A Finite Element Solution of the Mathematical Model for Smoke Dispersion from Two Sources

Nopparat Pochai

Abstract—Smoke discharging is a main reason of air pollution problem from industrial plants. The obstacle of a building has an affect with the air pollutant discharge. In this research, a mathematical model of the smoke dispersion from two sources and one source with a structural obstacle is considered. The governing equation of the model is an isothermal mass transfer model in a viscous fluid. The finite element method is used to approximate the solutions of the model. The triangular linear elements have been used for discretising the domain, and time integration has been carried out by semi-implicit finite difference method. The simulations of smoke dispersion in cases of one chimney and two chimneys are presented. The maximum calculated smoke concentration of both cases are compared. It is then used to make the decision for smoke discharging and air pollutant control problems on industrial area.

Keywords—Air pollution, Smoke dispersion, Finite element method, Stream function, Vorticity equation, Convection-diffusion equation, Semi-implicit method

I. Introduction

The air pollution problem is still important for human health. This research is considered the air pollution problem using the mass transfer model. In [8], a nonlinear mathematical model is proposed and analyzed to study the removal of gaseous pollutants and particulate matters from the atmosphere of a city by precipitation. The three pollution indexes (SO_2, NO_2 and PM10) and the daily air pollution indexes (APIs) of Shanghai in China are analyzed by rescaled range analysis (R/S), detrended fluctuation analysis (DFA) and spectral analysis is presented in [4]. In [1], they study the air flow using a lubricated system consisting of two bodies in proximity. The Poincare compactification to the dynamical system to get a complete qualitative analysis of the global flow is applied in [11].

In [10], [12] and [5], they used the finite difference method in the air pollution model of two dimensional spaces with single point source. In [6], they are consider the air pollution problem in three dimensional spaces. The initial conditions in the domains were assumed to be zero everywhere without structural obstacles. In [7] and [13], the fractional steps method, Carlson method and Crank-Nicolson method [2] are used to approximate the concentration of smoke dispersion. The diffusion of dust particles from a point source above ground level is considered in [3].

In this research, we investigate the behavior of smoke release into the atmosphere with a structural obstacle in two-dimensional spaces and one-dimensional in time using the finite element method and finite difference method, respectively. The structural obstacle and chimneys are added in the simulation. The simulation attempts to predict the behavior of the dispersion of smoke in the problem.

[a] Department of Mathematics, Faculty of Science, King Mongkut's Institute of Technology Ladkrabang, 10520, Thailand e-mail: nop_math@yahoo.com.
[b] Centre of Excellence in Mathematics, CHE, Si Ayutthaya Rd., Bangkok 10400, Thailand

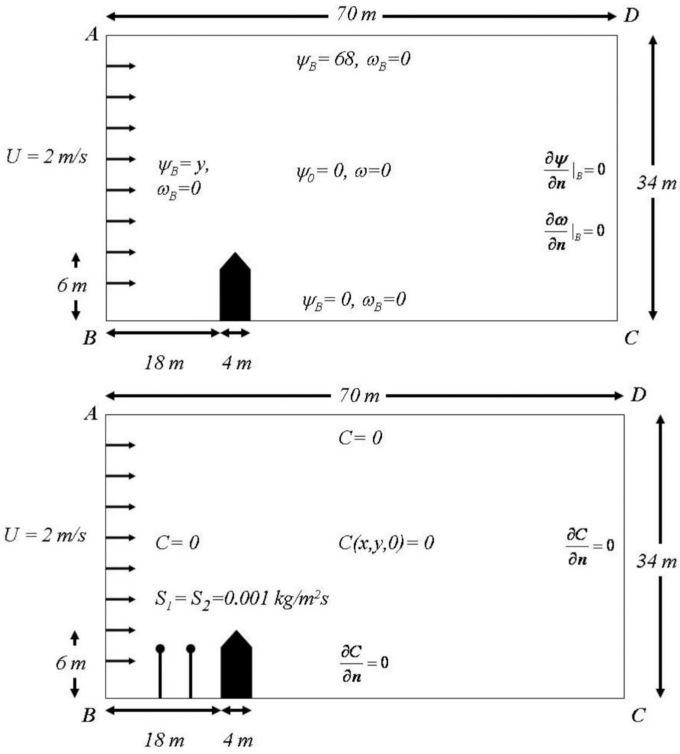

Fig. 1. The initial and boundary conditions of the problem

II. The Governing Equation

A. The mass transport model

Consider the equations describing the smoke dispersion in terms of the streamfunction, vorticity and convection-diffusion equations as follows [9],

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -\omega, \tag{1}$$

$$\frac{\partial \omega}{\partial t} + \frac{\partial \psi}{\partial y}\frac{\partial \omega}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \omega}{\partial y} = \nu \nabla^2 \omega, \tag{2}$$

$$\frac{\partial C}{\partial t} + \frac{\partial \psi}{\partial y}\frac{\partial C}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial C}{\partial y} = \eta \nabla^2 C + S, \tag{3}$$

where ν is a kinematic viscosity, ω is a vorticity, ψ is a stream function, C is a concentration, η is assumed to ba a constant of diffusion coefficient and S is intensity of mass source.

B. The boundary conditions

The boundary conditions of the streamfunction equation Eq.(1) are

$$\psi = \psi_B, \tag{4}$$

which is specified on the boundary Γ_ψ, and

$$\frac{\partial \psi}{\partial n} = -V_s, \tag{5}$$

with tangential flow velocity V_s specified on the rest of the boundary Γ_s. Here n denotes the outward unit normal to the boundary. Boundary conditions on the vorticity Eq.(2) are

$$\omega = \omega_B, \tag{6}$$

which is specified on the boundary Γ_ω, and

$$\frac{\partial \omega}{\partial n} = \chi_n, \tag{7}$$

with the value of the normal derivative specified on the other part of the boundary Γ_χ. The boundary conditions of the convective-diffusion Eq.(3) are

$$C = C_B \text{ on } \Gamma_C, \tag{8}$$

$$-\frac{\partial C}{\partial n} = j_B \text{ on } \Gamma_j. \tag{9}$$

Since boundary is a non-absorbing, we can put $j_B = 0$.

III. The Numerical Technique

A. The finite element discretisation

The Galerkin finite element method to discretising of the streamfunction Eq.(1), the vorticity transport Eq.(2), and the convection-diffusion equation Eq.(3) will be used. Let W_ψ, W_ω and W_C be weighting functions. The finite element formulation with the weighted residual forms of these equations:

$$\int_\Omega W_\psi (\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}) d\Omega + \int_\Omega W_\psi \omega d\Omega = 0, \tag{10}$$

$$\int_\Omega W_\omega \frac{\partial \omega}{\partial t} d\Omega + \int_\Omega W_\omega (\frac{\partial \psi}{\partial y}\frac{\partial \omega}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \omega}{\partial y}) d\Omega - \int_\Omega W_\omega \nu \nabla^2 \omega d\Omega = 0, \tag{11}$$

$$\int_\Omega W_C \frac{\partial C}{\partial t} d\Omega + \int_\Omega W_C (\frac{\partial \psi}{\partial y}\frac{\partial C}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial C}{\partial y}) d\Omega - \int_\Omega W_C \eta \nabla^2 C d\Omega - \int_\Omega W_C S d\Omega = 0. \tag{12}$$

Integration by parts of terms concerning the Laplacian, we get

$$\int_\Omega (\frac{\partial W_\psi}{\partial x} + \frac{\partial W_\psi}{\partial y})(\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}) d\Omega + \int_\Omega W_\psi \omega d\Omega - \int_{\Gamma_s} W_\psi \frac{\partial \psi}{\partial n} d\Gamma = 0, \tag{13}$$

$$\int_\Omega W_\omega \frac{\partial \omega}{\partial t} d\Omega + \int_\Omega W_\omega (\frac{\partial \psi}{\partial y}\frac{\partial \omega}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \omega}{\partial y}) d\Omega + \int_\Omega \nu \nabla W_\omega \nabla \omega d\Omega - \int_{\Gamma_\chi} \nu W_\omega \frac{\partial \omega}{\partial n} d\Gamma = 0, \tag{14}$$

$$\int_\Omega W_C \frac{\partial C}{\partial t} d\Omega + \int_\Omega W_C (\frac{\partial \psi}{\partial y}\frac{\partial C}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial C}{\partial y}) d\Omega + \int_\Omega \eta \nabla W_C \nabla C d\Omega - \int_\Omega W_C S d\Omega - \int_{\Gamma_j} j_B W_C d\Gamma = 0. \tag{15}$$

The domain Ω is divided into triangular elements with local node numbers 1, 2 and 3 with nodal coordinates $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3), respectively. The unknown streamfunction, vorticity and concentration are linearly interpolated as follows,

$$\psi = \sum_{\alpha=1}^{3} \phi_\alpha \psi_\alpha, \qquad W_\psi = \sum_{\alpha=1}^{3} \phi_\alpha W_{\psi_\alpha}, \tag{16}$$

$$\omega = \sum_{\alpha=1}^{3} \phi_\alpha \omega_\alpha, \qquad W_\omega = \sum_{\alpha=1}^{3} \phi_\alpha W_{\omega_\alpha}, \tag{17}$$

$$C = \sum_{\alpha=1}^{3} \phi_\alpha C_\alpha, \qquad W_C = \sum_{\alpha=1}^{3} \phi_\alpha W_{C_\alpha}, \tag{18}$$

where linear interpolation functions, $\phi_\alpha = \frac{1}{2\Delta^e}(a_\alpha + b_\alpha x + c_\alpha y)$ with area of a triangular element e is $\Delta^e = \frac{b_2 c_3 - b_3 c_2}{2}$. For $\psi_\alpha, \omega_\alpha, C_\alpha$ are nodal values of the corresponding unknown, and $W_{\psi_\alpha}, W_{\omega_\alpha}, W_{C_\alpha}$ are their arbitrary variations.

Substituting Eqs.(16-18) into Eqs.(13-15), it is obtain that

$$\sum_{\beta=1}^{3} D^e_{\alpha\beta} \psi_\beta - \sum_{\beta=1}^{3} M^e_{\alpha\beta} \omega_\beta - \Gamma^e_{s\alpha} = 0, \tag{19}$$

$$\sum_{\beta=1}^{3} M^e_{\alpha\beta} \dot{\omega}_\beta + \sum_{\beta=1}^{3} A^e_{\alpha\beta} \omega_\beta + \nu \sum_{\beta=1}^{3} D^e_{\alpha\beta} \omega_\beta - \Gamma^e_{\chi\alpha} = 0, \tag{20}$$

$$\sum_{\beta=1}^{3} M^e_{\alpha\beta} \dot{C}_\beta + \sum_{\beta=1}^{3} A^e_{\alpha\beta} C_\beta + \eta \sum_{\beta=1}^{3} D^e_{\alpha\beta} C_\beta - S^e_\alpha + \Gamma^e_{j\alpha} = 0, \tag{21}$$

for each $\alpha = 1, 2, 3$, where the coefficients are given by

$$M^e_{\alpha\beta} = \int_e \phi_\alpha \phi_\beta d\Omega = \frac{\Delta^e}{12}(1 + \delta_{\alpha\beta}), \tag{22}$$

$$D^e_{\alpha\beta} = \int_e (\frac{\partial \phi_\alpha}{\partial x}\frac{\partial \phi_\beta}{\partial x} + \frac{\partial \phi_\alpha}{\partial y}\frac{\partial \phi_\beta}{\partial y}) d\Omega = \frac{1}{4\Delta^e}(b_\alpha b_\beta + c_\alpha c_\beta), \tag{23}$$

$$A^e_{\alpha\beta} = \int_e \phi_\alpha (\sum_{\gamma=1}^{3} \frac{\partial \phi_\gamma}{\partial y} \psi_\gamma \frac{\partial \phi_\beta}{\partial x} - \sum_{\gamma=1}^{3} \frac{\partial \phi_\gamma}{\partial x} \phi_\gamma \frac{\partial \phi_\beta}{\partial y}) d\Omega, = \frac{1}{12\Delta^e} \sum_{\gamma=1}^{3} (c_\gamma b_\beta - b_\gamma c_\beta) \psi_\gamma, \tag{24}$$

$$S^e_\alpha = \int_e \phi_\alpha (\sum_{\gamma=1}^{3} \phi_\gamma S_\gamma) d\Omega = \frac{1}{12\Delta^e}(S_\alpha + \sum_{\gamma=1}^{3} S_\gamma), \tag{25}$$

$$\Gamma^e_{s\alpha} = \int_{\Gamma^e_s} \phi_\alpha \frac{\partial \psi}{\partial n} d\Gamma = -\frac{V_s}{2} |\Gamma^e_s| \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \tag{26}$$

$$\Gamma^e_{\chi\alpha} = \int_{\Gamma^e_\chi} \nu\phi_\alpha \frac{\partial\omega}{\partial n} d\Gamma = \nu\frac{\chi_n}{2}|\Gamma^e_\chi| \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \tag{27}$$

$$\Gamma^e_{j\alpha} = \int_{\Gamma^e_j} \phi_\alpha j_B d\Gamma = \frac{j_B}{2}|\Gamma^e_j| \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \tag{28}$$

with $\delta_{\alpha\beta}$ is the Kronecker delta function, $\Gamma^e_i = \Gamma_i \cap \partial e$ where ∂e is a boundary of an element e and $|\Gamma^e_i|$ is a length of adjacent nodes on a boundary element, for all $i = s, \chi, j$.

Assembling all element equations over domain Ω, the total equations becomes,

$$[D]\{\psi\} - [M]\{\omega\} - \{\Gamma_s\} = \{0\}, \tag{29}$$

$$[M]\{\dot\omega\} + [A(\psi)]\{\omega\} + \nu[D]\{\omega\} - \{\Gamma_\chi\} = \{0\}, \tag{30}$$

$$[M]\{\dot C\} + [A(\psi)]\{C\} + \eta[D]\{C\} + \{S\} + \{\Gamma_j\} = \{0\}, \tag{31}$$

where $[D], [M], [A]$ denote the total matrices, $\{\psi\}, \{\omega\}, \{C\}$ are the nodal unknown column vectors.

B. Time integration

We interprets Eqs.(29-31) as a nonlinear system of first-order ordinary differential equations for unknown $\{\psi\}, \{\omega\}$ and $\{C\}$. Consider the discretisation of the total equations with respect to time, the semi-implicit scheme to the initial value problem is applied. The time derivatives of the nodal vorticity and concentration are approximated by

$$\dot\omega_\beta = \frac{d\omega_\beta}{dt} \simeq \frac{\omega^{n+1}_\beta - \omega^n_\beta}{\Delta t}, \tag{32}$$

$$\dot C_\beta = \frac{dC_\beta}{dt} \simeq \frac{C^{n+1}_\beta - C^n_\beta}{\Delta t}, \tag{33}$$

where ω^n_β and C^n_β denote the nodal vorticity and concentration, respectively. For a time level t_n is defined by $t_{n+1} = t_n + \Delta t$ with the time increment Δt for each $n = 0, 1, 2, \ldots$ Substituting Eqs.(32-33) into Eqs.(29-31) at time level t_{n+1}, it follows that

$$[D]\{\psi^{n+1}\} = [M]\{\omega^n\} + \{\Gamma_s\}, \tag{34}$$

$$\frac{1}{\Delta t}[M]\{\omega^{n+1}\} + \nu[D]\{\omega^{n+1}\} = \frac{1}{\Delta t}[M]\{\omega^n\} - [A(\psi^{n+1})]\{\omega^n\} - \{\Gamma^{n+1}_\chi\}, \tag{35}$$

$$\frac{1}{\Delta t}[M]\{C^{n+1}\} + \eta[D]\{C^{n+1}\} = \frac{1}{\Delta t}[M]\{C^n\} - [A(\psi^{n+1})]\{C^n\} + \{S^{n+1}\} - \{\Gamma^{n+1}_j\}. \tag{36}$$

for all $n = 1, 2, 3, \ldots$

IV. Numerical Experiment

A. Topography of a domain

The dispersion of smoke from 2 chimneys into the atmosphere as shown in Fig.1 will be considered. Let C (kg/m^3) be the concentration of the substances in smoke. There are two chimney pots are located 7 m high above the ground. The wind at mean velocity $U = 2$ m/sec is assumed to flow horizontally from left to right. Downstream from the chimneys is located a structural obstacle with cross section of 6 m high and 4 m wide. In our assumption, we assume that the flow does not depend on the concentration.

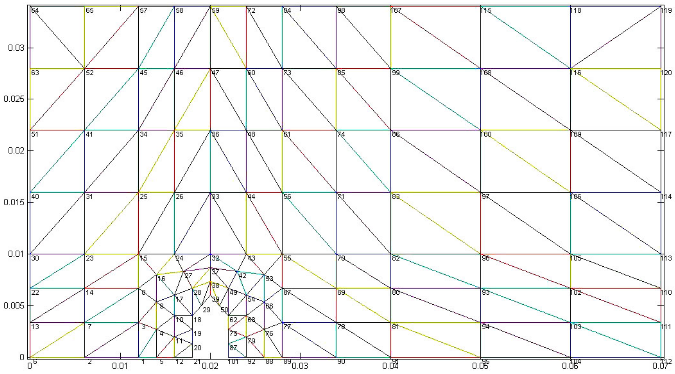

Fig. 2. The triangular finite element mesh with 190 elements and 120 nodes

Physical constants for the wind are; kinematic viscosity of the air $\nu = 1.45 \times 10^{-5}$ m^2/sec, diffusion coefficient of the substance is $\eta = 4.58 \times 10^{-5}$ m^2/sec, gravity acceleration $g = 9.81$ m/sec^2. Since the structural obstacle building height $L = 6$ m, the Reynolds and Peclet numbers for the problem are $Re = UL/\nu \simeq$ 827,586 and $Pe = \nu/\eta \simeq 0.9178$

B. The initial and boundary conditions

The initial and boundary conditions are indicated in Fig.1. The initial condition, the atmosphere is assumed to be motionless. The boundary conditions for $t > 0$ are

$$C = 0 \text{ at } 0 \le x \le 70, y = 34, \tag{37}$$

$$C = 0 \text{ at } x = 0, 0 \le y \le 34, \tag{38}$$

$$\frac{\partial C}{\partial n} = 0 \text{ at } 0 \le x \le 70, y = 0, \tag{39}$$

$$\frac{\partial C}{\partial n} = 0 \text{ at } x = 70, 0 \le y \le 34. \tag{40}$$

C. Finite element solutions

Assume the condition that the concentration contours perpendicularly intersect the efflux boundary CD on the right-hand side. The time increment $\Delta t = 0.1$ sec. is used. The triangular finite element mesh with 190 elements and 120 nodes is presented in Fig. 2 .

Consider the problem in 2 cases: 1 source and 2 sources. In 2 sources problem, the intensities of smoke discharging are $S_1 = 0.001$ kg/m^{2}sec and $S_2 = 0.001$ kg/m^{2}sec at element numbers 7th and 10th, respectively. The case of 1 source, the intensity of smoke has been increased to be 2 folds of the first case, $S_1 = 0.002$ kg/m^{2}sec only at element numbers 7th.

The approximation of smoke concentration at 4, 200 and 800 sec of both cases are summarized in Fig.3 - Fig.5. The maximum of smoke concentration at each time levels are given in Table I.

V. Discussion and Conclusions

In this research, two cases of smoke dispersion with a structural obstacle structure from 1 and 2 chimneys discharge

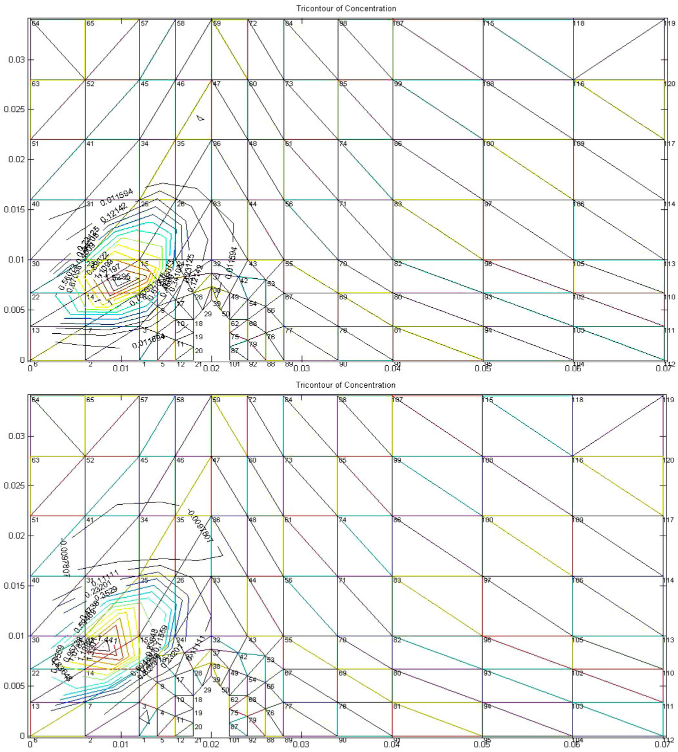

Fig. 3. The finite element approximations of smoke concentration in 2 cases: 2 points source and 1 point source at 4 sec

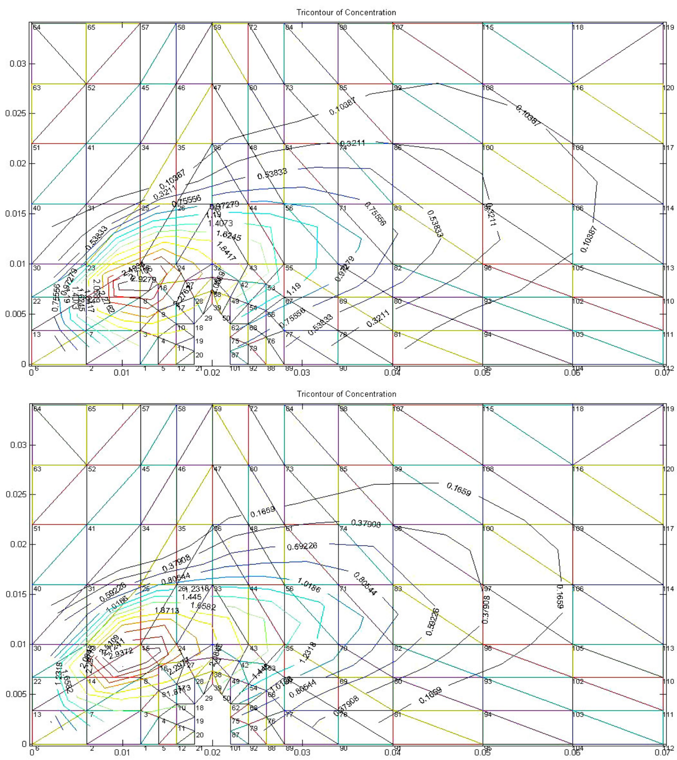

Fig. 4. The finite element approximations of smoke concentration in 2 cases: 2 points source and 1 point source at 3 min 20 sec

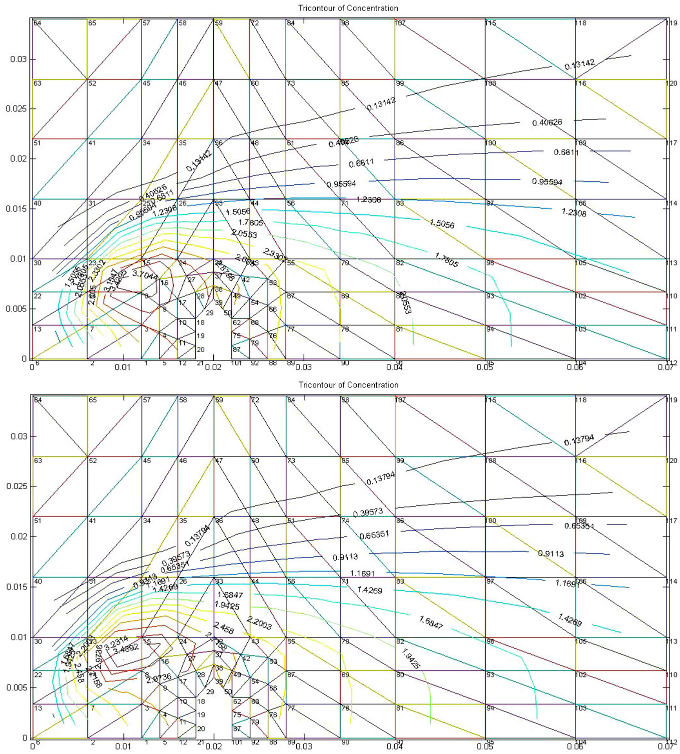

Fig. 5. The finite element approximations of smoke concentration in 2 cases: 2 points source and 1 point source at 13 min 20 sec

TABLE I
THE MAXIMUM OF SMOKE CONCENTRATION IN EACH TIME LEVELS

Element number of point sources	S_1 (kg/m^{2}sec)	S_2 (kg/m^{2}sec)	Time (sec)	Maximum of smoke concentration (mg/m^{2}sec)
E7, E10	0.001	0.001	4	1.3789
E7	0.002	0	4	1.4358
E7, E10	0.001	0.001	40	2.9256
E7	0.002	0	40	3.2673
E7, E10	0.001	0.001	400	3.9370
E7	0.002	0	400	3.7763
E7, E10	0.001	0.001	800	3.9181
E7	0.002	0	800	3.6947

into the atmosphere are presented. The smoke concentration measurement is simulated using the isothermal mass transfer in a viscous fluid. The diffusion of smoke dispersion is coupled to the equations of viscous fluid flow.

A model of type presented here, it is possible to determined velocity and vorticity of the air, and smoke concentration at every point of industrial area with simple structural obstacle. For the discretisation of the domain, triangular linear elements have been used. Time integration has been carried out by means of semi-implicit method.

The numerical example is supposed that concentration of smoke of 1 source is greater than 2 point sources. From the calculation results, it is obtained that the overall concentration by the case of 2 points discharging is lower than 1 point discharging in a short time. However, if we consider the maximum smoke concentration in a larger time, we found that the case of 1 source will give lower concentration than case of 2 sources. This mean that the case of 1 point source of concentration could be a better choice in this consideration problem.

For the real-world application, more complicate structural obstacle structures and several point sources should be consider. It is possible to take another indexes of air quality or concentration of toxic as the smoke concentration in this model.

ACKNOWLEDGMENT

The author would like to deeply appreciate the Centre of Excellence in Mathematics (CEM), a national centre under the Postgraduate Education and Research Development Office (PERDO), Commission on Higher Education of the Ministry of Education, Thailand, for the research grant in this research.

REFERENCES

[1] Ciuperca, I., Hafidi, I. and Jai, M., *Analysis of a parabolic compressible first-order slip Reynolds equation with discontinuous coefficients*, Nonlinear Analysis: Theory, Methods & Applications, **69(4)** (2008), 1219-1234.
[2] Crank, J., Nicolson, P., A practical method for numerical solution of partial differential equations of heat conduction type, *Proc. Cambridge Philos. Soc.*, 43:50-67, 1947.
[3] Hassan, M.H.A., Eltayeb, I.A., *Diffusion of dust particles from a point source above ground level and a line source at ground level*, Geophys. J. Int., **142** (2000), 426-438.
[4] Kai, S., Chun-qiong, L., Nan-shan, A. and Xiao-hong, Z., *Using three methods to investigate time-scaling properties in air pollution indexes time series*, Nonlinear Analysis: Real World Applications, **9(2)** (2008), 693-707.
[5] Konglok, S.A., Tangmanee, S., Numerical Solution of Advection-Diffusion of an Air Pollutant by the Fractional Step Method, *Proceeding in the 3rd national symposium on graduate research*, Nakonratchasrima, Thailand, July, 18th - 19th, 2002.
[6] Konglok, S.A., A K-model for simulating the dispersion of sulfur dioxide in a tropical area, *Journal of Interdisciplinary Mathematics*, 10:789-799, 2007.
[7] Konglok, S.A., Pochai, N, Tangmanee, S., A Numerical Treatment of the Mathematical Model for Smoke Dispersion from Two Sources, *Proceeding in International Conference in Mathematics and Applications (ICMA-MU 2009)*, Bangkok, Thailand, December 17th - 19th, 2009.
[8] Naresh, R., Sundar, S. and Shukla, J.B., *Modeling the removal of gaseous pollutants and particulate matters from the atmosphere of a city*, Nonlinear Analysis: Real World Applications, **8(1)** (2007), 337-344.
[9] Ninomiya, H. and Onishi, K., *Flow analysis using a PC*, CRC Press, 1991.
[10] Pasquill, F., *Atmospheric Diffusion*, 2nd edn., Horwood, Chicsester, 1974.
[11] Prez-Chavela, E. , Uribe, F.J. and Velasco, R.M., *The global flow in the Chapman mechanism* , Nonlinear Analysis: Theory, Methods & Applications, **71(1-2)** (2009), 88-95.
[12] Richtmyer, R.D., Morton, K.W., *Difference methods for initial-value problems*, Interscience, New York, 1967.
[13] Yanenko, N.N., *The Method of fractional Steps*, Springer-Verlag, 1971.

Nopparat Pochai received the B.Sc. and M.Sc. degree in mathematics from Khon Kaen University, Khon Kaen, Thailand, in 1999 and 2002 respectively, and Ph.D. degrees in applied mathematics from King Mongkut's University of Technology Thonburi, Bangkok, Thailand, in 2008. He got the scholarships from the University Development Commission Scholarship(U.D.C.), Royal Thai Government. He currently is an assistance professor in mathematics with Department of Mathematics, King Mongkut's Institute of Technology Ladkrabang. His research interests include finite element method, finite difference method, mathematical modelling of water pollution control and numerical computation of flow analysis. Email: konoppar@kmitl.ac.th