A New Method to Solve a Non Linear Differential System

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Abstract—In this article, our objective is the analysis of the resolution of non-linear differential systems by combining Newton and Continuation (N-C) method. The iterative numerical methods converge where the initial condition is chosen close to the exact solution. The question of choosing the initial condition is answered by N-C method.

Keywords—Continuation method, Newton method, finite difference method, numerical analysis and non-linear Partial Differential Equation.

I. INTRODUCTION

NUMERICAL continuation methods have provided important contributions toward the numerical solution of nonlinear system. The methods may be used not only to compute solutions, which might otherwise be hard to obtain, but also to gain insight into qualitative properties of the solutions [1]. In this study, our objective is the analysis of the resolution of finite non-linear algebraic systems. These systems are represented by the following shape:

$$F(u) = 0 \qquad (1)$$

Where, $F : IR^n \rightarrow IR^n$ is a non-linear function.

For $u = (u_1, u_2, ..., u_n)^T \in IR^n$,

$F(u) = (F_1(u), F_2(u), ..., F_n(u))^T \in IR^n$ each of the scalar functions $F_i(u)$, is non linear. Our goal is setting up efficient and powerful algorithms in order to solve (1) using the Continuation method. The origin of the non-linear discreet systems is varied. These systems arise frequently in engineering and scientific problems, because those problems are often formulated as determining a function that satisfies some set of equations, for example, the Navier-Stokes equations, Maxwell's equations, or Newton's law [2].

Without loss of generality, we consider the non-linear Poisson equation [3]-[8], (has one or two dimensional space). For $\overline{\Omega}$ opened on *IR,* search for $u : \overline{\Omega} \rightarrow IR, (\overline{\Omega} = \Omega \cup \partial\Omega)$, which verifies:

$$\begin{cases} -\dfrac{d^2u}{dx^2} + f(u) = g(x), \, in \, \Omega \\ u = 0 \quad on \, \partial\Omega \end{cases} \qquad (2)$$

$f : IR \rightarrow IR$, is generally a real function that has a real non-linear variable. The equation (2) models the bending of a beam.

II. DISCRETIZATION OF THE PROBLEM

Let's consider the following problem: Being given a non-linear function f, with one real variable. Find a U function two times continuously derivable on [0, 1] as:

$$\begin{cases} -\dfrac{d^2u}{dx^2} + c(x)u = g(x) \quad 0 < x < 1 \\ u(0) = \alpha \, , \, u(1) = \beta \qquad \alpha, \beta \in IR \end{cases} \qquad (3)$$
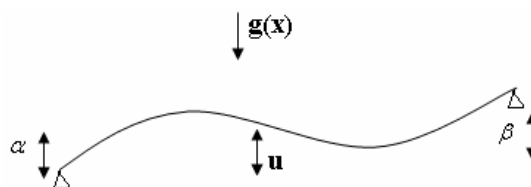
This mechanical situation problem is the one of bending the beam, stretched according to its axis by a linear load strength g(x) and merely supported its ends. Then the moment of bending non-linear f (u) at the point of abscissa x is the solution of the problem (3) with c(x) = f/EI(x), E the Young modulus of the material, I(x) is the principal moment of inactivity of the beam section at the x point.

Except for some rare cases, a formula that permits to get u(x) explicitly doesn't exist for all $x \in [0,1]$. It therefore requests to find a means to approach the values of the solution of the problem (3) more accurately. A method to reach this goal consists in finding a number of finite parameters $\{u_i / i = 1, ..., n\}$, as either an approximation of $u(x_i), i = 1, ..., n$. We are interested in the method of finite differences.

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III. METHOD OF THE FINITE DIFFERENCES

Let **n** be positive, put $h = \dfrac{1}{n+1}$ with **h** is the step of discretization (supposed to be uniform here); $x_i = ih$ for $i = 0,...,n+1, \{x_i\}$ are the discretization nodes.
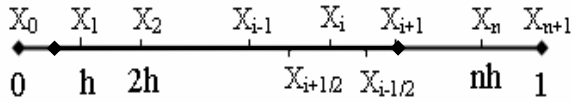
Besides, it's possible to demonstrate that u is a regular function (for example u is class C^4) that [9]:

$$\frac{du}{dx_i} = \frac{u(x_{i+1/2}) - u(x_{i-1/2})}{h} + O(h^2)$$

then

$$\frac{d^2 u}{dx_i^2} = \frac{u(x_{i+1}) - 2u(x_i) + u(x_{i-1})}{h^2} + O(h^2) \tag{4}$$

To solve (3) numerically, based on (4), and calculate the values u_i (we note $u_i \cong u(x_i)$) with $1 \leq i \leq n$, cautious to be approximately $u(x_i)$ (after replacing the formula with the differences (4) in (3) :

$$\begin{cases} \dfrac{-u_{i-1} + 2u_i - u_{i+1}}{h^2} + c(x_i)u_i = g(x_i) \\ u_0 = \alpha, \quad u_{n+1} = \beta \end{cases} \tag{5}$$

The problem (5) is called **approach problem** (or discreet problem) gotten, by a method of finite differences, by opposition to the problem (3) declares a continuous problem. The vector shape of (5) is presented as follows:

Find $u = (u_1,...,u_n)^T$ $(with \ u_0 = \alpha \ et \ u_{n+1} = \beta)$, as $F(u) = 0$ with (6):

$$F(u) = \begin{bmatrix} F_1(u) \\ F_2(u) \\ . \\ . \\ . \\ F_{n-1}(u) \\ F_n(u) \end{bmatrix} = \begin{bmatrix} \dfrac{-\alpha + (2+c_1)u_1 - u_2}{h^2} - f(u_1) \\ \dfrac{-u_1 + (2+c_2)u_2 - u_3}{h^2} - f(u_2) \\ . \\ . \\ . \\ \dfrac{-u_{n-2} + (2+c_{n-1})u_{n-1} - u_n}{h^2} - f(u_{n-1}) \\ \dfrac{-u_{n-1} + (2+c_n)u_n - \beta}{h^2} - f(u_n) \end{bmatrix}$$

We, therefore, represent by (6) a non-linear system of n equations for the n unknown $(u_1,...,u_n)$. Hence, to solve this system, it is necessary to linearize while using one of the following iterative methods: method of the successive approximations, Newton's method, Newton-cord and Shamanski's methods [7]. These methods look for a linearization by a highly determined procedure. For example, the use of Newton's method is based on the following formula:

$$u^{j+1} = u^j - [DF(u^j)]^{-1} F(u^j), j = 0,1,2,3,... \tag{7}$$

by means of DF which is the Jacobian matrix (Jacobian). Therefore DF (u) of (6) =

$$\frac{1}{h^2} \begin{bmatrix} 2 + c_1 - h^2 \frac{\partial f(u_1)}{\partial u_1} & -1 & 0 & . & . & . & 0 \\ -1 & 2 + c_2 - h^2 \frac{\partial f(u_2)}{\partial u_2} & -1 & 0 & . & . & 0 \\ 0 & & & & & & . \\ . & & & & & & . \\ . & & & & & & . \\ . & & & & & & . \\ . & & & & & & 0 \\ . & & & & & & -1 \\ 0 & & . & & . & . & 0 & -1 & 2 + c_n - h^2 \frac{\partial f(u_n)}{\partial u_n} \end{bmatrix}$$

which is a sparse matrix (three diagonals) dependent on the solution of problem (6). Consequently, calculating u^{j+1} from u^j, is done by solving the following system: $DF(u^n)(u^n - u^{n+1}) = F(u^n)$

The algorithm of resolution proceeds as follows:
$j = 0$

$s = 1 \quad \{s \ is \ a \ mesure \ of \ convergence\}$

$while \ s > \varepsilon \ and \ j \leq n \max\{\max imal \ number \ of \ iteration\}$

$Calculate \ F(u^j)$

$Calculate \ DF(u^j)$

$Solve \ DF(u^j)y = F(u^j) \ \{by \ a \ linear \ solver\}$

$Calculate \ u^{j+1} = u^j - y$

$Calculate \ s = \dfrac{\|F(u^{j+1})\|}{\|F(u^0)\|}$

$j = j+1$

end

For such an iterative method (Newton's method), it is normal to ask the following questions:

1) **Existence:** Is the method well defined? $DF^{-1}(u^j)$ exists at every iteration?

2) **Convergence**: The continuation $\{u^j\}$ is its convergent in the way that $\lim_{j \to \infty} u^j = u$ where u verifies $F(u) = 0$?

For example, for the problem (5), we demonstrate the following:

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Theorem 3.1. *Let u be a solution of the system (5). Let*

$$\overline{f} = maz_{1 \le i \le n} \left| \frac{\partial f(u_i)}{u_i} \right|. \; \textit{if c, h and } \overline{f} \; \textit{verify the condition}$$

$c > h^2 \overline{f}, \forall i, 1 \le i \le n,$ *then* $DF^{-1}(u)$ *exists.*

Proof. It is obvious that the matrix $DF(u)$ is symmetrical, $DF(u)$ is also defined positive. Indeed:

$$\forall x \in IR^n \; we \; have \; x^t DF(u)x =$$

$$\frac{1}{h^2} \left[\begin{matrix} (2 + c_1 - h^2 \frac{\partial f(u_1)}{\partial u_1} x_1 - x_2, -x_1 + (2 + c_2 - h^2 \frac{\partial f(u_2)}{\partial u_2} x_2 - x_3, \dots, \\ - x_{n-1} + (2 + c_n - h^2 \frac{\partial f(u_n)}{\partial u_n} x_n \end{matrix} \right] \begin{bmatrix} x_1 \\ . \\ . \\ . \\ x_n \end{bmatrix}$$

With $2 + c_i - h^2 \frac{\partial f(u_i)}{\partial u_i} > 2 \Rightarrow x^t DF(u)x \ge \frac{1}{h^2} \left[x_1^2 + x_n^2 + \sum_{i=2}^{n} (x_i - x_{i-1})^2 \right] \ge 0 \therefore$

The outcome of this result is that if the initial condition u^0 of the iterative algorithm is chosen close to u then $DF^{-1}(u^0)$ exists on one hand. If the sequence of the iterations $\{u^j\}$ exists in the region, then $DF^{-1}(u^j)$ exist. In this concern, we can find more precise results on the convergence of Newton's method, for example in [3]**:**

Theorem 3.2. *If F is continuously derivable two times in relation to the variables* u_j, $1 \le j \le n$, *and if* $\overline{u}$ *is as* $F(\overline{u}) = 0$ *and if* $DF(\overline{u})$ *is then regular, the continuation* (u^j) *defined by Newton's method converges towards u when* $n \to \infty$ *provided that* u^0 *is chosen sufficiently close to u..* It appears therefore that the choice of u^0 sufficiently close to u is fundamental.

IV. METHOD OF CONTINUATION

The above stated iterative methods give the solutions that converge locally toward the solution a of $F(u) = 0$, so when the initial condition is chosen close to the exact solution. The question of choosing the initial condition is asked therefore to find an efficient method permitting to guarantee a perceptive choice. It is then the method of **Continuation** that permits to introduce a precise approach proposes forcing recognition to a parameter $t \in [0,1]$ and hence of $t = 0$, to make $t = 1$ at the solution a of $F(u)$. A new algorithm for the partition of $[0,1]$ was studied in [6]. This method has been introduced by [4]. From that time, it has been used by several authors **Avila** [5]-[7]. Reference [7] introduces it in the setting of the equations in the non-linear partial differential, while leaning on the method of the topological degree of Leray-Schauder. This technique demonstrated all its power in the analysis of the solutions existence of the non-linear **PDE**.

V. PRINCIPLE OF THE CONTINUATION METHOD

The Continuation method consists of proposing the problem $F(u) = 0$, in the setting of a related problems parameterized by a variable $t \in [0,1]$.

Either
$$F : [0,1] \times IR^n \to IR^n$$
$$(t, u) \to F(t, u)$$

As $F(1, u) = F(u), \forall u \in IR^n$.

We also suppose that for $t = 0$, the problem $F(0, u^0) = 0$ admits a unique solution u^0, capable to be calculated by the use of a simple algorithm. The choice of the initial condition to calculate the answer for $t = 1$, depend undoubtedly on u^0. So while following the related solutions $F(t, u^t) = 0$, from $t = 0$, one can make at $t = 1$ the solution of $F(u) = 0$ under precise hypotheses of the related functions $F(., .)$. It is the principle of the **Continuation** method. Numerically, it results by the successive research of the problem solutions, $F(t_j, a_j) = 0, 0 < j < m$ with $0 = t_0 < t_1 < t_2 < \dots < t_m = 1$, $\{t_i\}$ is a discretization of $[0,1]$. The ultimate phase of this process is finite for $t_m = 1$, giving then, $a_m = a$ verifying: $F(t_m, a_m) = F(1, a_m) = F(a_m) = 0.$

Suppose the existence of the solution a_j (where $\overline{a_j}$ approximation **of** a_j) as: $F(t_j, a_j) = 0$ (and $\|F(t_j, \overline{a_j})\| \le \varepsilon_j, \varepsilon_j$ small). Search for the pair $\{t_{j+1}, a_{j+1}\}$ with $t_j < t_{j+1} \le 1, a_{j+1} \in IR^n$, as $F(t_{j+1}, a_{j+1}) = 0$. Conduct the resolution of the non-linear system $F(t_{j+1}, a_{j+1}) = 0$ by a method of Newton type, Newton cord or Shamanski getting the continuation therefore: $\{a_{j+1}^k / k = 0,1,2,\dots, k_{j+1}\}$ with $a_{j+1}^0 = a_j^{k_j}, k_j$ denoting the indication of the last applied iteration in the approximation of a_j.

Example 5.1. (The equation of Poisson)

Let the following equation of Poisson: $-\Delta u + f(u) = g$ is solved thus by a method of Continuation. To the parameter $t \in [0,1]$, look for the solution u_t of the problem: $-\Delta u_t + t f(u_t) = g$. The discretization of this model by the method of the **finite Elements** [10] gives then $A u_t + t I f(u_t) = G$, with A is the matrix of rigidity, $I f(u_t)$ is the diagonal matrix:

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$$\begin{pmatrix} f(u_{t,1}) & 0 & . & . & & 0 \\ 0 & . & . & . & & . \\ . & . & . & . & & . \\ . & & . & . & . & 0 \\ 0 & & . & . & 0 & f(u_{t,n}) \end{pmatrix}$$

When $t = 0$, the previous model is linearized to give u_0 the solution of: $Au_0 = G$ resolute by an adequate linear solver. As indicated above, we pass from $t_0 = 0$ to $t_m = 1$, by the resolution of n non-linear systems. The method performance is going to be bound therefore to a reduction of the number of step m with minimum iterations to every t_j, $j = 1,...,m$.

VI. METHOD OF NEWTON - CONTINUATION

An application of Newton method to every step t_j, $j = 1,...,m$. gives:

$$\left.\begin{array}{l} x^{j,k+1} = x^{j,k} - DF_x(x^{j,k},t_j).F(x^{j,k},t_j), \\ x^{1,0} = x^0, x^{j+1,0} = x^{j,m_j} \end{array}\right\} k = 0,...,m_j-1; j = 1,...,N-1$$

$$x^{N,k+1} = x^{N,k} - DF_x(x^{N,k},1).F(x^{N,k},1)$$

with DF_x the Jacobean of F in relation to x and $0 = t_0 < t_1 < ... < t_m = 1$.

VII. ALGORITHM (MATLAB)

```
function [u,i,k] = continuation(tol,tol0,F,DF,A,f,g,kmax)
% input:
%- tol is the precision of calculation to every t
%- tol0 is the precision asked to the choice of ti
%- F=F(t,u)=Au + tf - g = 0
%- DF is the Jacobean matrix in relation to u
% output: - u the solution of F(1,u) = 0, i the number of
%step, k the total number of iterations, kmax the maximal
%number of iterations, Resolution of F(0,u) = 0
u0 = A\g ;

t = min(1,t + tau);
i = i + 1;
[u,verif,ki] = newton(F,DF,t,u,tol,kmax);
tau = tol0*norm(f(u0)) / norm(f(u));
k = k + ki;
if (ki > 1)
tol0 = tol0/2;
end
if (i > 1)
i = i-1;
end
end
```

VIII. APPLICATIONS

In this paragraph, we will validate the theories exposed to the previous paragraph. We solve the non-linear Poisson equation that has one dimension. The discrete system obtained by approach-finished differences is solved by the method of Newton and the method of Continuation. The choice of initial condition in this last method illustrates its importance.

In this application we will solve the following non-linear differential equation:

$$\begin{cases} -\dfrac{d^2u}{dx^2} + e^u = 2 + e^{x(1-x)} \\ u(0) = u(1) = 0 \end{cases}$$
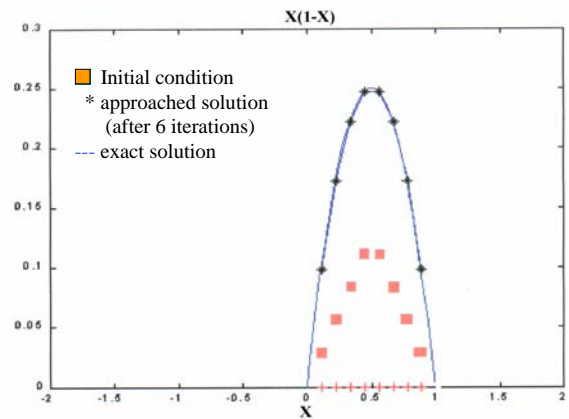
where, $u = x(1 - x)$ is an exact solution.

Fig. 1 Method of Newton (case of convergence)

The Fig. 1 gives the solution of this model with a discretization of 8 points of the domain [0, 1], by using the method of Newton (■ represent the initial solution, * the approached solution, the continuous curve represents the exact solution).

We notice that the method of Newton converges in this example that is because the initial condition is chosen well. On the other hand if this condition is far from the exact solution (u_0 = (-3.4, 9.75, 2.875, -37.5, 0.375, 2.5, -0.1038, -0.0331), the method of Newton diverge (Fig. 2).
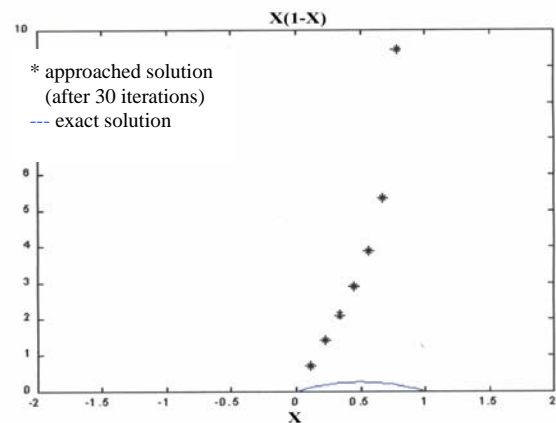

Fig. 2 Method of Newton (case of divergence after 30 iterations)

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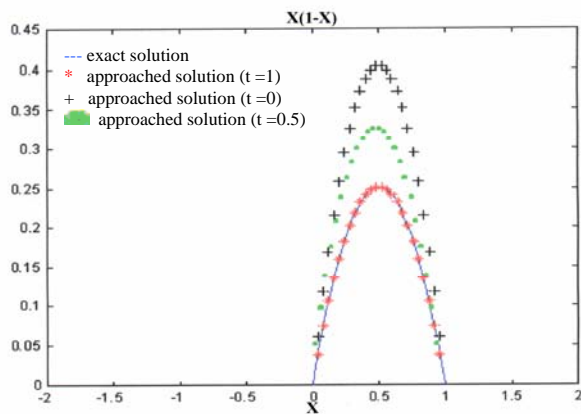International Journal of Mathematical and Computational Sciences
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Fig. 3 Method of N-C

The Fig. 3 represents the solution while using the method of continuation with a discretization of 24 points of the domain [0, 1], and a step of t=0.1 and k=1(number of iterations for this step), immediately we notice the convergence of this method.

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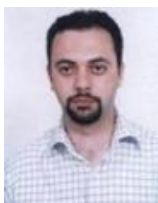
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