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On Generalized New Class of Matrix Polynomial Set

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***Abstract*—**New generalization of the new class matrix polynomial set have been obtained. An explicit representation and an expansion of the matrix exponential in a series of these matrix are given for these matrix polynomials.

***Keywords*—**Generating functions, Recurrences relation and Generalization of the new class matrix polynomial set.

I. Introduction

RECENTLY, Hermite matrix polynomials have been introduced and studied in [3]-[4] and generalization of Hermite matrix polynomials are given [6] for matrix in $C^{N\times N}$ whose eigenvalues are all situated in the right open half-plane. Moreover, some properties of the Hermite matrix polynomials have been presented in [1]-[2]. If D_0 is the complex plane cut along the negative, real axis and $\log(z)$ denotes the principal logarithm of z, then $z^{1/2}$ represents $\exp\left(\frac{1}{2}\log(z)\right)$. If A is a matrix in $C^{N\times N}$ .the set of all the eigenvalues of A is denoted by $\sigma(A)$ .if $f(z)$ and $g(z)$ are holomorphic functions of the complex variable z ,which are defined in an open set Ω of the complex plane, and A is a matrix in $C^{N\times N}$ such that $\sigma(A)\subset\Omega$. Then from the properties of the matrix functional calculus $[7]$. It follows that $f(A)g(A)=g(A)f(A)$. If A is a matrix with $\sigma(A)\subset D_0$, then $A^{1/2}=\sqrt{A}$ denotes the a image by $z^{1/2}$ of the matrix functional calculus acting on the matrix A . We say that A is a positive stable matrix

$$if\ \ \mathrm{Re}(z)>0 \quad for\ all\ \ z\in\sigma(A) \tag{1}$$

If $A(k,n)$ are matrix in $C^{N\times N}$ for $n\geq 0\, and\ k\geq 0$ then it follows that [1].

$$\sum_{n=o}^{\infty}\sum_{k=0}^{\infty}A(k,n)=\sum_{n=o}^{\infty}\sum_{k=0}^{n}A(k,n-k) \tag{2}$$

$$\sum_{n=o}^{\infty}\sum_{k=0}^{\infty}A(k,n)=\sum_{n=o}^{\infty}\sum_{k=0}^{[n/2]}A(k,n-2k) \tag{3}$$

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For m is a positive, similarly to (2) one can find [6].

$$\sum_{n=o}^{\infty}\sum_{k=0}^{\infty}A(k,n)=\sum_{n=o}^{\infty}\sum_{k=0}^{[n/m]}A(k,n-mk)\ ;\ n>m \tag{4}$$

a new class of matrix polynomial $k_n(x,A)$ suggested by Hermite polynomials $H_n(x,A)$ have been introduced and discussed in " as discussed by Kahmmash $[5]$" as follows.

$$k_n(x,A)=\sum_{r=0}^{\left[\frac{n}{2}\right]}\sum_{s=0}^{\left[\frac{n-2r}{3}\right]}\frac{(-1)^r(A/2)^{n-(3/2)r-3s}\,n!x^{2n-3r-6s}3^{n-r-3s}}{(n-2r-3s)!r!s!} \tag{5}$$

Where $k_n(x,A)$ is a polynomial of degree precisely $2n$ in x and that

$$k_n(x,A)=(3(A/2))^n x^{2n}+\pi_{2n-3} in\ x\,.$$

The aim of this paper is to derive the generalization of new class of matrix polynomials set , an explicit representation ,expand the matrix exponential in a series of the generalized new class of matrix polynomial set with some recurrence relations in particular the four terms recurrence relation for theses matrix polynomials.

II. Definition of Generalization of a New Class of Matrix Polynomials Set

Let A be a matrix in $C^{N\times N}$ such that $\mathrm{Re}(\mu)>0$ for every eigenvalue $\mu\in\sigma(A)$. For $n=0,1,2,.....\lambda\in R^+$ and m is a positive integer, we define the generalized anew of matrix polynomials by

$$F(x,t)=e^{\left(\lambda\left(3(A/2)x^2t-3\sqrt{A/2}\,x\,t^{m-1}+t^mI\right)\right)} \tag{6}$$
$$=\sum_{n=0}^{\infty}k_{n,m}^{\lambda}(x,A)t^n$$

Since

$$\exp\left(\lambda\left(3(A/2)x^2t-3\sqrt{A/2}\,x\,t^{m-1}+t^mI\right)\right)$$
$$=e^{3\lambda(A/2)x^2t}\ e^{-3\lambda\sqrt{A/2}\,x\,t^{m-1}}\ e^{\lambda t^m I}$$
$$=\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{\left(3(A/2)x^2\right)^n}{n!r!s!}$$

$$. \ \lambda^{n+r+s}(-1)^r\left(3\sqrt{A/2}x\right)^r t^{n+mr+ms-r}.$$

$$=\sum_{n=0}^{\infty}\sum_{r=0}^{\left[\frac{n}{m-1}\right]}\sum_{s=0}^{\infty}\frac{\left(3(A/2)x^2\right)^{n-2r-3s}}{(n-(m-1)r)!r!s!}$$

$$. \ \lambda^{n-rm+2r+s}(-1)^r\left(3\sqrt{A/2}x\right)^r t^{n+ms}.$$

$$=\sum_{n=0}^{\infty}\sum_{r=0}^{\left[\frac{n}{m-1}\right]}\sum_{s=0}^{\left[\frac{n-(m-1)r}{m}\right]}\frac{\left(3(A/2)x^2\right)^{n-(m-1)r-ms}}{(n-(m-1)r-ms)!r!s!}$$

$$. \ \lambda^{n-rm+2r-ms+s}(-1)^r\left(3\sqrt{A/2}x\right)^r t^{n}.$$

$$k^{\lambda}_{n,m}(x,A)$$

$$=\sum_{r=0}^{\left[\frac{n}{m-1}\right]}\sum_{s=0}^{\left[\frac{n-(m-1)r}{m}\right]}\frac{(-1)^r n!x^{2n-2(m-1)r-2ms+r}\lambda^{n-(m+2)r-(m-1)s}}{(n-(m-1)r-ms)!r!s!}$$

$$.(A/2)^{n-(m-1)r-ms+(1/2)r}\,3^{n-(m-1)r-ms+r}.$$

$$=\lambda^n\sum_{r=0}^{\left[\frac{n}{m-1}\right]}\sum_{s=0}^{\left[\frac{n-(m-1)r}{m}\right]}\frac{(-1)^r\lambda^{n-(m-2)r-ms}x^{2n-(2m-3)r-2ms}}{\lambda^{(m+2)r+(m-1)s}(n-(m-1)r-ms)!r!s!} \quad (7)$$

$$.(A/2)^{n-(m-3/2)r-ms}\,3^{n-(m-2)r-ms}.$$

Where $k^{\lambda}_{n,m}(x,A)$ is a polynomial of degree precisely $2n$ in x and that

$$k^{\lambda}_{n,m}(x,A)=\left((A/2)\right)^n x^{2n}+\pi_{2n-m}(x)$$

Where $\pi_{2n-m}(x)$ is a matrix polynomial of degree $(2n-m)$ in x .

For simplicity we denote $k^{\lambda}_{n,m}(x,A)$ for the generalized new class of matrix polynomials when $\lambda=1$ it should be observed that ,in view of the explicit representation (6),the generalized new class of matrix polynomials $k^{\lambda}_{n,3}(x,A)$ reduces to the new class of matrix polynomials $k_n(x,A)/n!$ as given in (5)

Note that

$$\left(3Ax-3\sqrt{A/2}\,t^{m-2}\right)^{-1}\frac{d}{dx}e^{3(A/2)x^2t-3\sqrt{A/2}x\,t^{m-1}}$$

$$=t\ e^{3(A/2)x^2t-3\sqrt{A/2}x\,t^{m-1}}$$

This ,

$$\exp\left(\left(3Ax-3\sqrt{A/2}\,t^{m-2}\right)^{-m}\frac{d^m}{dx^m}\right)$$

$$. \exp 3(A/2)x^2t-3\sqrt{A/2}x\,t^{m-1}.$$

$$=\sum_{n=0}^{\infty}\frac{1}{n!}\left(3Ax-3\sqrt{A/2}\,t^{m-2}\right)^{-1}$$

$$. \frac{d}{dx}\Big|^{mn}\exp\left(3(A/2)x^2t-3\sqrt{A/2}x\,t^{m-1}\right).$$

$$=\sum_{n=0}^{\infty}\frac{1}{n!}t^{mn}\exp\left(3(A/2)x^2t-3\sqrt{A/2}x\,t^{m-1}\right)$$

$$=\exp\left(3(A/2)x^2t-3\sqrt{A/2}x\,t^{m-1}+t^m I\right)$$

There fore, we have

$$\exp\left(\left(3Ax-3\sqrt{A/2}\,t^{m-2}\right)^{-m}\frac{d^m}{dx^m}\right)\exp 3(A/2)x^2t$$

$$-3\sqrt{A/2}x\,t^{m-1}=\sum_{n=0}^{\infty}k_{n.m}(x,A)\,t^n.$$

$$\exp\left(\left(3Ax-3\sqrt{A/2}\,t^{m-2}\right)^{-m}\frac{d^m}{dx^m}\right)$$

$$. \sum_{n=0}^{\infty}\frac{x^n}{n!}\left(3(A/2)x-3\sqrt{A/2}\,t^{m-2}\right)^n t^n$$

$$=\sum_{n=0}^{\infty}k_{n.m}(x,A)\,t^n.$$

Identification of the coefficients of t^n in both sides gives a new representation for the generalized new class matrix polynomials for $\lambda=1$, in the form:

$$k_{n,m}(x,A)=\frac{1}{n!}\exp\left(\left(3Ax-3\sqrt{A/2}\,t^{m-2}\right)^{-m}\frac{d^m}{dx^m}\right) \quad (8)$$

$$. \left(3(A/2)x-3\sqrt{A/2}\,t^{m-2}\right)^n x^n.$$

For $m=2$,the expression (8) gives another representation for the new class matrix polynomials in the form .

$$k_{n,m}(x,A)=\frac{1}{n!}\exp\left(\left(3Ax-3\sqrt{A/2}\ \right)^{-2}\frac{d^2}{dx^2}\right) \quad (9)$$

$$. \left(3(A/2)x-3\sqrt{A/2}\ \right)^2 x^n.$$

Let B be a matrix in $C^{N\times N}$ satisfies the spectral property .

$$\left|\mathrm{Re}(\mu)\right|>\left|\mathrm{Im}(\mu)\right|,\ \forall\ \mu\in\sigma(B) \quad (10)$$

Suppose that $A=2B^2$ in view of the spectral mapping theorem $[7]$. it is easy to find that

$\sigma(A)=\left\{2b^2 : b\in\sigma(B)\right\}$ and by (10) we have

$$\mathrm{Re}\left(2b^2\right)=2\left[\left(\mathrm{Re}(b)\right)^2-\left(\mathrm{Im}(b)\right)^2\right]>0\ ,\ b\in\sigma(B)$$

That is, A is appositive stable matrix .In (2.1), putting $t=1$ and $B=\sqrt{A/2}$ gives .

$$\exp\left(\lambda\left(3(AB)^2-3Bx+I\right)\right)=\sum_{n=0}^{\infty}k_{n,m}^{\lambda}\left(x,2B^2\right)$$

Therefore for the matrix B satisfies (10),an expansion of $\exp\left(3\lambda x^2B-3x\right)B$ in a series of the new class matrix polynomials is obtained in the form:

$$\exp\left(3\lambda x^2B-3x\right)B$$

$$=\exp(\lambda)\sum_{n=0}^{\infty}k_{n,m}^{\lambda}\left(x,2B^2\right),\ -\infty<x<\infty.$$

III. Recurrence Relations

Now, since

$$F(x,t)=e^{\left(\lambda\left(3(A/2)x^2t-3\sqrt{A/2}\,x\,t^{m-1}+t^mI\right)\right)} \tag{11}$$

$$=\sum_{n=0}^{\infty}k_{n,m}^{\lambda}(x,A)t^n.$$

Differentiating (3.1) with respect to x yields .

$$\lambda\left(\left(3Axt-3\sqrt{A/2}\,t^2\right)\right)e^{\left(\lambda\left(3(A/2)x^2t-3\sqrt{A/2}\,x\,t^{m-1}+t^mI\right)\right)} \tag{12}$$

$$=\sum_{n=0}^{\infty}D\,k_{n-mk,m}^{\lambda}(x,A)t^n.$$

By (11) and (12),we have

$$\lambda\left(3Axt-3\sqrt{A/2}\,t^{m-1}\right)$$

$$.\sum_{n=0}^{\infty}k_{n,m}^{\lambda}(x,A)t^n=\sum_{n=0}^{\infty}D\,k_{n,m}^{\lambda}(x,A)t^n.$$

$$\lambda\left(3Ax-3\sqrt{A/2}\,t^{m-2}\right)$$

$$.\sum_{n=0}^{\infty}k_{n,m}^{\lambda}(x,A)t^{n+1}=\sum_{n=0}^{\infty}D\,k_{n,m}^{\lambda}(x,A)t^n.$$

Since $D\,k_{0,m}^{\lambda}(x,A)=0$, then for $n\geq 1$ one obtains

$$\lambda\left(3Ax-3\sqrt{A/2}\,t^{m-2}\right)k_{n-1,m}^{\lambda}(x,A)=D\,k_{n,m}^{\lambda}(x,A) \tag{13}$$

Iteration (13) ,for $0\leq k\leq n$ gives.

$$D^k\,k_{n,m}^{\lambda}(x,A) \tag{14}$$

$$=\left[\lambda\left(3Ax-3\sqrt{A/2}\,t^{m-2}\right)\right]^k k_{n-k,m}^{\lambda}(x,A)$$

Differentiating (11) with respect to x and t, we find

$$\partial f/\partial x=\lambda\left(3Axt-3\sqrt{A/2}\,t^{m-1}\right)$$

$$.\exp\left(\lambda\left(3(A/2)x^2t-3\sqrt{A/2}\,x\,t^{m-1}+t^mI\right)\right).$$

And

$$\partial f/\partial t=\lambda\left(3(A/2)x^2-3(m-1)\sqrt{A/2}\,x\,t^{m-2}+mt^{m-1}I\right)$$

$$.\,e^{\left(\lambda\left(3(A/2)x^2t-3\sqrt{A/2}\,x\,t^{m-1}+t^mI\right)\right)}.$$

Therefore, $F(x,t)$ satisfies the partial matrix differential equation.

$$\left((A/2)x^2I-(m-1)\sqrt{A/2}\,x\,t^{m-2}+\frac{m}{3}t^{m-1}I\right)\frac{\partial f}{\partial x}$$

$$-\left(A\,x\,t-\sqrt{A/2}\,x\,t^{m-1}\right)\frac{\partial f}{\partial t}=0.$$

Which by using (11)

$$\left(\frac{A}{2}x^2I-(m-1)\sqrt{A/2}\,x\,t^{m-2}+\frac{m}{3}t^{m-1}I\right)\sum_{n=0}^{\infty}Dk_{n,m}^{\lambda}(x,A)t^n$$

$$-\left(Axt-\sqrt{A/2}\,t^{m-1}\right)\sum_{n=1}^{\infty}n\,k_{n,m}^{\lambda}(x,A)t^{n-1}=0.$$

Or

$$\sum_{n=1}^{\infty}Axnk_{n,m}^{\lambda}(x,A)t^n=\sum_{n=1}^{\infty}\sqrt{A/2}\,nk_{n,m}^{\lambda}(x,A)t^{n+m-2}$$

$$+\sum_{n=1}^{\infty}(A/2)x^2Dk_{n,m}^{\lambda}(x,A)t^n-\sum_{n=1}^{\infty}(m-1)\sqrt{A/2}\,xt^{n+m-2}Dk_{n,m}^{\lambda}(x,A)$$

$$+\sum_{n=1}^{\infty}\frac{m}{3}D\,k_{n,m}^{\lambda}(x,A)t^{n+m-1}.$$

Or

$$\sum_{n=1}^{\infty}n\,k_{n,m}^{\lambda}(x,A)t^n=\sum_{n=1}^{\infty}\sqrt{1/2A}\,x\,n\,k_{n,m}^{\lambda}(x,A)t^{n+m-2}$$

$$+\sum_{n=1}^{\infty}\frac{x}{2}Dk_{n,m}^{\lambda}(x,A)t^n-\sum_{n=1}^{\infty}(m-1)\sqrt{1/2A}\,(1/x)t^{n+m-2}$$

$$.\,Dk_{n,m}^{\lambda}(x,A)+\sum_{n=1}^{\infty}\frac{m}{3Ax}D\,k_{n,m}^{\lambda}(x,A)t^{n+m-1}.$$

Since $k_{n,m}^{\lambda}(x,A)=\left(\lambda x\sqrt{2A}\right)^n/n!,$ for $0\leq n\leq m-2$,then we get .

$$n\,k_{n,m}^{\lambda}(x,A)=\frac{x}{2}D\,k_{n,m}^{\lambda}(x,A) \tag{15}$$

$$+\left((n/x)-(m-1)\sqrt{1/2A}\,(1/x)\right)Dk_{n-m+2,m}^{\lambda}(x,A)$$

$$+\frac{m}{3}(A/x)D\,k_{n-m+1}^{\lambda}(x,A).$$

For $\lambda=1$, from (6),

$$\exp\left(\left(3(A/2)x^2t-3\sqrt{A/2}\,x\,t^{m-1}+t^mI\right)\right)$$

$$=e^{3\sqrt{A/2}\,x\,t^{m-1}}e^{-t^mI}\sum_{n=0}^{\infty}k_{n,m}(x,A)t^n.$$

$$= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{\left(3\sqrt{A/2}\,x\right)^r(-1)^s}{r!s!} .t^{n+(m-1)r+ms}\, k_{n,m}(x,A).$$

$$\sum_{n=0}^{\infty}\frac{\left(3(A/2)\right)^n\left(x^2t\right)^n}{n!} = \sum_{n=0}^{\infty}\sum_{r=0}^{\left[\frac{n}{m-1}\right]}\sum_{s=0}^{\left[\frac{n-(m-1)r}{m}\right]}\frac{(-1)^s}{r!s!} .\left(3\sqrt{A/2}x\right)^r t^n k_{n-(m-1)r-ms,m}(x,A).$$

By equating of the coefficients of t^n, one gets

$$\frac{x^{2n}}{n!}I = (3A/2)^{-n}\sum_{r=0}^{\left[\frac{n}{m-1}\right]}\sum_{s=0}^{\left[\frac{n-(m-1)r}{m}\right]}\frac{(-1)^s}{r!s!} .\left(3\sqrt{A/2}\;x\right)^r k_{n-(m-1)r-ms,m}(x,A). \quad (16)$$

Since

$$\exp\left(\left(3(A/2)x^2t - 3\sqrt{A/2}\,x\,t^{m-1} + t^m u^m I\right)\right)$$

$$= \exp\left(3(A/2)x^2t - 3\sqrt{A/2}\,x\,t^{m-1} + t^m I\right) .\exp\left(-t^m + t^m u^m I\right).$$

Then

$$\sum_{n=0}^{\infty}k_{n.m}(x,A)(tu)^n = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k\left(1-u^m\right)^k}{k!} .t^{mk}\,k_{n,m}(x,A)t^n.$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{[n/m]}\frac{(-1)^k\left(1-u^m\right)^k}{k!}k_{n-mk,m}(x,A)t^n$$

Which , by comparing the coefficients of t^n ,we get

$$u^n k_{n.m}(x,A) = .\sum_{n=0}^{[n/m]}\frac{(-1)^k\left(1-u^m\right)^k}{k!}k_{n-mk,m}(x,A). \quad (17)$$

We thus arrive to the following result.
Theorem 3.1. The generalized new class of matrix polynomials set. Satisfy the following relations:

1- $D^k\,k^{\lambda}_{n,m}(x,A) = \left[\lambda\left(3Ax - 3\sqrt{A/2}\,t^{m-2}\right)\right]^k k^{\lambda}_{n-k,m}(x,A)$

2- $n k^{\lambda}_{n,m}(x,A) = \frac{x}{2}D\,k^{\lambda}_{n,m}(x,A) + \left((n/x) - (m-1)\sqrt{1/2A}\,(1/x)\right) .Dk^{\lambda}_{n-m+2,m}(x,A) + \frac{m}{3}(A/x)D\,k^{\lambda}_{n-m+1}(x,A).$

3- $\frac{x^{2n}}{n!}I = (3A/2)^{-n}\sum_{r=0}^{\left[\frac{n}{m-1}\right]}\sum_{s=0}^{\left[\frac{n-(m-1)r}{m}\right]}\frac{(-1)^s}{r!s!} .\left(3\sqrt{A/2}\;x\right)^r k_{n-(m-1)r-ms,m}(x,A).$

4- $u^n k_{n.m}(x,A) = \sum_{n=0}^{[n/m]}\frac{(-1)^k\left(1-u^m\right)^k}{k!}k_{n-mk,m}(x,A).$

Now, inserting (13) in (15), yields

$$n\,k^{\lambda}_{n,m}(x,A) = \frac{x}{2}\left(3\lambda Ax - 3\lambda\sqrt{A/2}\,t^{m-2}\right) \quad (18)$$

$$.\,k^{\lambda}_{n-1,m}(x,A) + \left((n/x) - (m-1)\sqrt{1/2A}(1/x)\right)k^{\lambda}_{n-m+2,m}(x,A) + \frac{m}{3}(A/x)D\,k^{\lambda}_{n-m+1}(x,A).$$

Replacing n by $n-m+1$ in (3.3), gives

$$D\,k^{\lambda}_{n-m+1,m}(x,A) = \lambda\left(3Ax - 3\sqrt{A/2}\,t^{m-2}\right)k^{\lambda}_{n-m,m}(x,A). \quad (19)$$

Substituting from (19) into (18), yields
The four terms recurrence relation as given in the following theorem:

Theorem 3.2. The generalized new class matrix polynomials $k^{\lambda}_{n,m}(x,A)$, satisfy the four terms recurrence relation:

$$nk^{\lambda}_{n,m}(x,A) = \frac{3\lambda x}{2}\left(Ax - \sqrt{A/2}\,t^{m-2}\right)k^{\lambda}_{n-1,m}(x,A) \quad (20)$$
$$+\left((n/x) - (m-1)\sqrt{1/2A}(1/x)\right)k^{\lambda}_{n-m+2,m}(x,A)$$
$$+\frac{\lambda m}{Ax}\left(Ax - \sqrt{A/2}\,t^{m-2}\right)k^{\lambda}_{n-m}(x,A),\; n \ge m.$$

With initial values

$k^{\lambda}_{n,m}(x,A) = \left(\lambda x\sqrt{2A}\right)/n!\,, 0 \le n \le m-2.$

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