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A note on the minimum cardinality of critical sets of inertias for irreducible zero-nonzero patterns of order 4

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Abstract—If there exists a nonempty, proper subset $\mathcal{S}$ of the set of all $(n+1)(n+2)/2$ inertias such that $\mathcal{S} \subseteq i(\mathcal{A})$ is sufficient for any $n \times n$ zero-nonzero pattern $\mathcal{A}$ to be inertially arbitrary, then $\mathcal{S}$ is called a critical set of inertias for zero-nonzero patterns of order n. If no proper subset of $\mathcal{S}$ is a critical set, then $\mathcal{S}$ is called a minimal critical set of inertias. In [Kim, Olesky and Driessche, Critical sets of inertias for matrix patterns, Linear and Multilinear Algebra, 57 (3) (2009) 293-306], identifying all minimal critical sets of inertias for $n \times n$ zero-nonzero patterns with $n \geq 3$ and the minimum cardinality of such a set are posed as two open questions by Kim, Olesky and Driessche. In this note, the minimum cardinality of all critical sets of inertias for 4×4 irreducible zero-nonzero patterns is identified.

Keywords—Zero-nonzero pattern, Inertia, Critical set of inertias, Inertially arbitrary.

I. Introduction

An $n \times n$ zero-nonzero pattern is a matrix $\mathcal{A} = [\alpha_{ij}]$ with entries in $\{*, 0\}$ where $*$ denotes a nonzero real number. The set of all real matrices $A = [a_{ij}]$ such that $a_{ij} \neq 0$ if and only if $\alpha_{ij} = *$ for all i and j. If $A \in Q(\mathcal{A})$, then A is a realization of $\mathcal{A}$. A subpattern of an $n \times n$ zero-nonzero pattern $\mathcal{A} = [\alpha_{ij}]$ is an $n \times n$ zero-nonzero pattern $\mathcal{B} = [\beta_{ij}]$ such that $\beta_{ij} = 0$ whenever $\alpha_{ij} = 0$. If $\mathcal{B}$ is a subpattern of $\mathcal{A}$, then $\mathcal{A}$ is a superpattern of $\mathcal{B}$. A zero-nonzero pattern $\mathcal{A}$ is reducible if there is a permutation matrix $\mathcal{P}$ such that

$$\mathcal{P}\mathcal{A}\mathcal{P}^T = \begin{pmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ 0 & \mathcal{A}_{22} \end{pmatrix}$$

where $\mathcal{A}_{11}$ and $\mathcal{A}_{22}$ are square matrices of order at least one. A pattern is irreducible if it is not reducible.

Recall that the inertia of a matrix A is an ordered triple $i(A) = (n_+, n_-, n_0)$ where n_+ is the number of eigenvalues of A with positive real part, n_- is the number of eigenvalues of A with negative real part, and n_0 is the number of eigenvalues of A with zero real part. The inertial of zero-nonzero pattern $\mathcal{A}$ is $i(\mathcal{A}) = \{i(A)|\ A \in Q(\mathcal{A})\}$. An $n \times n$ zero-nonzero pattern $\mathcal{A}$ is an inertially arbitrary pattern (IAP) if given any ordered triple (n_+, n_-, n_0) of nonnegative integers with $n_+ + n_- + n_0 = n$, there exists a real matrix $A \in Q(\mathcal{A})$ such that $i(A) = (n_+, n_-, n_0)$. Equivalently, $\mathcal{A}$ is an inertially arbitrary pattern if all the $(n+1)(n+2)/2$ ordered triples (n_+, n_-, n_0) of nonnegative integers with $n_+ + n_- + n_0 = n$ are in $i(\mathcal{A})$; see, e.g., [2-4].

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Let S be a nonempty, proper subset of the set of all $(n+1)(n+2)/2$ inertias for any $n \times n$ zero-nonzero pattern $\mathcal{A}$. If $S \subseteq i(\mathcal{A})$ is sufficient for $\mathcal{A}$ to be inertially arbitrary, then S is said to be a critical set of inertias for zero-nonzero patterns of order n and if no proper subset of S is a critical set of inertias, S is said to be a minimal critical set of inertias for zero-nonzero patterns of order n; see, e.g., [3]. All minimal critical sets of inertias for irreducible zero-nonzero patterns of order 2 are identified. But as posed in [3], identifying all minimal critical sets of inertias for irreducible zero-nonzero patterns of order $n \geq 3$ is an open question. Also open is the minimum cardinality of such a set.

In this note, we concentrate on the minimum cardinality of all critical sets of inertias for irreducible zero-nonzero patterns of order 4. It is shown that the minimum cardinality of all critical sets of inertias for 4×4 irreducible zero-nonzero patterns is 3.

II. Preliminaries and main results

A zero-nonzero pattern $\mathcal{A} = [\alpha_{ij}]$ has an associated digraph $D(\mathcal{A})$ with vertex set $\{1, 2, \ldots, n\}$ and for all i and j, an arc from i to j if and only if α_{ij} is $*$. A (directed) simple cycle of length k is a sequence of k arcs $(i_1, i_2), (i_2, i_3), \ldots, (i_k, i_1)$ such that the vertices $i_1, \ldots, i_k$ are distinct. The digraph of a matrix is defined analogously; see, e.g., [1]. A digraph is strongly connected if for each vertex i and every other vertex j ($\neq i$), there is an oriented path from i to j. A zero-nonzero pattern $\mathcal{A}$ is irreducible if and only if its digraph, $D(\mathcal{A})$, is strongly connected. For any digraph D, let $G(D)$ denote the underlying multigraph of D, i.e., the multigraph obtained from D by ignoring the direction of each arc; see, e.g., [2].

The following lemma 1 was stated as Proposition 2 in [2], which is useful to determine whether a zero-nonzero pattern is inertially arbitray or not.

Lemma 1. *Let $\mathcal{A}$ be an irreducible $n \times n$ zero-nonzero pattern and let $A \in Q(\mathcal{A})$. If T is a direct subgraph of $D(\mathcal{A})$ such that $G(T)$ is a tree, then $\mathcal{A}$ has a realization that is diagonally similar to A such that each entry corresponding to an arc of T is 1.*

We proceed by showing the following zero-nonzero pattern is nearly inertially arbitrary.

Theorem 1 *Let*

$$\mathcal{N} = \begin{pmatrix} * & * & 0 & * \\ * & * & * & 0 \\ 0 & 0 & 0 & * \\ * & 0 & * & 0 \end{pmatrix}.$$

Then the zero-nonzero pattern $\mathcal{N}$ allows all inertias (n_1, n_2, n_3) with nonnegative integers n_1, n_2 and n_3 such that $n_1 + n_2 + n_3 = 4$ except inertia $(0, 0, 4)$.

Proof. Since $(0, 0, 4) \in i(\mathcal{N})$ if and only if $\mathcal{N}$ allows some characteristic polynomial of the form

$$x^4 + (p + q)x^2 + pq$$

for p, $q \geq 0$. Suppose A is a realization of $\mathcal{N}$. By Lemma 1, without loss of generality, let

$$A = \begin{pmatrix} a & 1 & 0 & b \\ c & d & 1 & 0 \\ 0 & 0 & 0 & 1 \\ e & 0 & f & 0 \end{pmatrix}$$

for some nonzero real numbers a, b, c, d, e and f. Then the characteristic polynomial of A is

$$p_A(x) = x^4 - (a + d)x^3 + (ad - c - be - f)x^2$$

$$+[(a + d)f + bde]x + cf - adf - e.$$

Suppose

$$p_A(x) = x^4 + (p + q)x^2 + pq$$

Then

$$a + d = 0$$

and

$$(a + d)f + bde = 0$$

It follows that

$$bde = 0.$$

It is a contradiction. Hence, $\mathcal{N}$ does not allow $(0, 0, 4)$.

Next we show that the zero-nonzero pattern $\mathcal{N}$ allows all the remaining inertias. Note that for an arbitrary zero-nonzero pattern $\mathcal{N}$, $(n_+, n_-, n_0) \in i(\mathcal{N})$ if and only if $(n_-, n_+, n_0) \in i(\mathcal{N})$. So to complete the proof, it suffices to show that $\mathcal{N}$ allows inertias $(1, 0, 3)$, $(2, 0, 2)$, $(1, 1, 2)$, $(3, 0, 1)$, $(2, 1, 1)$, $(4, 0, 0)$, $(3, 1, 0)$ and $(2, 2, 0)$.

Consider realizations of $\mathcal{N}$

$$\begin{pmatrix} -2 & 1 & 0 & \frac{1}{2} \\ -\frac{22}{3} & 3 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{4}{3} & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & \frac{4}{3} \\ -\frac{1}{2} & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{3}{4} & 0 & \frac{1}{2} & 0 \end{pmatrix},$$

$$\begin{pmatrix} \frac{1}{2} & 1 & 0 & 2 \\ \frac{1}{4} & \frac{1}{2} & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & -3 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 0 & -2 \\ 4 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 4 & 0 & 2 & 0 \end{pmatrix},$$

$$\begin{pmatrix} \frac{1}{2} & 1 & 0 & \frac{2}{3} \\ -\frac{3}{4} & \frac{1}{2} & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -3 & 0 & 3 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 0 & \frac{11}{2} \\ 4 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 0 & 4 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 1 & 0 & -2 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & 3 & 0 \end{pmatrix} \text{and} \begin{pmatrix} 1 & 1 & 0 & 1 \\ \frac{1}{2} & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{3}{2} & 0 & 1 & 0 \end{pmatrix}$$

with inertias $(1, 0, 3)$, $(2, 0, 2)$, $(1, 1, 2)$, $(3, 0, 1)$, $(2, 1, 1)$, $(4, 0, 0)$, $(3, 1, 0)$ and $(2, 2, 0)$, respectively. It follows that $\mathcal{N}$ allows all inertias except $(0, 0, 4)$.

Corollary 1. *Let S be a nonempty, proper subset of the set of all $(n + 1)(n + 2)/2$ inertias for 4×4 irreducible zero-nonzero patterns. If S is a critical set of inertias, then $(0, 0, 4) \in S$.*

Proof. By a way of contradiction assume that $(0, 0, 4)$ does not belong to S. Then S must contain some of the rest of inertias. By Theorem 1, $S \subseteq i(\mathcal{N})$ and $\mathcal{N}$ is not inertially arbitrary. It follows that S is not a critical set of inertias; a contradiction.

The following result was stated as Theorem 4 in [2].

Lemma 2. *Let the zero-nonzero pattern of order 4*

$$\mathcal{M} = \begin{pmatrix} 0 & * & 0 & 0 \\ * & 0 & * & 0 \\ 0 & 0 & * & * \\ * & 0 & 0 & * \end{pmatrix}.$$

Then $\mathcal{M}$ allows all inertias (n_1, n_2, n_3) with nonnegative integers n_1, n_2 and n_3 such that $n_1 + n_2 + n_3 = 4$ except $(1, 0, 3)$, $(0, 1, 3)$, $(2, 0, 2)$ and $(0, 2, 2)$.

The following corollary indicates that the minimum cardinality of critical sets of inertias for irreducible 4×4 zero-nonzero patterns is at least 2.

Corollary 2. *There is no critical set of inertias with a single inertia for irreducible 4×4 zero-nonzero patterns. Moreover, if S is a critical set of inertias for irreducible 4×4 zero-nonzero patterns, then S must contain $(0, 0, 4)$ and one of the inertias $(1, 0, 3)$, $(0, 1, 3)$, $(2, 0, 2)$ and $(0, 2, 2)$.*

Proof. The first part of Corollary 2 follows directly from Theorem 1 and Lemma 2. If S is a critical set of inertias, then $(0, 0, 4) \in S$ by Corollary 1. If none of the inertias $(1, 0, 3)$, $(0, 1, 3)$, $(2, 0, 2)$ and $(0, 2, 2)$ is in S, the $S \subseteq i(\mathcal{M})$ in Lemma 2. But it is clear that $\mathcal{M}$ is not inertially arbitrary. It follows that S is not a critical set of inertias; a contradiction.

Theorem 2. *Let the zero-nonzero pattern of order 4*

$$\mathcal{P} = \begin{pmatrix} * & * & * & * \\ * & * & 0 & 0 \\ * & 0 & 0 & 0 \\ * & 0 & 0 & 0 \end{pmatrix}.$$

Then $\mathcal{P}$ allows all inertias (n_1, n_2, n_3) with nonnegative integers n_1, n_2 and n_3 such that $n_1 + n_2 + n_3 = 4$ except the only inertias $(4,0,0)$, $(0,4,0)$, $(3,1,0)$, $(1,3,0)$ and $(2,2,0)$.

Proof. Since $\mathcal{P}$ requires singularity, it follows that all of the inertias $(4,0,0)$, $(0,4,0)$, $(3,1,0)$, $(1,3,0)$ and $(2,2,0)$ are not allowed by $\mathcal{P}$.

Consider realizations of $\mathcal{P}$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 1 & 1 \\ -3 & -2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 2 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \text{and} \begin{pmatrix} -1 & 1 & 1 & 1 \\ -3 & 2 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{pmatrix}$$

with inertias $(0,0,4)$, $(1,0,3)$, $(2,0,2)$, $(1,1,2)$, $(3,0,1)$ and $(2,1,1)$, respectively. It follows that the zero-nonzero pattern $\mathcal{P}$ allows all inertias except $(4,0,0)$, $(0,4,0)$, $(3,1,0)$, $(1,3,0)$ and $(2,2,0)$.

It was known that the set $\{(0,0,4),(1,0,3),(4,0,0)\}$ is a minimal critical set of inertias for irreducible zero-nonzero patterns of order 4. Other minimal critical sets on inertias can be obtained by replacing $(4,0,0)$ or $(1,0,3)$ by its reversal; see, e.g., [3, Theorem 7]. As mentioned in Section 6 in [3], for $n = 4$, it is unknown that whether there are other critical sets of inertias. Also mentioned is that the minimum cardinality of all critical sets of inertias for 4×4 irreducible zero-nonzero patterns is at most 3. The next theorem answers this problem completely.

Theorem 3. *The minimum cardinality of all critical sets of inertias for irreducible 4×4 zero-nonzero patterns is 3.*

Proof. By a way of contradiction suppose that the minimum cardinality of all critical sets of inertias is 2. Let S be an arbitrary critical set of inertias with cardinality 2. Then, by Corollary 2, S must contain $(0,0,4)$ and only one of the inertias $(1,0,3)$, $(0,1,3)$, $(2,0,2)$ and $(0,2,2)$.

Case 1. *S contains inertias $(0,0,4)$ and $(1,0,3)$ or its reversal.* Then S does not contain all the inertias $(4,0,0)$, $(0,4,0)$, $(3,1,0)$, $(1,3,0)$ and $(2,2,0)$. By Theorem 2, we have $S \subseteq i(\mathcal{P})$ and $\mathcal{P}$ is not inertially arbitrary. It follows that S is not a critical set of inertias for irreducible zero-nonzero patterns of order 4, which is a contradiction.

Case 2. The case that S contains inertias $(0,0,4)$ and $(2,0,2)$ or its reversal is similar to Case 1. We omit its proof.

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