

# A Note on Toeplitz Matrices

Hsuan-Chu Li

**Abstract**—In this note, we demonstrate explicit LU factorizations of Toeplitz matrices for some small sizes. Furthermore, we obtain the inverse of referred Toeplitz matrices by applying the above-mentioned results.

**Keywords**—Toeplitz matrices, LU factorization, inverse of a matrix.

## I. INTRODUCTION

LET  $T_n$  be a Toeplitz matrix with size  $n$ , i.e.

$$T_n := T_{\{n, a_{-(n-1)}, a_{-(n-2)}, \dots, a_{n-2}, a_{n-1}\}}$$

$$= \begin{bmatrix} a_0 & a_{-1} & a_{-2} & \cdots & a_{1-n} \\ a_1 & a_0 & a_{-1} & \cdots & a_{2-n} \\ a_2 & a_1 & a_0 & \cdots & a_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_0 \end{bmatrix}$$

where  $a_{-(n-1)}, a_{-(n-2)}, \dots, a_{n-2}, a_{n-1}$  are complex numbers [3, p.1189]. In recent papers, many authors discussed some properties of Toeplitz matrices. Trench[5] firstly established the inversion of a Toeplitz matrix by a low number of its columns and the entries of the original Toeplitz matrix. Gohberg and Semencul [1], Gohberg and Krupnik [4] and Heinig and Rost [2] expressed the inverse of a Toeplitz matrix in different forms.

In this work, we try to obtain the explicit LU factorization of a Toeplitz matrix with size  $n = 2, 3, 4, 5$  respectively. By applying these results, we can easily calculate the determinants of the referred Toeplitz matrix by multiplying the main diagonals of the corresponding upper triangular matrix. Finally, we seek to establish the inverses of Toeplitz matrices by using the following elementary fact:

For any  $n \times n$  matrix  $A$ , let  $\tilde{A}_{ij}$  be the matrix obtained from  $A$  by deleting the  $i$ -th row and the  $j$ -th column. Then

$$A^{-1} = \text{Transpose of } \frac{(-1)^{i+j} \det(\tilde{A}_{ij})}{\det(A)}.$$

Hsuan-Chu Li is with the Center for General Education, Jen-Teh Junior College of Medicine, Nursing and Management, Miaoli, Taiwan, as Assistant Professor (e-mail: k0401001@ms4.kntech.com.tw).

## II. MAIN RESULT AND AN EXAMPLE

Now we are in a position to state the main theorem.

**Theorem:** For  $n = 2, 3, 4, 5$ ,  $T_n$  can be factored as  $T_n = L_n U_n$ , where  $L_n = [L_n(i, j)]$  is a lower triangular matrix with unit main diagonal and  $U_n = [U_n(i, j)]$  is an upper triangular matrix, whose entries are defined as follows:  
 $n = 2$ :

$$T_2 = L_2 U_2, \text{ where } T_2 = \begin{bmatrix} a_0 & a_{-1} \\ a_1 & a_0 \end{bmatrix}, L_2 = \begin{bmatrix} 1 & 0 \\ \frac{a_1}{a_0} & 1 \end{bmatrix} \text{ and}$$

$$U_2 = \begin{bmatrix} a_0 & a_{-1} \\ 0 & \frac{a_0^2 - a_1 a_{-1}}{a_0} \end{bmatrix}.$$

$n = 3$ :

$$T_3 = L_3 U_3, \text{ where } T_3 = \begin{bmatrix} a_0 & a_{-1} & a_{-2} \\ a_1 & a_0 & a_{-1} \\ a_2 & a_1 & a_0 \end{bmatrix};$$

$$L_3 = \begin{bmatrix} 1 & 0 & 0 \\ \frac{a_1}{a_0} & 1 & 0 \\ \frac{a_2}{a_0} & \frac{a_1 a_0 - a_2 a_{-1}}{a_0^2 - a_1 a_{-1}} & 1 \end{bmatrix};$$

$$U_3 = \begin{bmatrix} a_0 & a_{-1} & a_{-2} \\ 0 & \frac{a_0^2 - a_1 a_{-1}}{a_0} & \frac{a_0 a_{-1} - a_1 a_{-2}}{a_0} \\ 0 & 0 & U_3(3,3) \end{bmatrix} \text{ and}$$

$$U_3(3,3) = \frac{(a_0^2 - a_1 a_{-1})(a_0^2 - a_2 a_{-2}) - (a_1 a_0 - a_2 a_{-1})(a_0 a_{-1} - a_1 a_{-2})}{a_0(a_0^2 - a_1 a_{-1})}.$$

$n = 4$ :

$$T_4 = L_4 U_4, \text{ where } T_4 = \begin{bmatrix} a_0 & a_{-1} & a_{-2} & a_{-3} \\ a_1 & a_0 & a_{-1} & a_{-2} \\ a_2 & a_1 & a_0 & a_{-1} \\ a_3 & a_2 & a_1 & a_0 \end{bmatrix};$$

$$L_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{a_1}{a_0} & 1 & 0 & 0 \\ \frac{a_2}{a_0} & \frac{a_1 a_0 - a_2 a_{-1}}{a_0^2 - a_1 a_{-1}} & 1 & 0 \\ \frac{a_3}{a_0} & \frac{a_2 a_0 - a_3 a_{-1}}{a_0^2 - a_1 a_{-1}} & L_4(4,3) & 1 \end{bmatrix}; \text{ where}$$

$$L_4(4,3) = \frac{(a_0^2 - a_1 a_{-1})(a_1 a_0 - a_2 a_{-1}) - (a_2 a_0 - a_3 a_{-1})(a_0 a_{-1} - a_1 a_{-2})}{(a_0^2 - a_1 a_{-1})(a_0^2 - a_2 a_{-2}) - (a_1 a_0 - a_2 a_{-1})(a_0 a_{-1} - a_1 a_{-2})};$$

$$U_4 = \begin{bmatrix} a_0 & a_{-1} & a_{-2} & a_{-3} \\ 0 & \frac{a_0^2 - a_1 a_{-1}}{a_0} & \frac{a_0 a_{-1} - a_1 a_{-2}}{a_0} & \frac{a_0 a_{-2} - a_1 a_{-3}}{a_0} \\ 0 & 0 & U_4(3,3) & U_4(3,4) \\ 0 & 0 & 0 & U_4(4,4) \end{bmatrix};$$

$$U_4(3,3) = \frac{(a_0^2 - a_1 a_{-1})(a_0^2 - a_2 a_{-2}) - (a_1 a_0 - a_2 a_{-1})(a_0 a_{-1} - a_1 a_{-2})}{a_0(a_0^2 - a_1 a_{-1})};$$

$$U_4(3,4) = \frac{(a_0^2 - a_1 a_{-1})(a_0^2 - a_2 a_{-3}) - (a_1 a_0 - a_2 a_{-1})(a_0 a_{-2} - a_1 a_{-3})}{a_0(a_0^2 - a_1 a_{-1})};$$

$$U_4(4,4) = \{ [(a_0^2 - a_1 a_{-1})(a_0^2 - a_2 a_{-2}) - (a_1 a_0 - a_2 a_{-1})(a_0 a_{-1} - a_1 a_{-2})] \\ \times [(a_0^2 - a_1 a_{-1})(a_0^2 - a_3 a_{-3}) - (a_2 a_0 - a_3 a_{-1})(a_0 a_{-2} - a_1 a_{-3})] \\ - [(a_0^2 - a_1 a_{-1})(a_1 a_0 - a_2 a_{-2}) - (a_2 a_0 - a_3 a_{-1})(a_0 a_{-1} - a_1 a_{-2})] \\ \times [(a_0^2 - a_1 a_{-1})(a_0 a_{-1} - a_2 a_{-3}) - (a_1 a_0 - a_2 a_{-1})(a_0 a_{-2} - a_1 a_{-3})] \} \\ \div \{ a_0(a_0^2 - a_1 a_{-1}) [(a_0^2 - a_1 a_{-1})(a_0^2 - a_2 a_{-2}) - (a_1 a_0 - a_2 a_{-1})(a_0 a_{-1} - a_1 a_{-2})] \}$$

$n = 5 :$

$$T_5 = L_5 U_5, \text{ where } T_5 = \begin{bmatrix} a_0 & a_{-1} & a_{-2} & a_{-3} & a_{-4} \\ a_1 & a_0 & a_{-1} & a_{-2} & a_{-3} \\ a_2 & a_1 & a_0 & a_{-1} & a_{-2} \\ a_3 & a_2 & a_1 & a_0 & a_{-1} \\ a_4 & a_3 & a_2 & a_1 & a_0 \end{bmatrix};$$

$$L_5 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ L_5(2,1) & 1 & 0 & 0 & 0 \\ L_5(3,1) & L_5(3,2) & 1 & 0 & 0 \\ L_5(4,1) & L_5(4,2) & L_5(4,3) & 1 & 0 \\ L_5(5,1) & L_5(5,2) & L_5(5,3) & L_5(5,4) & 1 \end{bmatrix};$$

where

$$L_5(2,1) = \frac{a_1}{a_0};$$

$$L_5(3,1) = \frac{a_2}{a_0};$$

$$L_5(4,1) = \frac{a_3}{a_0};$$

$$L_5(5,1) = \frac{a_4}{a_0};$$

$$L_5(3,2) = \frac{a_1 a_0 - a_2 a_{-1}}{a_0^2 - a_1 a_{-1}};$$

$$L_5(4,2) = \frac{a_2 a_0 - a_3 a_{-1}}{a_0^2 - a_1 a_{-1}};$$

$$L_5(5,2) = \frac{a_3 a_0 - a_4 a_{-1}}{a_0^2 - a_1 a_{-1}};$$

$$L_5(4,3) = \frac{(a_0^2 - a_1 a_{-1})(a_1 a_0 - a_2 a_{-1}) - (a_2 a_0 - a_3 a_{-1})(a_0 a_{-1} - a_1 a_{-2})}{(a_0^2 - a_1 a_{-1})(a_0^2 - a_2 a_{-2}) - (a_1 a_0 - a_2 a_{-1})(a_0 a_{-1} - a_1 a_{-2})};$$

$$L_5(5,3) = \frac{(a_0^2 - a_1 a_{-1})(a_2 a_0 - a_3 a_{-1}) - (a_3 a_0 - a_4 a_{-1})(a_0 a_{-1} - a_1 a_{-2})}{(a_0^2 - a_1 a_{-1})(a_0^2 - a_2 a_{-2}) - (a_1 a_0 - a_2 a_{-1})(a_0 a_{-1} - a_1 a_{-2})};$$

$$L_5(5,4) = \{ [(a_0^2 - a_1 a_{-1})(a_0^2 - a_2 a_{-2}) - (a_1 a_0 - a_2 a_{-1})(a_0 a_{-1} - a_1 a_{-2})] \\ \times [(a_0^2 - a_1 a_{-1})(a_0^2 - a_3 a_{-3}) - (a_2 a_0 - a_3 a_{-1})(a_0 a_{-2} - a_1 a_{-3})] \\ - [(a_0^2 - a_1 a_{-1})(a_2 a_0 - a_3 a_{-2}) - (a_3 a_0 - a_4 a_{-1})(a_0 a_{-1} - a_1 a_{-2})] \\ \times [(a_0^2 - a_1 a_{-1})(a_0 a_{-1} - a_2 a_{-3}) - (a_1 a_0 - a_2 a_{-1})(a_0 a_{-2} - a_1 a_{-3})] \} \\ \div \{ [(a_0^2 - a_1 a_{-1})(a_0^2 - a_2 a_{-2}) - (a_1 a_0 - a_2 a_{-1})(a_0 a_{-1} - a_1 a_{-2})] \\ \times [(a_0^2 - a_1 a_{-1})(a_0^2 - a_3 a_{-3}) - (a_2 a_0 - a_3 a_{-1})(a_0 a_{-2} - a_1 a_{-3})] \\ - [(a_0^2 - a_1 a_{-1})(a_1 a_0 - a_2 a_{-2}) - (a_2 a_0 - a_3 a_{-1})(a_0 a_{-1} - a_1 a_{-2})] \\ \times [(a_0^2 - a_1 a_{-1})(a_0 a_{-1} - a_2 a_{-3}) - (a_1 a_0 - a_2 a_{-1})(a_0 a_{-2} - a_1 a_{-3})] \};$$

$$U_5 = \begin{bmatrix} a_0 & a_{-1} & a_{-2} & a_{-3} & a_{-4} \\ 0 & U_5(2,2) & U_5(2,3) & U_5(2,4) & U_5(2,5) \\ 0 & 0 & U_5(3,3) & U_5(3,4) & U_5(3,5) \\ 0 & 0 & 0 & U_5(4,4) & U_5(4,5) \\ 0 & 0 & 0 & 0 & U_5(5,5) \end{bmatrix};$$

where

$$U_5(2,2) = \frac{a_0^2 - a_1 a_{-1}}{a_0};$$

$$U_5(2,3) = \frac{a_0 a_{-1} - a_1 a_{-2}}{a_0};$$

$$U_5(2,4) = \frac{a_0 a_{-2} - a_1 a_{-3}}{a_0};$$



$$\begin{aligned} & \times [(a_0^2 - a_1 a_{-1})(a_0 a_{-2} - a_2 a_{-4}) - (a_1 a_0 - a_2 a_{-1})(a_0 a_{-3} - a_1 a_{-4})] \} \\ & \div \{ a_0^3 (a_0^2 - a_1 a_{-1})^2 [(a_0^2 - a_1 a_{-1})(a_0^2 - a_2 a_{-2}) - (a_1 a_0 - a_2 a_{-1})(a_0 a_{-1} - a_1 a_{-2})] \} \\ & \times [(a_0^2 - a_1 a_{-1})(a_0 a_{-2} - a_2 a_{-4}) - (a_1 a_0 - a_2 a_{-1})(a_0 a_{-3} - a_1 a_{-4})] \} \\ & \div \{ a_0^3 (a_0^2 - a_1 a_{-1})^2 [(a_0^2 - a_1 a_{-1})(a_0^2 - a_2 a_{-2}) - (a_1 a_0 - a_2 a_{-1})(a_0 a_{-1} - a_1 a_{-2})] \} \end{aligned}$$

**Corollary:** The inverses of  $T_2, T_3, T_4, T_5$  are listed as follows:

$$T_2^{-1} = T \text{ ranspose of } \frac{(-1)^{i+j} \det((\tilde{T}_2)_{ij})}{a_0^2 - a_1 a_{-1}};$$

$$T_3^{-1} = T \text{ ranspose of } \frac{(-1)^{i+j} \det((\tilde{T}_3)_{ij})}{(a_0^2 - a_1 a_{-1})(a_0^2 - a_2 a_{-2}) - (a_1 a_0 - a_2 a_{-1})(a_0 a_{-1} - a_1 a_{-2})};$$

$$T_4^{-1} = T \text{ ranspose of } \frac{(-1)^{i+j} \det((\tilde{T}_4)_{ij})}{\det(T_4)}; \text{ where}$$

$$\begin{aligned} \det \tilde{T}_4 = & \{ [(a_0^2 - a_1 a_{-1})(a_0^2 - a_2 a_{-2}) - (a_1 a_0 - a_2 a_{-1})(a_0 a_{-1} - a_1 a_{-2})] \\ & \times [(a_0^2 - a_1 a_{-1})(a_0^2 - a_3 a_{-3}) - (a_2 a_0 - a_3 a_{-1})(a_0 a_{-2} - a_1 a_{-3})] \\ & - [(a_0^2 - a_1 a_{-1})(a_1 a_0 - a_3 a_{-2}) - (a_2 a_0 - a_3 a_{-1})(a_0 a_{-1} - a_1 a_{-2})] \\ & \times [(a_0^2 - a_1 a_{-1})(a_0 a_{-1} - a_2 a_{-3}) - (a_1 a_0 - a_2 a_{-1})(a_0 a_{-2} - a_1 a_{-3})] \} \\ & \div \{ a_0^2 (a_0^2 - a_1 a_{-1}) \}; \end{aligned}$$

$$T_5^{-1} = T \text{ ranspose of } \frac{(-1)^{i+j} \det((\tilde{T}_5)_{ij})}{\det(T_5)}; \text{ where}$$

$$\begin{aligned} \det \tilde{T}_5 = & \{ \{ [(a_0^2 - a_1 a_{-1})(a_0^2 - a_2 a_{-2}) - (a_1 a_0 - a_2 a_{-1})(a_0 a_{-1} - a_1 a_{-2})] \\ & \times [(a_0^2 - a_1 a_{-1})(a_0^2 - a_3 a_{-3}) - (a_2 a_0 - a_3 a_{-1})(a_0 a_{-2} - a_1 a_{-3})] \\ & - [(a_0^2 - a_1 a_{-1})(a_1 a_0 - a_3 a_{-2}) - (a_2 a_0 - a_3 a_{-1})(a_0 a_{-1} - a_1 a_{-2})] \\ & \times [(a_0^2 - a_1 a_{-1})(a_0 a_{-1} - a_2 a_{-3}) - (a_1 a_0 - a_2 a_{-1})(a_0 a_{-2} - a_1 a_{-3})] \} \\ & \times \{ [(a_0^2 - a_1 a_{-1})(a_0^2 - a_2 a_{-2}) - (a_1 a_0 - a_2 a_{-1})(a_0 a_{-1} - a_1 a_{-2})] \\ & \times [(a_0^2 - a_1 a_{-1})(a_0^2 - a_4 a_{-4}) - (a_3 a_0 - a_4 a_{-1})(a_0 a_{-3} - a_1 a_{-4})] \\ & - [(a_0^2 - a_1 a_{-1})(a_2 a_0 - a_4 a_{-2}) - (a_3 a_0 - a_4 a_{-1})(a_0 a_{-1} - a_1 a_{-2})] \\ & \times [(a_0^2 - a_1 a_{-1})(a_0 a_{-2} - a_2 a_{-4}) - (a_1 a_0 - a_2 a_{-1})(a_0 a_{-3} - a_1 a_{-4})] \} \\ & - \{ [(a_0^2 - a_1 a_{-1})(a_0^2 - a_2 a_{-2}) - (a_1 a_0 - a_2 a_{-1})(a_0 a_{-1} - a_1 a_{-2})] \\ & \times [(a_0^2 - a_1 a_{-1})(a_1 a_0 - a_4 a_{-3}) - (a_3 a_0 - a_4 a_{-1})(a_0 a_{-2} - a_1 a_{-3})] \\ & - [(a_0^2 - a_1 a_{-1})(a_2 a_0 - a_4 a_{-2}) - (a_3 a_0 - a_4 a_{-1})(a_0 a_{-1} - a_1 a_{-2})] \\ & \times [(a_0^2 - a_1 a_{-1})(a_0 a_{-1} - a_2 a_{-3}) - (a_1 a_0 - a_2 a_{-1})(a_0 a_{-2} - a_1 a_{-3})] \} \\ & \times \{ [(a_0^2 - a_1 a_{-1})(a_0^2 - a_2 a_{-2}) - (a_1 a_0 - a_2 a_{-1})(a_0 a_{-1} - a_1 a_{-2})] \\ & \times [(a_0^2 - a_1 a_{-1})(a_0 a_{-1} - a_3 a_{-4}) - (a_2 a_0 - a_3 a_{-1})(a_0 a_{-3} - a_1 a_{-4})] \\ & - [(a_0^2 - a_1 a_{-1})(a_1 a_0 - a_3 a_{-2}) - (a_2 a_0 - a_3 a_{-1})(a_0 a_{-1} - a_1 a_{-2})] \} \end{aligned}$$

**Example:** Let  $a_4 = 2, a_3 = 7, a_2 = 4, a_1 = 15, a_0 = 6, a_{-1} = 21, a_{-2} = 8, a_{-3} = 28, a_{-4} = 10$ , then  $T_5 = L_5 U_5$ , where

$$T_5 = \begin{bmatrix} 6 & 21 & 8 & 28 & 10 \\ 14 & 6 & 21 & 8 & 28 \\ 4 & 14 & 6 & 21 & 8 \\ 7 & 4 & 14 & 6 & 21 \\ 2 & 7 & 4 & 14 & 6 \end{bmatrix};$$

$$L_5 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 7 & 1 & 0 & 0 & 0 \\ 3 & 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 & 0 \\ 7 & 41 & 917 & 1 & 0 \\ 6 & 86 & 258 & 1 & 0 \\ 1 & 0 & 2 & 0 & 1 \\ 3 & 0 & 2 & 0 & 1 \end{bmatrix};$$

$$U_5 = \begin{bmatrix} 6 & 21 & 8 & 28 & 10 \\ 0 & -43 & 7 & -172 & 14 \\ 0 & 0 & 3 & 3 & 3 \\ 0 & 0 & 2 & 7 & 4 \\ 0 & 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & -5903 & 917 \\ 0 & 0 & 0 & 774 & 387 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix};$$

and

$$\begin{aligned} \det T_5 &= \prod_{i=1}^5 U_5(i, i) \\ &= 6 \times (-43) \times \frac{2}{3} \times (-\frac{5903}{774}) \times 0 \\ &= 0. \end{aligned}$$

#### REFERENCES

- [1] I. Gohberg, A. Semencul, On the inversion of finite Toeplitz matrices and their continuous analogues, Mat. Issled. 7 (1972) 201-223 (in Russian).
- [2] G. Heinig, K. Rost, Algebraic methods for Toeplitz-like matrices and operators, in: Operator Theory: Advances and Applications, vol. 13, Birkhäuser, Basel, 1984.
- [3] Xiao-Guang Lv, Ting-Zhu Huang, A note on inversion of Toeplitz matrices, Applied Mathematics Letters 20 (2007) 1189-119.
- [4] I. Gohberg, N. Krupnik, A formula for the inversion of finite Toeplitz matrices, Mat. Issled. 7 (12) (1972) 272-283 (in Russian).
- [5] W. F. Trench, An algorithm for the inversion of finite Toeplitz matrix, J. SIAM 13 (3) (1964) 515-522.