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Generalised Slant Weighted Toeplitz Operator

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Abstract—A slant weighted Toeplitz operator A_ϕ is an operator on $L^2(\beta)$ defined as $A_\phi = WM_\phi$ where M_ϕ is the weighted multiplication operator and W is an operator on $L^2(\beta)$ given by $We_{2n} = \frac{\beta_n}{\beta_{2n}} e_n$, $\{e_n\}_{n\in Z}$ being the orthonormal basis. In this paper, we generalise A_ϕ to the k-th order slant weighted Toeplitz operator U_ϕ and study its properties.

Keywords—Slant weighted Toeplitz operator, weighted multiplication operator.

I. Introduction

O. Toeplitz [5] introduced the Toeplitz operators in the year 1911 and later many mathematicians came up with different generalisations of the Toeplitz operators. In 1995, Ho [2] introduced the class of slant Toeplitz operators having the property that the matrices with respect to the standard orthonormal basis could be obtained by eliminating every alternate row of the matrices of the corresponding Toepltiz operators. All these operators arise in plenty of applications like prediction theory [6], solution of differential equations [7] and wavelet analysis [8]. However, these studies were made in context of the usual Hardy spaces and Lorentz spaces. In the mean time, the notion of weighted sequence spaces $H^2(\beta)$ and $L^2(\beta)$ also gained momentum. Shields [4] made a systematic study of the multiplication operator on $L^2(\beta)$, while Lauric [3] studied particular cases of Toeplitz operators on $H^2(\beta)$. Motivated by these studies and the various applications of slant Toeplitz operators, we introduced and studied [1] the notion of a slant weighted Toeplitz operator on $L^2(\beta)$. In this paper we extend our study to the k-th order slant weighted Toeplitz operator. We now begin with the notations and preliminaries.

Let $\beta = \{\beta_n\}_{n\in Z}$ be a sequence of positive numbers such that $\beta_0 = 1$, $0 < \frac{\beta_n}{\beta_{n+1}} \leq 1$ for all $n \geq 0$ and $0 < \frac{\beta_n}{\beta_{n-1}} \leq 1$ for all $n \leq 0$. Consider the spaces [4], [3]

$$L^2(\beta) = \left\{ f(z) = \sum_{n=-\infty}^{\infty} a_n z^n | a_n \in \mathbb{C} \text{ and } \|f\|_\beta^2 = \sum_{n=-\infty}^{\infty} |a_n|^2 \beta_n^2 < \infty \right\}$$

and

$$H^2(\beta) = \left\{ f(z) = \sum_{n=0}^{\infty} a_n z^n | a_n \in \mathbb{C} \text{ and } \|f\|_\beta^2 = \sum_{n=0}^{\infty} |a_n|^2 \beta_n^2 < \infty \right\}.$$

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Then $(L^2(\beta), \|\cdot\|_\beta)$ is a Hilbert space with an orthonormal basis $\left\{e_n(z) = \frac{z^n}{\beta_n}\right\}_{n\in Z}$ and with respect to the inner product defined by $\left\langle \sum a_n z^n, \sum b_n z^n \right\rangle = \sum a_n \bar{b}_n \beta_n^2$.

Also, $H^2(\beta)$ is a subspace of $L^2(\beta)$. If,

$$L^\infty(\beta) = \left\{ \phi(z) = \sum_{n=-\infty}^{\infty} a_n z^n | \phi L^2(\beta) \subseteq L^2(\beta) \text{ and } \exists\, c \in \mathbb{R} \text{ such that } \|\phi f\|_\beta \leq c\|f\|_\beta \text{ for all } f \in L^2(\beta) \right\},$$

then $L^\infty(\beta)$ is a Banach space with a norm defined by

$$\|\phi\|_\infty = \inf\{c | \|\phi f\|_\beta \leq c\|f\|_\beta \quad \text{for all} \quad f \in L^2(\beta)\}.$$

If $P : L^2(\beta) \to H^2(\beta)$ is the orthogonal projection of $L^2(\beta)$ onto $H^2(\beta)$, then the weighted Toeplitz operator T_ϕ on $H^2(\beta)$ [3] with symbol ϕ in $L^\infty(\beta)$ is defined by

$$T_\phi f = P(\phi f) \quad \text{for all} \quad f \in H^2(\beta).$$

Let M_ϕ denote the weighted multiplication operator on $L^2(\beta)$. Then,

$$M_\phi(f) = \phi f \quad \text{for all} \quad f \in L^2(\beta),$$

and,

$$M_\phi e_k(z) = \sum_{n=-\infty}^{\infty} a_{n-k} \frac{\beta_n e_n(z)}{\beta_k} \quad \text{for all} \quad k \in Z.$$

If $W : L^2(\beta) \to L^2(\beta)$ is defined as $We_{2n}(z) = \frac{\beta_n}{\beta_{2n}} e_n(z)$ and $We_{2n-1}(z) = 0$, then a slant weighted Toeplitz operator [1] $A_\phi : L^2(\beta) \to L^2(\beta)$ is given by: $A_\phi(f) = WM_\phi f$ for all $f \in L^2(\beta)$.

Hence $A_\phi e_k(z) = \sum \frac{a_{2n-k}}{\beta_k} \beta_n e_n(z)$. The matrix of A_ϕ is as follows:

$$\left[\begin{array}{c|cccc}
\cdots & \cdots & \cdots & \cdots & \cdots \\ \hline
\cdots & a_0\frac{\beta_0}{\beta_0} & a_{-1}\frac{\beta_0}{\beta_1} & a_{-2}\frac{\beta_0}{\beta_2} & \cdots \\
\cdots & a_2\frac{\beta_1}{\beta_0} & a_1\frac{\beta_1}{\beta_1} & a_0\frac{\beta_1}{\beta_2} & \cdots \\
\cdots & a_4\frac{\beta_2}{\beta_0} & a_3\frac{\beta_2}{\beta_1} & a_2\frac{\beta_2}{\beta_2} & \cdots \\
\cdots & a_6\frac{\beta_3}{\beta_0} & a_5\frac{\beta_3}{\beta_1} & a_4\frac{\beta_3}{\beta_2} & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots
\end{array}\right]$$

In the following section we introduce the k-th order slant weighted Toeplitz operator U_ϕ which is a generalisation of A_ϕ. A_ϕ is the particular case of U_ϕ for $k = 2$. We have studied some properties of A_ϕ in [1]. In this paper we investigate those results and many more other properties for the generalised operator U_ϕ. Henceforth we assume that $\frac{\beta_{kn}}{\beta_n} \le M < \infty$.

II. k-TH ORDER SLANT WEIGHTED TOEPLITZ OPERATOR

Let $W_k : L^2(\beta) \to L^2(\beta)$ be defined as

$$W_k e_n(z) = \begin{cases} \frac{\beta_{n/k}}{\beta_n} e_{n/k}(z) & \text{if } n \text{ is divisible by } k \\ 0 & \text{otherwise.} \end{cases}$$

The adjoint of W_k is given by

$$W_k^* e_n(z) = \frac{\beta_n}{\beta_{kn}} e_{kn}(z) \quad \text{for all} \quad n \in Z.$$

Also, the matrix of W_k is

$$\left[\begin{array}{c|cccccccccccc} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \hline \cdots & \frac{\beta_0}{\beta_0} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & \\ \cdots & \underbrace{0 \quad 0 \quad 0 \quad 0 \quad \cdots}_{k\text{-times}} & & & & & \frac{\beta_1}{\beta_k} & 0 & \cdots & \cdots & \cdots & \cdots & \\ \cdots & 0 & 0 & 0 & 0 & \cdots\cdots & 0 & \cdots & 0 & \frac{\beta_2}{\beta_{2k}} & \cdots & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{array}\right]$$

Therefore $\|W_k\| = \sup \frac{\beta_n}{\beta_{kn}} \le 1$.

Definition II.1. For an integer $k \ge 2$, we define the k-th order slant weighted Toeplitz operator $U_\phi : L^2(\beta) \to L^2(\beta)$ as $U_\phi(f) = W_k M_\phi(f)$ for all $f \in L^2(\beta)$. The effect of U_ϕ on the orthonormal basis $\{e_i(z) = \frac{z^i}{\beta_i}\}_{i \in Z}$ can be given by: $U_\phi e_i(z) = \frac{1}{\beta_i} \sum_{n=-\infty}^{\infty} a_{nk-i} \beta_n e_n(z)$. Also, the (i,j)th element of U_ϕ is given by $\langle U_\phi e_j, e_i\rangle = \left\langle \frac{1}{\beta_j} \sum_{n=-\infty}^{\infty} a_{nk-j}\beta_n e_n(z), e_i(z)\right\rangle = a_{ik-j}\frac{\beta_i}{\beta_j}$. Therefore the matrix of U_ϕ with respect to this basis is as follows:

$$\left[\begin{array}{cc|cccc} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & a_{-k+1}\frac{\beta_{-1}}{\beta_{-1}} & a_{-k}\frac{\beta_{-1}}{\beta_0} & a_{-k-1}\frac{\beta_{-1}}{\beta_1} & a_{-k-2}\frac{\beta_{-1}}{\beta_2} & \cdots \\ \hline \cdots & a_1\frac{\beta_0}{\beta_{-1}} & a_0\frac{\beta_0}{\beta_0} & a_{-1}\frac{\beta_0}{\beta_1} & a_{-2}\frac{\beta_0}{\beta_2} & \cdots \\ \cdots & a_{k+1}\frac{\beta_1}{\beta_{-1}} & a_k\frac{\beta_1}{\beta_0} & a_{k-1}\frac{\beta_1}{\beta_1} & a_{k-2}\frac{\beta_1}{\beta_2} & \cdots \\ \cdots & a_{2k+1}\frac{\beta_2}{\beta_{-1}} & a_{2k}\frac{\beta_2}{\beta_0} & a_{2k-1}\frac{\beta_2}{\beta_1} & a_{2k-2}\frac{\beta_2}{\beta_2} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{array}\right]$$

The adjoint of U_ϕ, denoted by U_ϕ^* is given by

$$\langle U_\phi^* e_j, e_i\rangle = \langle e_j, U_\phi e_i\rangle = \bar{a}_{jk-i}\frac{\beta_j}{\beta_i}.$$

The matrix of U_ϕ^* can be obtained now.

III. ALGEBRAIC PROPERTIES

We make the following observations:

Theorem III.1. (i) $W_k = U_1$
(ii) *U_ϕ is bounded*
(iii) *P reduces W_k*
(iv) $W_k M_{z^p} W_k^* = 0$ *for* $p = 1, 2, \ldots, k-1$.

Proof: (i) Take $\phi = 1$ in $U_\phi = W_k M_\phi$.

(ii)
$$\begin{aligned} \|U_\phi\| &= \|W_k M_\phi\| \\ &\le \|W_k\|\|M_\phi\| \\ &\le \|\phi\|_\infty \end{aligned}$$

since $\|M_\phi\| = \|\phi\|_\infty$ as shown by Shields [4].
(iii) Case (a) $i = kn$, $n \in \mathbb{N} \cup \{0\}$.

$$\begin{aligned} PW_k e_i(z) = PW_k e_{kn}(z) &= P\frac{\beta_n}{\beta_{kn}} e_n(z) \\ &= \frac{\beta_n}{\beta_{kn}} e_n(z) \\ &= W_k e_{kn}(z) \\ &= W_k P e_{kn}(z) \\ &= W_k P e_i(z). \end{aligned}$$

Case (b) $i = kn$, n is a negative integer.
Then

$$PW_k e_i(z) = P\frac{\beta_n}{\beta_{kn}} e_n(z) = 0 = W_k P e_i(z).$$

Case (c) i is not a multiple of k.

$$PW_k e_i(z) = 0 = W_k P e_i(z) \quad \text{for all} \quad i \in Z.$$

Hence we conclude that

$$PW_k = W_k P \tag{1}$$

Now for all $n \in \mathbb{N} \cup \{0\}$,

$$PW_k^* e_n(z) = W_k^* e_n(z) = W_k^* P e_n(z).$$

For negative integers n,

$$PW_k^* e_n(z) = W_k^* P e_n(z)$$

Thus

$$PW_k^* = W_k^* P \tag{2}$$

From (1) and (2) we get that P reduces W_k.
(iv) Consider $f \in L^2(\beta)$. Then $W_k^* f$ lies in the closed span of $\{e_{kn}(z) : n \in Z\}$. Hence $M_z(W_k^* f)$ belongs to the closed span of $\{e_{kn+1}(z) : n \in Z\}$. Also, $M_{z^2}(W_k^* f)$ belongs to the closed span of $\{e_{kn+2}(z) : n \in Z\}$ and so on. In fact, for all $p = 1, 2, \ldots k-1$, $M_{z^p}(W_k^* f)$ belongs to the closed span of $\{e_{kn+p}(z) : n \in Z\}$. Hence $W_k M_{z^p} W_k^*(f) = 0$ for all $f \in L^2(\beta)$. Therefore $W_k M_{z^p} W_k^* = 0$ for all $p = 1, 2, \ldots k-1$. ■

Lemma III.2. *If $h(z)$ is an $L^2(\beta)$ function and for a fixed integer $k \ge 2$, $\frac{\beta_{kn}}{\beta_n} \le M < \infty$, then $h(z^k)$ is also an $L^2(\beta)$ function.*

Proof: Let $h(z) = \sum_{n=-\infty}^{\infty} \alpha_n z^n$ be an $L^2(\beta)$ function.
Then

$$\|h(z)\|_\beta^2 = \sum_{n=-\infty}^{\infty} |\alpha_n|^2 \beta_n^2 < \infty \tag{3}$$

Also, then

$$h(z^k) = \sum_{n=-\infty}^{\infty} \alpha_n z^{kn} = \sum_{n=-\infty}^{\infty} \alpha_n \beta_{kn} e_{kn}(z) \, .$$

Hence

$$\begin{aligned}\|h(z^k)\|_\beta^2 &= \sum_{n=-\infty}^{\infty} |\alpha_n|^2 \beta_{kn}^2 \\ &= \sum_{n=-\infty}^{\infty} |\alpha_n|^2 \beta_n^2 \times \frac{\beta_{kn}^2}{\beta_n^2} \\ &\leq M^2 \sum_{n=-\infty}^{\infty} |\alpha_n|^2 \beta_n^2 < \infty\end{aligned}$$

from (3).
Therefore $h(z^k)$ is also an $L^2(\beta)$ function. ■

Theorem III.3. *For all f in $L^2(\beta)$,*
(i) $W_k^* f \in L^2(\beta)$ *if* $\frac{\beta_{kn}}{\beta_n}$ *is bounded .*
(ii) $W_k f \in L^2(\beta)$.

Proof: (i) Let $f(z) = \sum a_n z^n$ be in $L^2(\beta)$. Then

$$\begin{aligned}W_k^* f(z) &= W_k^* \sum a_n z^n \\ &= W_k^* \sum a_n \beta_n e_n(z) \\ &= \sum a_n \beta_n W_k^* e_n(z) \\ &= \sum a_n \beta_n \frac{\beta_n}{\beta_{kn}} e_{kn}(z) \\ &= \sum a_n z^{kn} \frac{\beta_n^2}{\beta_{kn}^2} \\ &= f_k(z^k)\end{aligned}$$

where

$$f_k(z) = \sum a_n \frac{\beta_n^2}{\beta_{kn}^2} z^n.$$

We claim that $f_k(z) \in L^2(\beta)$.
Since

$$\begin{aligned}\|f_k(z)\|_\beta^2 &= \sum |a_n|^2 \frac{\beta_n^4}{\beta_{kn}^4} \beta_n^2 \\ &\leq \sum |a_n|^2 \beta_n^2 < \infty.\end{aligned}$$

Hence $f_k(z)$ is in $L^2(\beta)$. Also, then by Lemma III.2, $f_k(z^k)$ is in $L^2(\beta)$. This implies that $W_k^* f(z) \in L^2(\beta)$.
(ii) Let $f = \sum a_n z^n \in L^2(\beta)$. Then

$$\begin{aligned}W_k(f) &= W_k \sum a_n z^n \\ &= \sum a_n W_k \beta_n e_n(z).\end{aligned}$$

Since W_k eliminates all other terms, we consider only those terms for which n is a multiple of k. That is $n = km$ (say). Then

$$\begin{aligned}W_k(f) &= \sum_{m=-\infty}^{\infty} a_{km} \beta_m e_m(z) \\ \Rightarrow \quad \|W_k(f)\|_\beta^2 &= \sum_{m=-\infty}^{\infty} |a_{km}|^2 \beta_m^2 \\ &= \sum_{m=-\infty}^{\infty} |a_{km}|^2 \beta_{km}^2 \times \frac{\beta_m^2}{\beta_{km}^2} \qquad (4) \\ &\leq \sum_{m=-\infty}^{\infty} |a_{km}|^2 \beta_{km}^2 < \infty \qquad (5)\end{aligned}$$

Now,

$$\begin{aligned}&\sum_n |a_n|^2 \beta_n^2 < \infty \\ \Rightarrow \quad &\sum |a_{km}|^2 \beta_{km}^2 + \sum_{n \neq km} |a_n|^2 \beta_n^2 < \infty \\ \Rightarrow \quad &\sum_m |a_{km}|^2 \beta_{km}^2 < \infty \qquad (6)\end{aligned}$$

Hence, we conclude that $W_k f \in L^2(\beta)$. ■

Theorem III.4. $W_k(f(z^k)) = f(z)$ *for all* $f \in L^2(\beta)$.

Proof: Let $f = \sum a_n z^n$ be in $L^2(\beta)$. Then

$$\begin{aligned}W_k(f(z^k)) &= W_k \sum a_n z^{kn} \\ &= \sum a_n W_k(\beta_{kn} e_{kn}(z)) \\ &= \sum a_n \beta_n e_n(z) \\ &= \sum a_n z^n \\ &= f(z).\end{aligned}$$

■

Corollary III.5. $W_k \phi(z^k) = \phi(z)$ *for all* $\phi \in L^\infty(\beta)$.

Lemma III.6. (i) $W_k W_k^* f(z) = f_k(z)$ *where* $f_k(z) = \sum a_n \frac{\beta_n^2}{\beta_{kn}^2} z^n$.
(ii) $W_k^* W_k f(z) = h(z^k)$ *where* $h(z) = \sum a_{kn} \frac{\beta_n^2}{\beta_{kn}^2} z^n$, $n \in Z$.

Proof: (i) $W_k W_k^* f(z) = W_k f_k(z^k) = f_k(z)$.
Thus $W_k W_k^* \neq I$ as in the case of ordinary space L^2 [2].

(ii)
$$\begin{aligned}W_k^* W_k f(z) &= W_k^* \sum a_{kn} z^n \\ &= \sum a_{kn} W_k^* \beta_n e_n(z) \\ &= \sum a_{kn} \beta_n \frac{\beta_n}{\beta_{kn}} e_{kn}(z) \\ &= \sum a_{kn} \frac{\beta_n^2}{\beta_{kn}^2} z^{kn}\end{aligned}$$

Thus $W_k^* W_k f(z) = h(z^k)$, where

$$h(z) = \sum a_{kn} \frac{\beta_n^2}{\beta_{kn}^2} z^n, \quad n \in Z\,.$$

■

Theorem III.7. *If $\beta = \{\beta_n\}_{n \in Z}$ is a sequence of positive numbers such that $\beta_0 = 1$, $0 < \frac{\beta_n}{\beta_{n+1}} \leq 1$ for $n \geq 0$, $0 < \frac{\beta_n}{\beta_{n-1}} \leq 1$ for $n \leq 0$ and $\frac{\beta_{kn}}{\beta_n} \leq M < \infty$, then a bounded operator U on $L^2(\beta)$ is a k-th order slant weighted Toeplitz operator if and only if $M_z U = U M_{z^k}$.*

Proof: For necessity: Let $U = U_\phi$ be a slant weighted

Toeplitz operator. Then,

$$\begin{aligned} M_z U e_i(z) &= M_z U_\phi e_i(z) \\ &= M_z\left(\sum_{n=-\infty}^{\infty} \frac{a_{kn-i}\beta_n e_n(z)}{\beta_i}\right) \\ &= \sum_{n=-\infty}^{\infty} \frac{a_{kn-i}\beta_{n+1} e_{n+1}(z)}{\beta_i} \\ &= \sum_{n=-\infty}^{\infty} \frac{a_{k(n-1)-i}\beta_n e_n(z)}{\beta_i} \qquad (7) \end{aligned}$$

On the other hand,

$$\begin{aligned} & UM_{z^k} e_i(z) \\ &= U_\phi M_{z^k} e_i(z) \\ &= U_\phi\left(\frac{\beta_{i+k}}{\beta_i} e_{i+k}(z)\right) = \frac{\beta_{i+k}}{\beta_i} U_\phi e_{i+k}(z) \\ &= \frac{\beta_{i+k}}{\beta_i}\frac{1}{\beta_{i+k}} \sum_{n=-\infty}^{\infty} a_{kn-i-k}\beta_n e_n(z) \qquad (8) \end{aligned}$$

Hence, from (7) and (8) we conclude that $M_z U = UM_{z^k}$.
For sufficiency: Suppose that U is a bounded operator on $L^2(\beta)$ which satisfies $M_z U = UM_{z^k}$ and let $f = \sum a_n z^n$ be in $L^2(\beta)$. Then,

$$\begin{aligned} U(f(z^k)) &= U\sum a_n z^{kn} \\ &= \sum a_n UM_{z^{kn}} 1 \\ &= \sum a_n M_{z^n} U1 \\ &= \sum a_n z^n U1 = f(z)U1 \end{aligned}$$

By Lemma III.2,

$$\begin{aligned} \|f(z)U1\|_\beta &= \|Uf(z^k)\|_\beta \\ &\le \|U\|\|f(z^k)\|_\beta \\ &\le M\|U\|\,\|f(z)\|_\beta < \infty\,. \end{aligned}$$

Now, take $U1 = \phi_0(z)$. Then $\phi_0(z)$ is bounded. Similarly, we can show that

$$\begin{aligned} U(zf(z^k)) &= f(z)Uz \\ U(z^2 f(z^k)) &= f(z)Uz^2 \\ &\vdots \\ U(z^{k-1} f(z^k)) &= f(z)Uz^{k-1}. \end{aligned}$$

Also then, on taking $\phi_1(z)=Uz$, $\phi_2(z)=Uz^2, \ldots, \phi_{k-1}(z)= Uz^{k-1}$ we get that $\phi_1(z), \phi_2(z) \ldots \phi_{k-1}(z)$ are all bounded. So, by Lemma III.2, each $\phi_j(z^k)$ is bounded for $j = 0, 1, 2, \ldots, k-1$. Hence the function

$$\phi(z) = \phi_0(z^k) + \bar{z}\phi_1(z^k) + \ldots + \bar{z}^{k-1}\phi_{k-1}(z^k)$$

is also bounded. Next we show that ϕ is, infact, the inducing function for the k-th order slant weighted Toeplitz operator $U_\phi = U$. Or, in other words, we show that $U = W_k M_\phi$. Since $f \in L^2(\beta)$, we can write

$$f(z) = f_0(z^k) + zf_1(z^k) + \ldots + z^{k-1} f_{k-1}(z^k),$$

where $f_0, f_1, \ldots, f_{k-1}$ are all in $L^2(\beta)$. Then,

$$\begin{aligned} & W_k M_\phi(f) \\ &= W_k[\phi f] \\ &= W_k[[\phi_0(z^k) + \bar{z}\phi_1(z^k) + \ldots + \bar{z}^{k-1}\phi_{k-1}(z^k)] \\ &\quad \times [f_0(z^k) + zf_1(z^k) + \ldots z^{k-1} f_{k-1}(z^k)]] \end{aligned}$$

Now, as W_k eliminates all other terms, we consider only the following terms. We get

$$\begin{aligned} & W_k M_\phi(f) \\ &= W_k[\phi_0(z^k) f_0(z^k)] + W_k[\phi_1(z^k) f_1(z^k)] + \ldots \\ &\quad + W_k[\phi_{k-1}(z^k) f_{k-1}(z^k)] \\ &= \phi_0(z) f_0(z) + \phi_1(z) f_1(z) + \ldots + \phi_{k-1}(z) f_{k-1}(z) \\ &= f_0(z)U1 + f_1(z)Uz + \ldots + f_{k-1}(z)Uz^{k-1} \\ &= Uf_0(z^k) + Uzf_1(z^k) + \ldots + Uz^{k-1} f_{k-1}(z^{k-1}) \\ &= Uf. \end{aligned}$$

■

Corollary III.8. *$M_\phi U_\psi$ is a k-th order slant weighted Toeplitz operator.*

Proof:

$$\begin{aligned} M_z M_\phi U_\psi &= M_\phi M_z U_\psi \\ &= M_\phi U_\psi M_{z^k}\,. \end{aligned}$$

■

References

[1] S.C. Arora and Ritu Kathuria, *Properties of slant weighted Toeplitz operator*. Annals of Functional Analysis, (to appear).
[2] M.C. Ho, *Properties of Slant Toeplitz Operators*. Indiana Univ. Math. J., **45**(3) (1996), 843–862.
[3] Vasile Lauric, *On a weighted Toeplitz operator and its commutant*. Int. J. Math. & Math. Sci., **6** (2005), 823–835.
[4] A.L. Shields, *Weighted shift operators and analytic function theory*. Topics in Operator Theory, Math. Surveys 13, Amer. Math. Soc. Providence, R.I., 1974.
[5] O. Toeplitz, *Zur theorie der quadratishen and bilinearan Formen Von unendlichvielen, Veranderlichen*. Math. Ann. 70 (1911), 351–376.
[6] Helson and Szego, *A problem in prediction theory*. Ann. Math. Pura Appl. 51 (1960), 107–138.
[7] T. Goodman, C. Micchelli and J. Ward. *Spectral radius formula for subdivision operators*. Recent Advances in Wavelet Analysis, ed. L. Schumaker and G. Webb, Academic Press (1994), 335–360.
[8] L. Villemoes, *Wavelet analysis of refinement equations*. SIAM J. Math. Analysis 25 (1994), 1433–1460.