\boldsymbol{W}(\boldsymbol{r}, t)$ is the so-called diffusive velocity, which is proportional to viscosity coefficient:

$$\boldsymbol{W}(\boldsymbol{r}, t) = \nu \frac{(\nabla \times \boldsymbol{\Omega}) \times \boldsymbol{\Omega}}{|\boldsymbol{\Omega}|^2}.$$

If vorticity distribution is known, flow velocity can be computed using Biot-Savart law:

$$\boldsymbol{V}(\boldsymbol{r}, t) = \boldsymbol{V}_\infty + \frac{1}{2\pi} \int_S \frac{\boldsymbol{\Omega}(\boldsymbol{\xi}, t) \times (\boldsymbol{r} - \boldsymbol{\xi})}{|\boldsymbol{r} - \boldsymbol{\xi}|^2} dS.$$

Equation (1) means that vorticity which exists in the flow moves and its velocity is $\boldsymbol{U}$. 'New' vorticity is being generated only on airfoil surface. This vortex layer influence on the flow is equivalent to streamlined airfoil influence, so vortex layer intensity can be found from boundary condition on airfoil surface. We assume that there is no vorticity in the flow and we need to compute vortex layer intensity on airfoil surface. From mathematical point of view this problem is equivalent to ideal incompressible steady flow simulation around the airfoil. In real unsteady viscous flow similar problem should be solved at every time step.

III. Exact solution for simplest airfoils

Using methods of complex analysis, exact solutions for vortex layer intensity in ideal incompressible steady flow could be found for some simplest airfoils (circular, elliptical, Zhukovsky airfoils) [6]. Vortex layer intensity is equal to tangent projection of velocity on airfoil surface. Complex value of flow velocity on airfoil surface can be found using the following formula:

$$V^*(p) = \frac{R\,|\boldsymbol{V}_\infty| \sin(\phi + \beta - p) + \dfrac{G}{2\pi}}{\dfrac{i\,R\,e^{i(p-\phi)}}{2} \left(1 - \dfrac{a^2}{(Re^{i(p-\phi)} + H)^2}\right)}.$$

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Here V^* means complex conjugate quantity to velocity V, $p \in [0, 2\pi)$ defines the point on airfoil surface, β — angle of incidence.

For elliptical airfoil

$$a = \sqrt{a_1^2 - b_1^2}, \quad R = a_1 + b_1, \quad \phi = 0, \quad H = 0,$$

a_1 and b_1 are major and minor semiaxes of the ellipse.

For Zhukovsky airfoil

$$R = \sqrt{(a + d\cos\phi)^2 + (h + d\sin\phi)^2},$$

$$\phi = \arctan\frac{h}{a}, \quad H = ih - de^{-i\phi},$$

a, d and h are arbitrary parameters, which correspond to length, width and curvature of the airfoil.

G is flow velocity circulation; for elliptical airfoil it can be chosen arbitrarily (from mathematical point of view), we assume it to be equal to zero independently on angle of incidence, while for Zhukovsky airfoil it is proportional to uniform flow velocity and depends on the airfoil shape and its angle of incidence:

$$G = -2\pi|\boldsymbol{V}_\infty|\sin(\beta + \phi)(\sqrt{h^2 + a^2} + d).$$

Using the previous formulae we can obtain the exact solution for vortex layer intensity. These exact solutions will be used for numerical schemes comparison and their accuracy estimation.

IV. Vortex element method

We consider model problem of ideal incompressible flow simulation around an airfoil. Vorticity is equal to zero everywhere in flow region and the airfoil is simulated with thin vortex layer with intensity $\gamma(p_0) = \gamma(x_0, y_0)$ on airfoil surface K. In this problem velocity vector $\boldsymbol{V} = (v_x, v_y, 0)^T$ can be determined in every point $\boldsymbol{r} = (x, y, 0)^T$ in flow region using Biot-Savart law (point $\boldsymbol{r}_0 = (x_0, y_0, 0)^T$ lies on the airfoil surface K, $\boldsymbol{\gamma}(\boldsymbol{r}_0) = \gamma(\boldsymbol{r}_0)\boldsymbol{k}$ is vortex layer intensity vector, $\boldsymbol{k} = (0, 0, 1)^T$):

$$\boldsymbol{V}(\boldsymbol{r}) = \boldsymbol{V}_\infty + \oint_K \frac{\boldsymbol{\gamma}(\boldsymbol{r}_0) \times (\boldsymbol{r} - \boldsymbol{r}_0)}{2\pi|\boldsymbol{r} - \boldsymbol{r}_0|^2} dl_{r_0}.$$

Limit values of flow velocity on the airfoil surface are equal to

$$\boldsymbol{V}_\pm(\boldsymbol{r}) = \boldsymbol{V}_\infty + \oint_K \frac{\boldsymbol{\gamma}(\boldsymbol{r}_0) \times (\boldsymbol{r} - \boldsymbol{r}_0)}{2\pi|\boldsymbol{r} - \boldsymbol{r}_0|^2} dl_{r_0} \pm \pm \left(\frac{\boldsymbol{\gamma}(\boldsymbol{r})}{2} \times \boldsymbol{n}(\boldsymbol{r})\right).$$

Here $\boldsymbol{n}(\boldsymbol{r})$ is unit normal vector on airfoil surface in point $\boldsymbol{r}$, $\boldsymbol{V}_+(\boldsymbol{r})$ corresponds to limit value of velocity from flow side, $\boldsymbol{V}_-(\boldsymbol{r})$ corresponds to limit value of velocity from airfoil side.

In order to determine vortex layer intensity γ we should solve equation $\boldsymbol{V}_-(\boldsymbol{r}) = 0$ on the airfoil surface. It can be easily shown [3] that we can solve scalar equation

$$\boldsymbol{n}(\boldsymbol{r}) \cdot \boldsymbol{V}_-(\boldsymbol{r}) = 0$$

or scalar equation

$$\boldsymbol{\tau}(\boldsymbol{r}) \cdot \boldsymbol{V}_-(\boldsymbol{r}) = 0$$

instead of vector equation $\boldsymbol{V}_-(\boldsymbol{r}) = 0$. From mathematical point of view there is no difference between solutions of these equations, but from computational point of view these approaches are very different.

A. Classical numerical scheme for vortex element method

In 'classical' approach [1], [5] vortex layer intensity on airfoil surface is assumed to be piecewise constant function and it satisfies equation $\boldsymbol{n} \cdot \boldsymbol{V}_- = 0$, which corresponds to zero velocity normal component on airfoil surface and leads to singular integral equation

$$\oint_K \frac{\boldsymbol{n}(\boldsymbol{r}) \cdot [\boldsymbol{k} \times (\boldsymbol{r} - \boldsymbol{r}_0)]}{2\pi|\boldsymbol{r} - \boldsymbol{r}_0|^2} \gamma(\boldsymbol{r}_0)\, dl_{r_0} = -\boldsymbol{n}(\boldsymbol{r}) \cdot \boldsymbol{V}_\infty. \quad (2)$$

It should be noted that solution of (2) certainly exists due to form of right side of this equation, but it is not unique. In order to select unique solution additional integral condition should be added:

$$\oint_K \gamma(\boldsymbol{r}) dl_r = G. \quad (3)$$

Kernel of equation (2) is unbounded and it has nonintegrable singularity when $|\boldsymbol{r} - \boldsymbol{r}_0| \to 0$, and special numerical schemes are used for Cauchy principal value computation. They allow to obtain the solution of linear system approximating (2) with high accuracy when number of collocating points on the airfoil is large and its surface is smooth curve. It is proved [1] that numerical solution converges to exact one in some integral norm. This 'classical' method we will call 'NVEM' (Vortex element method with normal components of velocity on airfoil surface).

At the same time if we simulate flow around the airfoil with angular points or sharp edges using NVEM, difference between numerical and exact solutions (in uniform norm) becomes significant and it increases proportionally to number of collocating points on airfoil surface. So it is impossible to determine vortex layer intensity with high accuracy. So well-known numerical schemes, which are effective in vortex element method for inviscous fluids, can be generalized for viscous case for smooth airfoils, but they can't be applied for 2D Navier-Stockes equations solution for airfoils with angular points and sharp edges. The main problem is that in viscous case all vortex elements generated on airfoil surface become part of vortex wake near the airfoil.

It also should be noted that linear algebraic system corresponding to (2) becomes ill-conditioned for airfoils with angular points or sharp edges.

B. Modified numerical schemes for vortex element method

In this paper some other approaches are developed which are equal to previous one from analytical point of view. Vortex layer intensity will be determined from solution of equation $\boldsymbol{\tau} \cdot \boldsymbol{V}_- = 0$, corresponding to zero tangent component of flow velocity limit value [2]. It leads to Fredholm integral equation with bounded (in case of smooth airfoils) kernel:

$$\oint_K \frac{\boldsymbol{\tau}(\boldsymbol{r}) \cdot [\boldsymbol{k} \times (\boldsymbol{r} - \boldsymbol{r}_0)]}{2\pi|\boldsymbol{r} - \boldsymbol{r}_0|^2} \gamma(\boldsymbol{r}_0) dl_{r_0} - \frac{\gamma(\boldsymbol{r})}{2} = = -\boldsymbol{\tau}(\boldsymbol{r}) \cdot \boldsymbol{V}_\infty. \quad (4)$$

Here $\boldsymbol{\tau}(\boldsymbol{r})$ is unit tangent vector on airfoil surface.

Solution of equation (4) is also non-unique, so the same additional condition (3) as in classical method is used. This method we will call 'TVEM' (Vortex element method with tangent components of velocity on airfoil surface).

Equation (4) also can be approximated with linear algebraic system which is well-conditioned both for smooth and non-smooth airfoils. Due to equation kernel boundness an arbitrary quadrature formula can be used for integral approximation in (4). In the simplest case we also can consider vortex layer intensity to be piecewise constant function.

Results of numerical experiments show that errors are sufficiently big, but they could be significantly decreased if we consider some 'weak' formulation of (4): integral equation (4) in discrete numerical scheme will be satisfied not in separate collocation points $\boldsymbol{r}_j$, $j = 1, \dots, N$, of airfoil surface, but on an average on airfoil surface parts (panels) K_p whose lengths are L_p, $p = 1, \dots, N$:

$$\frac{1}{L_p}\int_{K_p}\left[\oint_K \frac{\boldsymbol{\tau}(\boldsymbol{r})\cdot[\boldsymbol{k}\times(\boldsymbol{r}-\boldsymbol{r}_0)]}{2\pi|\boldsymbol{r}-\boldsymbol{r}_0|^2}\gamma(\boldsymbol{r}_0)dl_{r_0}\right]dl_r - \\ -\frac{1}{L_p}\int_{K_p}\frac{\gamma(\boldsymbol{r})}{2}dl_r = -\frac{1}{L_p}\int_{K_p}\big[\boldsymbol{\tau}(\boldsymbol{r})\cdot\boldsymbol{V}_\infty\big]dl_r, \\ p = 1, \dots, N. \quad (5)$$

The other modification consists of uniform vorticity distribution on airfoil surface. In 'classical' NVEM method intensity of vortex layer is assumed to be constant on every part (every panel) of the airfoil, but then all the vorticity from every panel concentrates in one point on the panel and integral in (2) transforms into a sum of influences of discrete (point) vortex elements. In suggested modified method TVEM vorticity is assumed to be uniformly distributed over every panel. Each panel on the airfoil is rectilinear segment, so the internal integral in the first term in (5) transforms into a sum of influences of panels with distributed vorticity (it should be noticed that in this case every panel doesn't influence on itself):

$$\sum_{\substack{q=1\\q\neq p}}^{N}\frac{1}{L_p}\int_{K_p}\left[\int_{K_q}\frac{\boldsymbol{\tau}(\boldsymbol{r})\cdot[\boldsymbol{k}\times(\boldsymbol{r}-\boldsymbol{r}_0)]}{2\pi|\boldsymbol{r}-\boldsymbol{r}_0|^2}\gamma(\boldsymbol{r}_0)dl_{r_0}\right]dl_r - \\ -\frac{1}{L_p}\int_{K_p}\frac{\gamma(\boldsymbol{r})}{2}dl_r = -\frac{1}{L_p}\int_{K_p}\big[\boldsymbol{\tau}(\boldsymbol{r})\cdot\boldsymbol{V}_\infty\big]dl_r, \\ p = 1, \dots, N. \quad (6)$$

As it was mentioned, vorticity is uniformly distributed over every panel, so unknown function $\gamma(\boldsymbol{r})$ is constant on every panel and (6) becomes linear algebraic system:

$$\sum_{j=1}^{N}\left(A_{ij}(1-\delta_{ij}) - \frac{1}{2}\delta_{ij}\right)\gamma_j = -B_i, \\ i = 1, \dots, N. \quad (7)$$

Here A_{ij} is matrix coefficient, δ_{ij} is Kronecker delta, B_i is stream influence on i-th panel, unknown variable γ_j is vortex layer intensity on j-th panel. Because of all the panels are rectilinear, $\boldsymbol{\tau}(\boldsymbol{r})$ is constant vector on i-th panel and we denote it $\boldsymbol{\tau}_i$:

$$A_{ij} = \frac{1}{L_i}\int_{K_i}\left[\int_{K_j}\frac{\boldsymbol{\tau}_i\cdot[\boldsymbol{k}\times(\boldsymbol{r}-\boldsymbol{r}_0)]}{2\pi|\boldsymbol{r}-\boldsymbol{r}_0|^2}dl_{r_0}\right]dl_r, \\ B_i = \boldsymbol{\tau}_i\cdot\boldsymbol{V}_\infty.$$

In order to write down final formula for matrix coefficient A_{ij}, we note that this coefficient is average tangent velocity on i-th panel, which is induced by j-th panel in assumption that j-th panel has unit vortex layer intensity. In order to compute it we firstly derive formula for average velocity vector on i-th panel, which is induced by j-th panel

$$\boldsymbol{V}_{ij} = \frac{1}{L_i}\int_{K_i}\left[\int_{K_j}\frac{\boldsymbol{k}\times(\boldsymbol{r}-\boldsymbol{r}_0)}{2\pi|\boldsymbol{r}-\boldsymbol{r}_0|^2}dl_{r_0}\right]dl_r,$$

then

$$A_{ij} = \boldsymbol{\tau}_i\cdot\boldsymbol{V}_{ij}.$$

On fig. 1 some auxiliary vectors are introduced: vectors $\boldsymbol{d}$ and $\boldsymbol{d}_0$ are codirectional with i-th and j-th panels, their lengths are L_i and L_j correspondingly; vectors $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ join the beginning of j-th with the beginning and the ending of i-th panel; vectors $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ join the ending of j-th with the beginning and the ending of i-th panel.

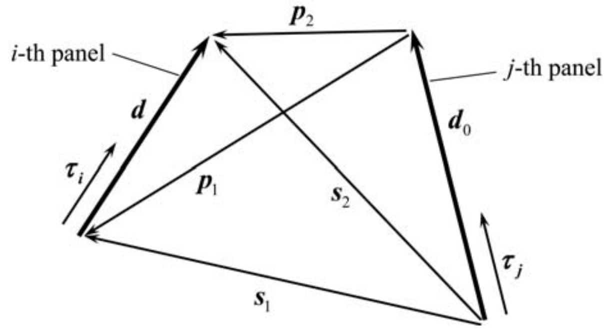

Fig. 1. Two panels from airfoil surface and auxiliary vectors

After some transformations we can obtain the following formula:

$$\boldsymbol{V}_{ij} = \frac{1}{2\pi|\boldsymbol{d}_0||\boldsymbol{d}|^2}\Big[q_1^{(1)}\boldsymbol{c}_1 + q_2^{(1)}\boldsymbol{c}_2 + q_3^{(1)}\boldsymbol{c}_3 + \\ + \Big(q_1^{(2)}\boldsymbol{c}_1 + q_2^{(2)}\boldsymbol{c}_2 + q_3^{(2)}\boldsymbol{c}_3\Big)\times\boldsymbol{k}\Big].$$

Here we denote:

$$\boldsymbol{c}_1 = (\boldsymbol{d}_0\cdot\boldsymbol{p}_1)\boldsymbol{d} + (\boldsymbol{d}\cdot\boldsymbol{s}_1)\boldsymbol{d}_0 - (\boldsymbol{d}\cdot\boldsymbol{d}_0)\boldsymbol{s}_1, \\ \boldsymbol{c}_2 = (\boldsymbol{d}_0\cdot\boldsymbol{s}_1)\boldsymbol{d} + (\boldsymbol{d}\cdot\boldsymbol{s}_1)\boldsymbol{d}_0 - (\boldsymbol{d}\cdot\boldsymbol{d}_0)\boldsymbol{s}_1 = \\ = \boldsymbol{c}_1 + (\boldsymbol{d}_0\cdot\boldsymbol{d}_0)\boldsymbol{d}, \\ \boldsymbol{c}_3 = (\boldsymbol{d}\cdot\boldsymbol{d})\boldsymbol{d}_0.$$

$$q_1^{(1)} = \arctan\frac{\boldsymbol{d}\cdot\boldsymbol{p}_1}{z_1} - \arctan\frac{\boldsymbol{d}\cdot\boldsymbol{p}_2}{z_1}, \\ q_2^{(1)} = \arctan\frac{\boldsymbol{d}\cdot\boldsymbol{s}_2}{z_2} - \arctan\frac{\boldsymbol{d}\cdot\boldsymbol{s}_1}{z_2}, \\ q_3^{(1)} = \arctan\frac{\boldsymbol{d}_0\cdot\boldsymbol{p}_2}{z_3} - \arctan\frac{\boldsymbol{d}_0\cdot\boldsymbol{s}_2}{z_3};$$

$$z_1 = (\boldsymbol{p}_1\times\boldsymbol{p}_2)_z, \quad z_2 = (\boldsymbol{s}_1\times\boldsymbol{s}_2)_z, \quad z_3 = (\boldsymbol{s}_2\times\boldsymbol{p}_2)_z;$$

$$q_1^{(2)} = \ln\frac{|\boldsymbol{p}_2|}{|\boldsymbol{p}_1|}, \quad q_2^{(2)} = \ln\frac{|\boldsymbol{s}_1|}{|\boldsymbol{s}_2|}, \quad q_3^{(2)} = \ln\frac{|\boldsymbol{p}_2|}{|\boldsymbol{s}_2|};$$

For neighboring panels when $\boldsymbol{p}_1 = 0$, $\boldsymbol{s}_2 \neq 0$, coefficients $q_1^{(1)}$ and $q_1^{(2)}$ vanish. For neighboring panels when $\boldsymbol{s}_2 = 0$, $\boldsymbol{p}_1 \neq 0$ vectors $\boldsymbol{d}_0$ and $\boldsymbol{d}$ should be replaced with $(-\boldsymbol{d}_0)$ and $(-\boldsymbol{d})$ correspondingly; then we obtain the previous case when $\boldsymbol{p}_1 = 0$, $\boldsymbol{s}_2 \neq 0$. If $\boldsymbol{s}_1 = 0$ or $\boldsymbol{p}_2 = 0$ only one of vectors $\boldsymbol{d}_0$ or $\boldsymbol{d}$ should be replaced with its opposite in order to obtain the first case.

The developed approach can also be generalized for nonstationary problems and for viscous incompressible flow simulation. It should be noticed, that in case of viscous flow the only difference is vortex elements motion equation [4], while vortex layer intensity computation procedure remains. When solving nonstationary aerodynamics problems (both for viscous and inviscid cases) vortex wake is simulated by a set of vortex elements with known intensities Γ_w and positions $\boldsymbol{r}_w$, $w = 1, \ldots, N_{wake}$. It's necessary to take into account vortex wake influence on vortex layer intensity computation. Additional terms corresponding to tangent velocity on i-th panel, which is induced by w-th vortex element of the wake, should be added to right side of equation (7):

$$B_i^{wake} = \boldsymbol{\tau}_i \cdot \sum_{w=1}^{N_{wake}} \boldsymbol{V}_w^{wake},$$

where

$$\boldsymbol{V}_w^{wake} = \frac{\Gamma_w}{L_i} \int_{K_i} \left[\frac{\boldsymbol{k} \times (\boldsymbol{r} - \boldsymbol{r}_w)}{2\pi |\boldsymbol{r} - \boldsymbol{r}_w|^2} \right] dl_r. \tag{8}$$

On fig. 2 by analogy with fig. 1 some auxiliary vectors are introduced for velocity $\boldsymbol{V}_w^{wake}$ computation: vectors $\boldsymbol{s}_0$ and $\boldsymbol{s}$ join the beginning and the ending of i-th panel with w-th vortex element placed in point $\boldsymbol{r}_w$.

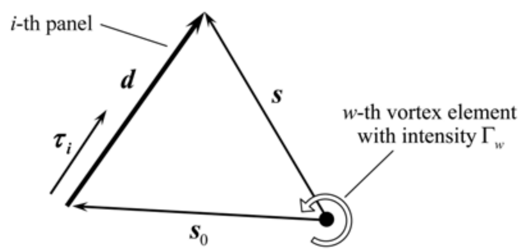

Fig. 2. i-th panel on airfoil surface, w-th vortex element in vortex wake and auxiliary vectors

Using these vectors after some transformations we can write down term (8) in the following form:

$$\boldsymbol{V}_w^{wake} = -\frac{\Gamma_w}{2\pi |\boldsymbol{d}|^2} \left[\alpha \boldsymbol{d} + \beta (\boldsymbol{d} \times \boldsymbol{k}) \right]. \tag{9}$$

Here we denote

$$\begin{gathered} \alpha = \arctan \frac{\boldsymbol{s} \cdot \boldsymbol{d}}{z_0} - \arctan \frac{\boldsymbol{s}_0 \cdot \boldsymbol{d}}{z_0}, \\ \beta = \ln \frac{|\boldsymbol{s}|}{|\boldsymbol{s}_0|}, \quad z_0 = (\boldsymbol{d} \times \boldsymbol{s}_0)_z. \end{gathered} \tag{10}$$

An important feature of methods based on integral equations (2), (4) and (5), is that the result of solving corresponding integral equation is the average value of vorticity layer intensity on every panel. If we need 'local' values of vorticity layer intensity, we can interpolate average values in some way, but results of test problems solving show that difference between obtained values and exact solution is sufficiently big, while average values are much closer.

We also developed another variant of numerical scheme for vortex element method which allows to find local values of vortex layer intensity more accurately in comparison with TVEM. Main idea of this modification (we call it LinTVEM) is the following. Vorticity distribution on the airfoil surface is assumed to be piecewise linear on every panel and it is continuous on the whole airfoil. At the ends of every panel vortex layer intensity is assumed to be equal to local values of vortex layer intensity. These values become unknown variables in linear algebraic system which approximates integral equation (5). This numerical scheme differs from the previous schemes, corresponding linear algebraic systems have some specific features, but from computational point of view this method is very similar to method TVEM. The main difference is that internal integral in first term in (5) consists of two parts: first part corresponds to half-sum of vortex layer intensities at the ends of panels, so this integral is absolutely the same as the internal integral in TVEM. The second part of the integral corresponds to linear distribution of vorticity with zero average value and this integral also can be calculated analytically. External integral in LinTVEM scheme also can be calculated analytically as in TVEM, but corresponding formulae become much more complicated even in comparison with TVEM, so in practice the most reasonable way is to use Gaussian quadrature formulae for numerical approximate computation of this integral.

We only note one significant feature of LinTVEM discrete schemes: if number of panels is even, corresponding linear system is singular, while for odd number of panels the matrix is well-conditioned and obtained accuracy is higher then in TVEM schemes (both for average values and especially for local values of vortex layer intensity). So in the examples which are shown below number of panels for LinTVEM schemes is always equal to $(N + 1)$ if N is even.

V. Numerical results

Now we investigate results which can be obtained using classical scheme NVEM and developed in this paper numerical schemes TVEM and LinTVEM. We consider some test problems for elliptical airfoils and Zhukovsky airfoils for which we know exact analytical solution for vortex layer intensity.

A. *Flow around an elliptical airfoil*

If we simulate flow around circular airfoil we obtain nearly the same results both for NVEM and TVEM numerical schemes.

Results of vortex layer intensity computation for steady flow around elliptical airfoil with major and minor semiaxes equal to $a_1 = 1{,}0$ and $b_1 = 0{,}1$ for angle of incidence $\beta = \frac{\pi}{6}$ with $N = 200$ panels on airfoil surface are shown on fig. 3. We can see that there is significant difference between exact solution and NVEM scheme solution near ends of ellipse major axis while TVEM scheme solution is much closer to exact solution for the same number of panels on the airfoil.

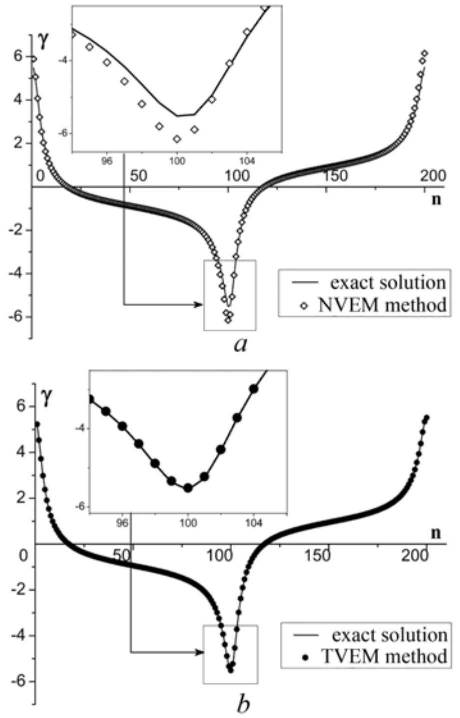

Fig. 3. Vortex layer intensity on elliptical airfoil with semiaxes ratio 10,0 at angle of incidence $\beta = \frac{\pi}{6}$ computed using NVEM (a) and TVEM (b)

If we now investigate thinner airfoil with semiaxes equal to $a_1 = 1{,}0$ and $b_1 = 0{,}05$ (axes ratio is 20) even without angle of incidence ($\beta = 0$) for $N = 200$ we find that NVEM allows to obtain solution with significant error along all the airfoil while TVEM scheme gives the solution which is very close to exact solution (fig. 4).

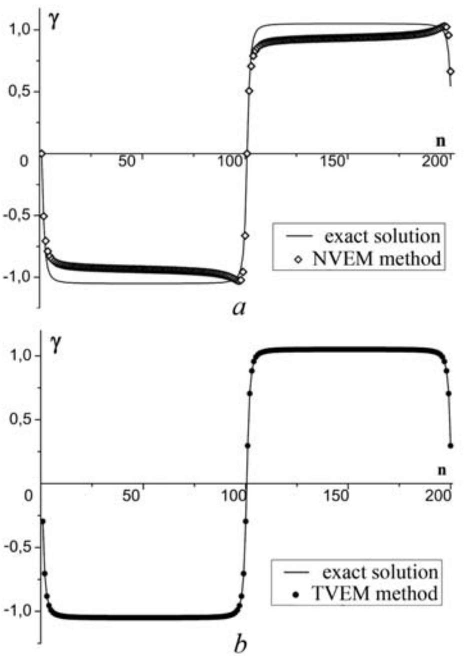

Fig. 4. Vortex layer intensity on elliptical airfoil with semiaxes ratio 20,0 at zero angle of incidence computed using NVEM (a) and TVEM (b)

B. Flow around Zhukovsky airfoil

Results of vortex layer intensity computation for steady flow around symmetrical Zhukovsky airfoil with relative width equal to 20 % (airfoil geometrical parameters are the following: $a = 1{,}0$, $d = 0{,}2$, $h = 0{,}0$) for angle of incidence $\beta = \frac{\pi}{6}$ using mentioned approaches (NVEM and TVEM numerical schemes) are shown on fig. 5. Number of panels on the airfoil $N = 50$.

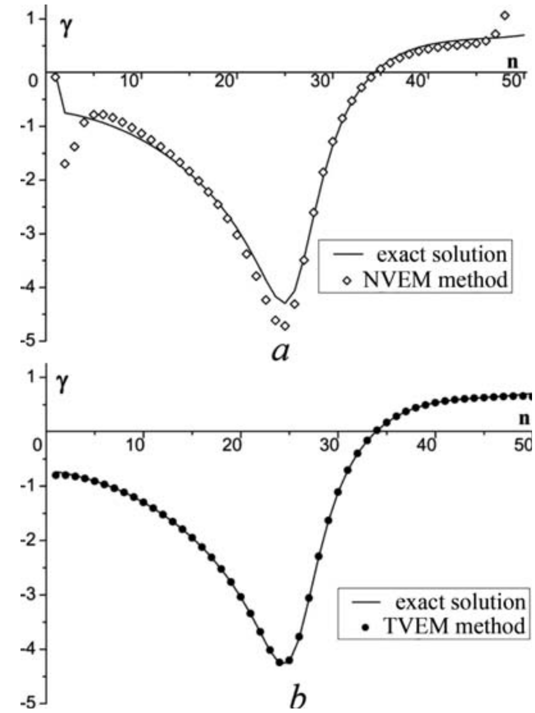

Fig. 5. Vortex layer intensity on symmetrical Zhukovsky airfoil computed using NVEM (a) and TVEM (b)

For greater values of N error in uniform norm becomes much bigger for 'classical' NVEM scheme, at the same time it vanishes in TVEM method developed in this paper.

Results for non-symmetrical Zhukovsky airfoil with relative width equal to 10 % (airfoil geometrical parameters: $a = 1{,}0$, $d = 0{,}1$, $h = 0{,}1$) for angle of incidence $\beta = \frac{\pi}{6}$ are nearly the same: NVEM scheme leads to significant errors near airfoil sharp edge while TVEM scheme gives solution very close to exact one (fig. 6).

Now we compare condition numbers for matrices of linear algebraic systems which are being solved in NVEM and TVEM methods. In table I condition number values are shown which were obtained in the same test problems for elliptical airfoils and Zhukovsky airfoils. Three variants for number of panels on the airfoil surfaces ($N = 50$, $N = 200$ and $N = 500$) have been considered for all cases.

So for smooth airfoils condition number for matrices in NVEM and TVEM are close and matrices are well-conditioned for both variants of vortex element method. But if we simulate flow around an airfoil with angular point or with sharp edge, matrices in TVEM are much better conditioned in comparison with NVEM.

TABLE I
CONDITION NUMBER FOR MATRICES IN NVEM AND TVEM

Cond A	$N=50$		$N=200$		$N=500$	
	NVEM	*TVEM*	*NVEM*	*TVEM*	*NVEM*	*TVEM*
Elliptical airfoil						
Semiaxes ratio 10,0	26	115	107	242	258	384
Semiaxes ratio 20,0	88	265	172	633	364	1012
Zhukovsky airfoil						
Symmetrical airfoil	$1.1 \cdot 10^4$	$3.0 \cdot 10^2$	$3.4 \cdot 10^5$	$2.0 \cdot 10^3$	$3.4 \cdot 10^6$	$7.0 \cdot 10^3$
Non-symmetrical airfoil	$9.0 \cdot 10^5$	$4.9 \cdot 10^2$	$1.7 \cdot 10^7$	$3.4 \cdot 10^3$	$1.5 \cdot 10^8$	$1.2 \cdot 10^4$

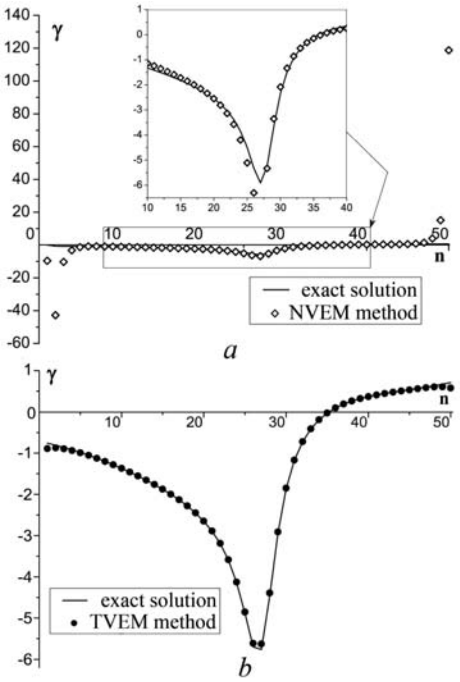

Fig. 6. Vortex layer intensity on non-symmetrical Zhukovsky airfoil computed using NVEM (a) and TVEM (b)

In order to compare different methods accuracy we firstly calculate 1-norm and ∞-norm of the errors for NVEM and TVEM numerical schemes. Results are shown on table II.

TABLE II
ERROR IN TEST PROBLEMS FOR NVEM AND TVEM

	$N=50$		$N=200$		$N=500$	
	NVEM	*TVEM*	*NVEM*	*TVEM*	*NVEM*	*TVEM*
Elliptical airfoil with semiaxes ratio 10,0						
$\|\Delta\|_1$	0.2594	0.0274	0.0681	0.0018	0.0275	0.0003
$\|\Delta\|_\infty$	0.5841	0.3769	0.1678	0.0387	0.0633	0.0059
Elliptical airfoil with semiaxes ratio 20,0						
$\|\Delta\|_1$	0.2659	0.0010	0.0655	0.0001	0.0263	0.0000
$\|\Delta\|_\infty$	0.9526	0.0084	0.3269	0.0060	0.1319	0.0011
Symmetrical Zhukovsky airfoil						
$\|\Delta\|_1$	1.3000	0.0149	0.4650	0.0010	0.2201	0.0002
$\|\Delta\|_\infty$	27.21	0.0563	113.70	0.0245	288.92	0.0141
Non-symmetrical Zhukovsky airfoil						
$\|\Delta\|_1$	27.60	0.0282	2.742	0.0020	0.5918	0.0003
$\|\Delta\|_\infty$	822.5	0.1809	1268.9	0.0641	1105.4	0.0384

Here $\|\cdot\|_1$ is a discrete analogue of norm in Banach space L_1 and $\|\cdot\|_\infty$ is a discrete analogue of uniform norm in Banach space C:

$$\|\Delta\|_1 = \sum_{i=1}^{N} |\gamma_i^0 - \gamma_i| L_i,$$
$$\|\Delta\|_\infty = \max_i |\gamma_i^0 - \gamma_i|,$$

γ_i^0 is exact solution (average vortex layer intensity on i-th panel), γ_i is computed value for it, L_i is i-th panel length.

If we analyze obtained results, we notice that for smooth airfoils errors of both numerical schemes become smaller when number of panels N increases, but error of TVEM scheme is many times smaller then error of NVEM scheme. At the same time for Zhukovsky airfoil error in NVEM method vanishes only in 1-norm while in ∞-norm error becomes larger when N increases. Error in TVEM schemes vanishes in both 1-norm and ∞-norm.

At last we compare TVEM scheme with LinTVEM scheme. In table III 1-norm and ∞-norm of error are shown for LinTVEM method. However, as opposed to table II here we can see error not for average values of vortex layer intensities but for local values of intensities on end points of panels on airfoil surface. It should be noted that LinTVEM allows also to calculate average values of vortex layer intensities on panels and they are very close to average values obtained using TVEM.

TABLE III
LOCAL ERRORS IN TEST PROBLEMS FOR LINTVEM

	$N=50$	$N=100$	$N=200$	$N=500$
Elliptical airfoil with semiaxes ratio 10,0				
$\|\Delta\|_1$	2.2083	0.2605	0.0046	0.0007
$\|\Delta\|_\infty$	0.8693	0.1489	0.0181	0.0038
Elliptical airfoil with semiaxes ratio 20,0				
$\|\Delta\|_1$	1.6062	0.6127	0.0638	0.0001
$\|\Delta\|_\infty$	0.5160	0.1848	0.0395	0.0038

We can see that for small number of panels N local errors are very big, but when N becomes equal to 200 and bigger, accuracy of LinTVEM for local values becomes approximately the same as accuracy of TVEM for average values. This result is very important for correct boundary condition satisfaction.

VI. Conclusion

The problem of 2D flow numerical simulation around airfoils is considered. For its solution two approaches are proposed: NVEM (Vortex element method with normal components of velocity on airfoil surface) and TVEM (Vortex element method with tangent components of velocity on airfoil surface). To compare effects of these methods vortex layer intensities are computed for some simplest airfoils with known exact solution (circular, elliptical and Zhukovsky airfoils). In method TVEM piecewise constant and piecewise linear functions are used to approximate intensity on the airfoil panels. Results for smooth airfoils (circular and elliptical) are qualitatively close in both methods (errors in TVEM much smaller then in NVEM), at the same time LinTVEM scheme is more effective in comparison with TVEM when number of panels N is more then 200.

Also for airfoils with sharp edge (Zhukovsky airfoil) for greater values of N error in ∞-norm becomes much bigger for NVEM scheme, while it vanishes in TVEM and LinTVEM methods developed in this paper. In 1-norm for airfoils with sharp edge numerical errors in TVEM method are smaller than in 'classical' NVEM method, however for both methods numerical error in 1-norm vanishes when N becomes bigger.

Developed method can be used for unsteady flow simulation and also for solving complicated aeroelastic problems.

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