

# The Game of Maundy Block

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**Abstract**—The game of Maundy Block is the three-player variant of Maundy Cake, a classical combinatorial game. Even though to determine the solution of Maundy Cake is trivial, solving Maundy Block is challenging because of the identification of *queer* games, i.e., games where no player has a winning strategy.

**Keywords**—Combinatorial game, Maundy Cake, Three-player partizan games.

## I. INTRODUCTION

THE game of Maundy Block is a three-player version of Maundy Cake [1]. Every instance of this game is defined as a set of blocks of integer side-lengths, with edges parallel to the  $x$ -,  $y$ -, and  $z$ - axes. A legal move for Left is to divide one of the blocks into *any* number of blocks of *equal* integer side-length by means of a certain number of cuts perpendicular to the  $x$ - axis; analogously, we define the legal moves for Center and Right. Players take turns making legal moves in cyclic fashion (. . . , Left, Center, Right, Left, Center, Right, . . .). When one of the three players is not more able to move, he/she leaves the game and the remaining players continue in alternation until one of them cannot move. Then that player leaves the game and the remaining player is the winner.

**Definition 1:** Given a positive integer  $n \geq 2$ , the prime factorization is written  $n = p_1 p_2 \dots p_k$  where the  $p_i$ s are the  $k$  prime factors. We define  $d(n) = k$  and  $d(1) = 0$ .

We recall that in the game of Maundy Cake the outcome for a  $l$  by  $r$  rectangle depends on the dimension of  $l$  and  $r$  as shown in Table I.

TABLE I

	Left starts	Right starts
$d(l) > d(r)$	Left wins	Left wins
$d(l) < d(r)$	Right wins	Right wins
$d(l) = d(r)$	Right wins	Left wins

## II. THREE-PLAYER PARTIZAN GAMES

For the sake of self-containment we recall the basic definitions and main results concerning a mathematical theory to classify three-player partizan games [2]. Such a theory is an extension of Conway's theory of partizan games [3] and, as a consequence, it is both a theory of games and a theory of numbers.

**Definition 2:** If  $L, C, R$  are any three sets of numbers previously defined and

- 1) no element of  $L$  is  $\geq_L$  any element of  $C \cup R$ , and

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- 2) no element of  $C$  is  $\geq_C$  any element of  $L \cup R$ , and
  - 3) no element of  $R$  is  $\geq_R$  any element of  $L \cup C$ ,
- then  $\{L|C|R\}$  is a number. All numbers are constructed in this way.

This definition for numbers is based on the definition and comparison operators for games given in the following two definitions.

**Definition 3:** If  $L, C, R$  are any three sets of games previously defined then  $\{L|C|R\}$  is a game. All games are constructed in this way.

**Definition 4:** We say that

- $x \geq_L y$  iff  $(y \geq_L \text{ no } x^C, y \geq_L \text{ no } x^R \text{ and no } y^L \geq_L x)$ ,
- $x \leq_L y$  iff  $y \geq_L x$ ,
- $x \geq_C y$  iff  $(y \geq_C \text{ no } x^L, y \geq_C \text{ no } x^R \text{ and no } y^C \geq_C x)$ ,
- $x \leq_C y$  iff  $y \geq_C x$ ,
- $x \geq_R y$  iff  $(y \geq_R \text{ no } x^L, y \geq_R \text{ no } x^C \text{ and no } y^R \geq_R x)$ ,
- $x \leq_R y$  iff  $y \geq_R x$ .

where  $x^L, x^C, x^R$  are respectively the typical elements of  $L, C$ , and  $R$ .

We write

- $x \not\geq_L y$  to mean that  $x \geq_L y$  does not hold,
- $x \not\geq_C y$  to mean that  $x \geq_C y$  does not hold,
- $x \not\geq_R y$  to mean that  $x \geq_R y$  does not hold.

**Definition 5:** We say that

- $x =_L y$  iff  $(x \geq_L y \text{ and } y \geq_L x)$ ,
- $x >_L y$  iff  $(x \geq_L y \text{ and } y \not\geq_L x)$ ,
- $x <_L y$  iff  $y >_L x$ ,
- $x =_C y$  iff  $(x \geq_C y \text{ and } y \geq_C x)$ ,
- $x >_C y$  iff  $(x \geq_C y \text{ and } y \not\geq_C x)$ ,
- $x <_C y$  iff  $y >_C x$ ,
- $x =_R y$  iff  $(x \geq_R y \text{ and } y \geq_R x)$ ,
- $x >_R y$  iff  $(x \geq_R y \text{ and } y \not\geq_R x)$ ,
- $x <_R y$  iff  $y >_R x$ ,
- $x = y$  if and only if  $(x =_L y, x =_C y, \text{ and } x =_R y)$ .

All the given definition are inductive, so that to decide whether  $x \geq_L y$  we check the pairs  $(x^C, y)$ ,  $(x^R, y)$ , and  $(x, y^L)$ .

**Theorem 1:** For any number  $x$

- $x^L <_L x, x <_L x^C, x <_L x^R$ ,
- $x^C <_C x, x <_C x^L, x <_C x^R$ ,
- $x^R <_R x, x <_R x^L, x <_R x^C$

and, for any two numbers  $x$  and  $y$

- either  $x \geq_L y$  or  $y \geq_L x$ ,
- either  $x \geq_C y$  or  $y \geq_C x$ ,
- either  $x \geq_R y$  or  $y \geq_R x$ .

Numbers are totally ordered with respect to  $\geq_L, \geq_C$ , and  $\geq_R$  but games are partially-ordered, i.e., there exist games  $x$  and  $y$  for which we have neither  $x \geq_L y$  nor  $y \geq_L x$ .

**Definition 6:** We define the sum of two numbers as follows

$$x + y = \{x^L + y, x + y^L | x^C + y, x + y^C | x^R + y, x + y^R\}.$$

TABLE II

Class	Left starts	Center starts	Right starts
=	Right wins	Left wins	Center wins
> <sub>L</sub>	Left wins	Left wins	Left wins
> <sub>C</sub>	Center wins	Center wins	Center wins
> <sub>R</sub>	Right wins	Right wins	Right wins
= <sub>LC</sub>	Center wins	Left wins	Center wins
= <sub>LR</sub>	Right wins	Left wins	Left wins
= <sub>CR</sub>	Right wins	Right wins	Center wins
< <sub>CR</sub>	?	Left wins	Left wins
< <sub>LR</sub>	Center wins	?	Center wins
< <sub>LC</sub>	Right wins	Right wins	?
<	?	?	?

All numbers can be classified in 11 outcome classes as shown in Table II. For further details, please refer to [2].

### III. CLASSIFYING THE INSTANCES OF MAUNDY BLOCK

*Theorem 2:* Let

$$G = [l_1, c_1, r_1] + \dots + [l_i, c_i, r_i] + \dots + [l_n, c_n, r_n]$$

be a general instance of Maundy Block. Then,  $G$  is a number.

*Proof:* Let  $G = \{G^L | G^C | G^R\}$  be a general instance of Maundy Block. By induction hypothesis,  $G^L$ ,  $G^C$ , and  $G^R$  are numbers; moreover, for every couple of options  $G^L$  and  $G^C$ , we can distinguish two different subcases:

- 1) if  $G^L$  and  $G^C$  concern the same block then

$$G^L <_L G^{LC} <_L \dots <_L G^{LC\dots C} \equiv G^{CL\dots L} <_L \dots <_L G^{CL} <_L G^C$$

It follows  $G^L <_L G^C$  where the number of center options following  $G^L$  is equal to the number of blocks created by  $G^L$  and the number of left options following  $G^C$  is equal to the number of blocks created by  $G^C$ .

- 2) if  $G^L$  and  $G^C$  concern two different blocks then

$$G^L <_L G^{LC} \equiv G^{CL} <_L G^C \Rightarrow G^L <_L G^C$$

In the same way, we prove that  $G^L <_L G^R$ ,  $G^C <_C G^L$ ,  $G^C <_C G^R$ ,  $G^R <_R G^L$ , and  $G^R <_R G^C$ . ■

*Example 1:* Let  $G = [3, 2, 4]$  be a block of Maundy Block.

We observe that

$$\begin{aligned} G^L &= [1, 2, 4] + [1, 2, 4] + [1, 2, 4] \\ <_L & [1, 1, 4] + [1, 1, 4] + [1, 2, 4] + [1, 2, 4] \\ <_L & [1, 1, 4] + [1, 1, 4] + [1, 1, 4] + [1, 1, 4] + [1, 2, 4] \\ <_L & [1, 1, 4] + [1, 1, 4] + [1, 1, 4] + [1, 1, 4] + [1, 1, 4] + [1, 1, 4] \\ <_L & [1, 1, 4] + [1, 1, 4] + [1, 1, 4] + [3, 1, 4] \\ <_L & [3, 1, 4] + [3, 1, 4] \\ &= G^C \end{aligned}$$

*Theorem 3:* In the game of Maundy Block

- 1)  $G = [1, 1, 1] = 0$
- 2)  $G = [l, 1, 1] >_L 0, l > 1$
- 3)  $G = [1, c, 1] >_C 0, c > 1$

- 4)  $G = [1, 1, r] >_R 0, r > 1$

*Proof:*

- 1) Trivial.
- 2) By induction hypothesis  $G^L \geq_L 0$  and  $G >_L 0$ .
- 3) Analogous to (2).
- 4) Analogous to (2).

*Theorem 4:* In the game of Maundy Block

- 1) if  $d(l) = d(c)$  then  $G = [l, c, 1] =_{LC} 0$ ,
- 2) if  $d(l) > d(c)$  then  $G = [l, c, 1] >_L 0$ ,
- 3) if  $d(l) < d(c)$  then  $G = [l, c, 1] >_C 0$ ,
- 4) if  $d(l) = d(r)$  then  $G = [l, 1, r] =_{LR} 0$ ,
- 5) if  $d(l) > d(r)$  then  $G = [l, 1, r] >_L 0$ ,
- 6) if  $d(l) < d(r)$  then  $G = [l, 1, r] >_R 0$ ,
- 7) if  $d(c) = d(r)$  then  $G = [1, c, r] =_{CR} 0$ ,
- 8) if  $d(c) > d(r)$  then  $G = [1, c, r] >_C 0$ ,
- 9) if  $d(c) < d(r)$  then  $G = [1, c, r] >_R 0$ ,

where  $l, c, r > 1$ .

*Proof:*

- 1) A generic left option  $G^L$  is represented by

$$[l_1, c, 1] + [l_2, c, 1] + \dots + [l_k, c, 1]$$

where  $l_1 = l_2 = \dots = l_k$  and  $d(l_i) < d(c)$  for all  $1 \leq i \leq k$ . By induction hypothesis,  $[l_i, c, 1] >_C 0$  for all  $1 \leq i \leq k$  therefore  $G^L >_C 0$ .

By similar reasoning, we can prove that  $G^C >_L 0$  therefore  $G =_{LC} 0$ .

- 2) We observe that there exists at least a left option

$$G^L = [l_1, c, 1] + \dots + [l_k, c, 1]$$

where  $d(l_i) \geq d(c)$  therefore, by induction hypothesis, either  $G^L >_L 0$  or  $G^L =_{LC} 0$ . In both cases we have  $G >_L 0$ .

- 3) Analogous to (2).

The other 6 cases can be proved analogously. ■

*Theorem 5:* Let  $G = [l, c, r]$  be a block of Maundy Block where  $l, c, r > 1$ . If

- $d(l) < d(c) + d(r)$
- $d(c) < d(l) + d(r)$
- $d(r) < d(l) + d(c)$

then  $G < 0$  else one of the following 6 cases occurs

- 1) if  $d(l) > d(c) + d(r)$  then  $G >_L 0$ ,
- 2) if  $d(l) = d(c) + d(r)$  then  $G <_{CR} 0$ ,
- 3) if  $d(c) > d(l) + d(r)$  then  $G >_C 0$ ,
- 4) if  $d(c) = d(l) + d(r)$  then  $G <_{LR} 0$ ,
- 5) if  $d(r) > d(l) + d(c)$  then  $G >_R 0$ ,
- 6) if  $d(r) = d(l) + d(c)$  then  $G <_{LC} 0$ .

*Proof:* Let's assume that  $d(l) < d(c) + d(r)$ ,  $d(c) < d(l) + d(r)$ , and  $d(r) < d(l) + d(c)$ . We have two subcases:

- $d(l) = 1$ . In this case,  $d(c) = d(r)$  therefore

$$G^L = [1, c, r] + \dots + [1, c, r] =_{CR} 0$$

as shown in the previous theorem.

- $d(l) > 1$ . In this case, there exist at least a left option

$$G^L = [l_1, c, r] + \dots + [l_k, c, r]$$

where  $d(l_i) = d(l) - 1$  for all  $1 \leq i \leq k$ . By induction hypothesis,  $[l_i, c, r]$  is

- $<_{LR} 0$  if  $d(c) = d(l_1) + d(r)$ ,
- $<_{LC} 0$  if  $d(r) = d(l_1) + d(c)$ ,
- $< 0$  otherwise.

Therefore  $G^L$  is  $<_{LR} 0$ ,  $<_{LC} 0$ , or  $< 0$ .

It follows that for each of the 2 aforementioned cases there exists at least a left option  $G^L \leq_C 0$  and  $G^L \leq_R 0$  therefore  $G <_C 0$  and  $G <_R 0$ . Analogously, we can prove that  $G <_L 0$  ( $G <_R 0$ ) considering

$$G^C = [l, c_1, r] + \dots + [l, c_k, r]$$

where  $d(c_i) = d(c) - 1$  for all  $1 \leq i \leq k$  therefore  $G < 0$ .

Now, let's suppose that the hypothesis  $d(l) < d(c) + d(r)$ ,  $d(c) < d(l) + d(r)$ , and  $d(r) < d(l) + d(c)$  is false. In this case, only one of the 6 cases mentioned previously can be true.

- 1) In this case, there exists at least a left option

$$G^L = [l_1, c, r] + \dots + [l_k, c, r]$$

where  $d(l_i) = d(l) - 1$  such that by induction hypothesis either  $G^L >_L 0$  or  $G^L <_{CR} 0$ . In both cases we have  $G >_L 0$ .

- 2) We observe that, for any center option

$$G^C = [l, c_1, r] + \dots + [l, c_k, r]$$

$d(c_i) < d(c) \Rightarrow d(l) > d(c_i) + d(r)$  for all  $1 \leq i \leq k$  therefore, by induction hypothesis,  $G^C >_L 0$ . In the same way, we prove that  $G^R >_L 0$ . Let's consider a generic left option

$$G^L = [l_1, c, r] + \dots + [l_k, c, r]$$

where  $d(l_i) < d(l)$  for all  $1 \leq i \leq k$ . It follows that  $d(l_i) \not\geq d(c) + d(r)$  therefore  $[l_i, c, r]$  can only be  $>_C 0$ ,  $>_R 0$ ,  $=_{CR} 0$ ,  $<_{LR} 0$ ,  $<_{LC} 0$ , or  $< 0$ . In any case,  $G^L <_L 0$  and therefore  $G <_{CR} 0$ .

We can prove the other 4 cases analogously. ■

*Theorem 6:* Let

$$G = [l_1, c_1, r_1] + \dots + [l_i, c_i, r_i] + \dots + [l_n, c_n, r_n]$$

be a general instance of Maundy Block. If  $d(l_i) \leq d(c_i)$  for all  $1 \leq i \leq n$  and Left has to play then Left has not a winning strategy.

*Proof:* Let's suppose that Left plays in the  $i$ -th block

$$[l_i, c_i, r_i] \rightarrow [l_{i_1}, c_i, r_i] + \dots + [l_{i_k}, c_i, r_i].$$

In every of these blocks  $d(l_{i_j}) < d(c_i)$  for all  $1 \leq j \leq k$  and Center can play in any of these blocks.

Successively, Right has to play but we observe that his/her move cannot affect the relation between Left and Center inside a block. When Left will move again, in every block  $[l, c, r]$ , we have  $d(l) \leq d(c)$  therefore, by induction hypothesis, Left has not a winning strategy. ■

The following theorem can be proven in the same way.

*Theorem 7:* Let

$$G = [l_1, c_1, r_1] + \dots + [l_i, c_i, r_i] + \dots + [l_n, c_n, r_n]$$

TABLE III

	Left starts	Center starts	Right starts
$G <_{CR} 0$	Left wins/q	Left wins	Left wins
$G <_{LR} 0$	Center wins	Center wins/q	Center wins
$G <_{LC} 0$	Right wins	Right wins	Right wins/q

q = queer.

TABLE IV

$G < 0$	Left starts	Center starts	Right starts
$L > C, L > R$	Left wins/q	Left wins/q	Left wins/q
$C > L, C > R$	Center wins/q	Center wins/q	Center wins/q
$R > L, R > C$	Right wins/q	Right wins/q	Right wins/q
$L = C, L > R$	Center wins/q	Left wins/q	Center wins/q
$L = R, L > C$	Right wins/q	Left wins/q	Left wins/q
$C = R, C > L$	Right wins/q	Right wins/q	Center wins/q
$L = C, L = R$	Right wins/q	Left wins/q	Center wins/q

$L = d(l), C = d(c), R = d(r), q = \text{queer}$ .

be a general instance of Maundy Block. If  $d(l_i) \leq d(r_i)$  for all  $1 \leq i \leq n$  and Left has to play then Left has not a winning strategy.

Analogously, we can get the same results for Center and Right. The previous theorems give us some further information about the outcome of  $G = [l, c, r] <_{CR} 0$ ,  $G = [l, c, r] <_{LR} 0$ ,  $G = [l, c, r] <_{LC} 0$ , and  $G = [l, c, r] < 0$  as shown in Table III and IV.

We briefly recall the definition of queer game introduced by Propp [4]:

*Definition 7:* A position in a three-player combinatorial game is called queer if no player can force a win.

#### IV. [25, 2, 2] IS A QUEER GAME

Let's consider the game  $G = [25, 2, 2]$ . We observe that  $d(25) = d(2) + d(2)$  therefore  $G <_{CR} 0$ . When Center or Right makes the first move Left has always a winning strategy. When Left makes the first move we know, by previous theorems, that neither Center nor Right has a winning strategy; therefore, we have two possible cases: either Left has a winning strategy or  $G$  is a queer game. We show that Left has not a winning strategy.

In the beginning, Left has only one plausible move:

$$[25, 2, 2] \rightarrow [5, 2, 2] + [5, 2, 2] + [5, 2, 2] + [5, 2, 2] + [5, 2, 2].$$

Successively, Center moves

$$[5, 2, 2] \rightarrow [5, 1, 2] + [5, 1, 2]$$

and Right moves

$$[5, 2, 2] \rightarrow [5, 2, 1] + [5, 2, 1]$$

obtaining the instance

$$[5, 1, 2] + [5, 1, 2] + [5, 2, 2] + [5, 2, 2] + [5, 2, 2] + [5, 2, 1] + [5, 2, 1].$$

Now, Left has 3 possible moves:

- If Left moves in  $[5, 1, 2]$  we have

$$[5, 1, 2] \rightarrow [1, 1, 2] + [1, 1, 2] + [1, 1, 2] + [1, 1, 2] + [1, 1, 2].$$

In this case, Center moves

$$[5, 2, 2] \rightarrow [5, 1, 2] + [5, 1, 2]$$

and Right moves

$$[1, 1, 2] \rightarrow [1, 1, 1] + [1, 1, 1].$$

Now, Left has to move and you can check easily that he/she has not a winning strategy.

- If Left moves in  $[5, 2, 2]$  we have

$$[5, 2, 2] \rightarrow [1, 2, 2] + [1, 2, 2] + [1, 2, 2] + [1, 2, 2] + [1, 2, 2].$$

In these 5 blocks, Center and Right can make 7 moves each one and Left can make only 6 moves in the other blocks therefore he/she has not a winning strategy.

- If Left moves in  $[5, 2, 1]$  we have

$$[5, 2, 1] \rightarrow [1, 2, 1] + [1, 2, 1] + [1, 2, 1] + [1, 2, 1] + [1, 2, 1].$$

Analogous to the first case.

It is amazing to observe that both  $[25, 2, 2]$  and  $[4, 2, 2]$  are  $<_{CR} 0$  but in  $[4, 2, 2]$  Left has still a winning strategy when he/she makes the first move.

#### REFERENCES

- [1] E. R. Berlekamp, J. H. Conway, R. K. Guy, *Winning Ways For Your Mathematical Plays*, 2nd ed. Natick, Massachusetts: A K Peters, 2001.
- [2] A. Cincotti, "Three-player partizan games," *Theoretical Computer Science*, vol. 332, pp. 367-389, 2005.
- [3] J. H. Conway, *On Numbers and Games*, 2nd ed. Natick, Massachusetts: A K Peters, 2001.
- [4] J. Propp, "Three-player impartial games," *Theoretical Computer Science*, vol. 233, pp. 263-278, 2000.