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Induced Graphoidal Covers in a Graph

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Abstract—An induced graphoidal cover of a graph G is a collection ψ of (not necessarily open) paths in G such that every path in ψ has at least two vertices, every vertex of G is an internal vertex of at most one path in ψ, every edge of G is in exactly one path in ψ and every member of ψ is an induced cycle or an induced path. The minimum cardinality of an induced graphoidal cover of G is called the induced graphoidal covering number of G and is denoted by $\eta_i(G)$ or η_i. Here we find induced graphoidal cover for some classes of graphs.

Keywords—Graphoidal cover, Induced graphoidal cover, Induced graphoidal covering number.

I. Introduction

A graph is a pair $G = (V, E)$, where V is the set of vertices and E is the set of edges. Here, we consider only nontrivial, finite, connected and simple graphs. The order and size of G are denoted by p and q respectively. The concept of graphoidal cover was introduced by B.D. Acharya and E. Sampathkumar [1] and the concept of semigraphoidal cover was introduced by S. Arumugram *et.al.* [3] (see also induced graphoidal cover by S. Arumugram [4]). The study of this parameter was initiated by S. Arumugram [4]. The reader may refer [3], [5] and [6] for the terms not defined here.

Definition I.1. *[1] A graphoidal cover of a graph G is a collection ψ of (not necessarily open) paths in G satisfying the following conditions:*
(i) Every path in ψ has at least two vertices.
(ii) Every vertex of G is an internal vertex of at most one path in ψ.
(iii) Every edge of G is in exactly one path in ψ.

The minimum cardinality of a graphoidal cover of G is called the graphoidal covering number of G and is denoted by $\eta(G)$.

Definition I.2. *[2] A simple graphoidal cover of a graph G is a collection ψ of (not necessarily open) paths in G satisfying the following conditions:*
(i) Every path in ψ has at least two vertices.
(ii) Every vertex of G is an internal vertex of at most one path in ψ.
(iii) Every edge of G is in exactly one path in ψ and any two paths in ψ have at most one vertex in common.

The minimum cardinality of a simple graphoidal cover of G is called the simple graphoidal covering number of G and is denoted by $\eta_s(G)$ or η_s.

Definition I.3. *[3] Let ψ be merely a partition of the edge-set $E(G)$ of G, each part of which, as an edge-induced subgraph of G, spans either a path or a cycle which is called semigraphoidal cover of G.*

Definition I.4. *[4] An induced graphoidal cover of a graph G is a collection ψ of (not necessarily open) paths in G satisfying the following conditions:*
(i) Every path in ψ has at least two vertices.
(ii) Every vertex of G is an internal vertex of at most one path in ψ.
(iii) Every edge of G is in exactly one path in ψ.
(iv) Every member of ψ is an induced cycle or an induced path.

The minimum cardinality of an induced graphoidal cover of G is called the induced graphoidal covering number of G and is denoted by $\eta_i(G)$ or η_i.

Remark I.5. *We observe that every path(cycle) in a simple graphoidal cover is an induced path(cycle) but every induced graphoidal cover is not necessarily a simple graphoidal cover of a graph.*

Definition I.6. *Let ψ be a collection of internally edge disjoint paths in G. A vertex of G is said to be an internal vertex of ψ if it is an internal vertex of some path in ψ, otherwise it is called an external vertex of ψ.*

II. Main Results

The following result for graphoidal covering number also holds for induced graphoidal covering number.

Theorem II.1. *([3],Theorem 3.6) For any induced graphoidal cover ψ of a (p,q)- graph G, let t_ψ denote the number of external vertices of ψ and let $t = min\ t_\psi$, where the minimum is taken over all induced graphoidal covers ψ of G then $\eta_i(G) = q - p + t$.*

Corollary II.2. *For any graph G, $\eta_i(G) \geq q - p$. Moreover, the following are equivalent*
(i) $\eta_i(G) = q - p$.
(ii) There exists an induced graphoidal cover of G without external vertices.
(iii) There exists a set Q of internally disjoint and edge disjoint induced graphoidal cycle or path without exterior vertices (From such a set Q of paths the required induced graphoidal cover can be obtained by adding the edges which are not covered by the paths in Q).

Corollary II.3. *If there exists an induced graphoidal cover ψ of a graph G such that every vertex of G with degree at least two is internal to ψ, then ψ is a minimum induced graphoidal cover of G and $\eta_i(G) = q - p + n$, where n is the number of pendant vertices of G.*

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Corollary II.4. *Since every graphoidal cover of a tree T is also an induced graphoidal cover of T, we have $\eta_i(T) = n-1$, where n is the number of pendant vertices of T.*

Theorem II.5. *Let G be a complete bipartite graph $K_{m,n}$, then*

$(i)\ \ \eta_i(K_{1,n}) = n-1, \ \ n \geq 2$

$(ii)\ \ \eta_i(K_{2,n}) = \begin{cases} q-p+1 & \text{if } n = 2,\ 3 \\ q-p & \text{if } n \geq 4 \end{cases}$

$(iii)\ \ \eta_i(K_{3,n}) = \begin{cases} q-p+1 & \text{if } n = 3 \\ q-p & \text{if } n \geq 4 \end{cases}$

$(iv)\ \ \eta_i(K_{m,n}) = q-p, \ \ n \geq m \text{ and } n \geq 4$

Proof: Let $X = \{v_1, v_2, v_3, \ldots, v_m\}$ and $Y = \{w_1, w_2, w_3, \ldots, w_n\}$ be a bipartition of $K_{m,n}$.

(i). When $n \geq 2$. $K_{1,n}$ is a tree with n pendant vertices and hence $\eta_i(K_{1,n}) = n-1$.

(ii). Case(a). When $n = 2$, i.e. $K_{2,2} = C_4$, we have $\eta_i(K_{2,2}) = 1$.

Case(b). When $n = 3$, let $X = \{v_1, v_2\}$ and $Y = \{w_1, w_2, w_3\}$ be a bipartition of $K_{2,3}$. Let $P_1 = (v_1, w_1, v_2, w_2, v_1)$ and $P_2 = (v_1, w_3, v_2)$ then $\psi = \{P_1, P_2\}$ is an induced graphoidal cover of $K_{2,3}$ with v_1 as its external vertex. Hence, $\eta_i(K_{2,3}) = q-p+1$.

Case(c). When $n \geq 4$, let $X = \{v_1, v_2\}$ and $Y = \{w_1, w_2, w_3, \ldots, w_n\}$ be a bipartition of $K_{2,n}$. Let $P_1 = (v_1, w_1, v_2, w_2, v_1)$, $P_2 = (v_2, w_3, v_1, w_4, v_2)$ and $P_{i-2} = (v_1, w_i, v_2), i = 5, 6, \ldots, n$. Then $\psi = \{P_1, P_2, P_{i-2} | i = 5, 6, \ldots, n\}$ is an induced graphoidal cover of $K_{2,n}$ and every vertex is an internal vertex of exactly one path in ψ. Hence, $\eta_i(K_{2,n}) = q-p$.

(iii). Case(a). When $n = 3$, let $X = \{v_1, v_2, v_3\}$ and $Y = \{w_1, w_2, w_3\}$ be a bipartition of $K_{3,3}$. Let $P_1 = (v_1, w_1, v_2, w_2, v_1)$, $P_2 = (v_1, w_3, v_2)$, $P_3 = (w_1, v_3, w_2)$, $P_4 = (v_3, w_3)$. Then $\psi = \{P_1, P_2, P_3, P_4\}$ is an induced graphoidal cover of $K_{3,3}$ with v_1 as its external vertex. Hence, $\eta_i(K_{3,3}) = q-p+1$.

Case(b). When $n \geq 4$, let $X = \{v_1, v_2, v_3\}$ and $Y = \{w_1, w_2, w_3, \ldots, w_n\}$ be a bipartition of $K_{3,n}$. Let $P_1 = (v_1, w_1, v_2, w_2, v_1)$, $P_2 = (v_2, w_3, v_1, w_4, v_2)$, $P_{i-2} = (v_1, w_i, v_2), i = 5, 6, \ldots, n$ and $Q = (w_1, v_3, w_2)$.
Then $\psi = \{P_1, P_2, P_3, P_4, \ldots, P_{n-2}, Q\} \cup S$, where S is the set of edges not covered by $P_1, P_2, P_3, \ldots, P_{n-2}, Q$ is an induced graphoidal cover of $K_{3,n}$ and every vertex is an internal vertex of a path in ψ. Hence, $\eta_i(K_{3,n}) = q-p$.

(iv). When $m \geq n$ and $n \geq 4$, let $X = \{v_1, v_2, v_3, \ldots, v_m\}$ and $Y = \{w_1, w_2, w_3, \ldots, w_n\}$ be a bipartition of $K_{m,n}$. Let $P_1 = (v_1, w_1, v_2, w_2, v_1)$, $P_2 = (v_2, w_3, v_1, w_4, v_2)$, $P_{i-2} = (v_1, w_i, v_2), i = 5, 6, \ldots, n$ and $Q_{j-2} = (w_1, v_j, w_2), j = 3, 4, \ldots, m$.
Then $\psi = \{P_1, P_2, P_3, \ldots, P_{n-2}, Q_1, Q_2, \ldots, Q_{m-2}\} \cup S$,where S is the set of edges not covered by $P_1, P_2, \ldots, P_{n-2}, Q_1, Q_2, \ldots, Q_{m-2}$ is an induced graphoidal cover of $K_{m,n}$ and every vertex is an internal vertex of a path in ψ. Hence, $\eta_i(K_{m,n}) = q-p$. ■

Theorem II.6. *If G is a unicyclic graph with n pendant vertices and the unique cycle C_k, and j denote the number of vertices of degree greater than 2 on C_k then*

$$\eta_i(G) = \begin{cases} 1 & \text{if } j = 0; \\ n+1 & \text{if } j = 1 \text{ and the unique vertex is of deg 3;} \\ & \text{or } j = 2 \text{ and the two vertices of deg 3 are} \\ & \text{adjacent on } C_k; \\ n & otherwise. \end{cases}$$

Proof: Let $C_k = \{v_1, v_2, v_3, \ldots, v_k, v_1\}$ be the unique cycle in G.

Case(a). When $j = 0$, then $G = C_k$ so that $\eta_i(G) = 1$.

Case(b). When $j = 1$ and the unique vertex, say u, of degree 3 on C_k. Let $T = G - C_k$ be the tree rooted at u, then T has $n+1$ pendant vertices so that $\eta_i(T) = n$. Let ψ_1 be a minimum induced graphoidal cover of T and so $|\psi_1| = n$. Then $\psi = \psi_1 \cup C_k$, where an arbitrary vertex of C_k is taken as an external vertex of C_k, is an induced graphoidal cover of G and so $|\psi| = n+1$. Hence, $\eta_i(G) \leq n+1$. Further, for any induced graphoidal cover ψ of G, the n pendant vertices of G and at least one vertex on C_k are external vertices in ψ so that $t \geq n+1$. Hence, $\eta_i(G) = q-p+t \geq q-p+n+1 = n+1$.

When $j = 2$ and v_1 and v_2 are adjacent vertices in C_k with deg v_1, deg $v_2 = 3$. Let T_1 be the tree rooted at v_1, then T_1 has n_1+1 pendant vertices so that $\eta_i(T_1) = n_1$. Let ψ_1 be a minimum induced graphoidal cover of T_1. Similarly, T_2 is also a tree rooted at v_2, then T_2 has n_2+1 pendant vertices so that $n = n_1+n_2$ and $\eta_i(T_2) = n_2$. Let ψ_2 be a minimum induced graphoidal cover of T_2. Then $\psi = \psi_1 \cup \psi_2 \cup C_k$ is an induced graphoidal cover of G and $|\psi| = |\psi_1| + |\psi_2| + 1 = n+1$. Hence, $\eta_i(G) \leq n+1$. Again, for any induced graphoidal cover ψ of G, the n pendant vertices of G and at least one vertex on C_k are external vertices in ψ so that $t \geq n+1$. Hence $\eta_i(G) = q-p+t \geq q-p+n+1 = n+1$.

Case(c). When $j = 1$ and the unique vertex u is of $deg\ u > 3$. Let T be the subgraph of G obtained by deleting all the vertices other than u of the unique cycle C_k in G so that $\eta_i(T) = n-1$, with ψ_1 as a minimum induced graphoidal cover. Then $\psi = \psi_1 \cup C_k$, where u is taken as an external vertex of C_k, is an induced graphoidal cover of G and every vertex of degree greater than 1 is an internal vertex of some path in ψ. Hence $\eta_i(G) = n$.

If $j = 2$ and the vertices of $deg \geq 4$ are adjacent then the result follows from case(c) with $j = 1$.

When $j = 2$ and v_1, v_3 are the two non adjacent vertices with each of $deg\ v_1, deg\ v_3 \geq 3$. Let $P = (v_1, v_k, v_{k-1}, \ldots, v_4, v_3)$ be an induced path of length $k-2$. Then $T = G - P$ is a tree with n pendant vertices so that $\eta_i(T) = n-1$, with ψ_1 as a minimum induced graphoidal cover. Then $\psi = \psi_1 \cup P$ is an induced graphoidal cover of G such that every vertex of degree greater than 1 is an internal vertex of some path in ψ. Hence, $\eta_i(G) = n$.

When $j \geq 3$ and deg $v_i \geq 3, i = 1, 2, \ldots, r$ and $r \leq k$, where $v_i, i = 1, 2, 3, \ldots, k$ are the vertices on C_k.
Suppose $k = 3$, Let $G_1 = G-(v_1v_2)$ be a tree with n pendant vertices. Also, let T be the induced subgraph of G_1 containing v_2 and v_3 such that v_3 is a pendant vertex in T. Then T has n_1+1 pendant vertices so that $\eta_i(T) = n_1$. Let ψ_1 be a

minimum induced graphoidal cover of T and so $|\psi_1| = n_1$. Again, $T' = G_1 - T$ is also a tree with n_2 pendant vertices so that $\eta_i(T') = n_2 - 1$, where $n_1 + n_2 = n$. Let ψ_2 be a minimum induced graphoidal cover of T' and so $|\psi_2| = n_2 - 1$. Then $\psi = \psi_1 \cup \psi_2 \cup (v_1 v_2)$, is an induced graphoidal cover of G and every vertex of degree greater than 1 is an internal vertex of some path in ψ. Hence, $\eta_i(G) = n_1 + n_2 - 1 + 1 = n$.

Next, suppose $k > 3$. let G' be the induced subgraph of G with vertex set $\{v_1, v_2, \ldots, v_s\}$, where $s < r$ and deg $v_1, v_s \geq 3$, on the cycle C_k. Then G' has n_1 pendant vertices so that $\eta_i(G') = n_1 - 1$. Let ψ_1 be the minimum induced graphoidal cover of G'. Then $T = G - G'$ is a tree with $n_2 + 2$ pendant vertices so that $n = n_1 + n_2$ and $\eta_i(T) = n_2 + 1$. Let ψ_2 be the minimum induced graphoidal cover of T. Then $\psi = \psi_1 \cup \psi_2$, is an induced graphoidal cover of G and every vertex of degree greater than 1 is an internal vertex of some path in ψ. Hence, $\eta_i(G) = n$. ■

Theorem II.7. *Let G be a bicyclic graph with n pendant vertices containing a $U(l;m)$ [see, [6]] and j be the number of vertices of degree greater than 2 in $U(l;m)$. Then*

$$\eta_i(G) = \begin{cases} 2 & \text{if } G = U(l;m); \\ n+2 & \text{if either } j = 1 \text{ and } deg\ u_0 = 5; \text{ or } j = 2, \\ & deg\ u_0 = 4 \text{ and the vertex of deg 3 in} \\ & U(l;m) \text{ is adjacent to } u_0; \text{ or } j = 3, \\ & deg\ u_0 = 4 \text{ such that } u_0 \text{ is adjacent to} \\ & \text{both vertices } v \text{ and } w \text{ of degree 3 in } C_l \\ & \text{and } C_m \text{ respectively}; \\ n+1 & otherwise. \end{cases}$$

Proof: Let the $l-$cycle be $C_l = \{u_0, u_1, \ldots, u_{l-1}, u_0\}$ and the $m-$cycle be $C_m = \{u_0, u_l, u_{l+1}, \ldots, u_{m+l-2}, u_0\}$ in G.

Case (a). Suppose $G = U(l;m)$. Then $\psi = \{C_l, C_m\}$ is an induced graphoidal cover of G with u_0 as the external vertex for both C_l and C_m. Hence, $\eta_i(G) = q - p + 1 = 2$.

Case(b). When $j = 1$ and $deg\ u_0 = 5$. Then $G_1 = G - C_m$ is a unicyclic graph with one vertex of deg 3 in the cycle so that $\eta_i(G_1) = n+1$. Let ψ_1 be a minimum induced graphoidal cover of G_1. Then $\psi = \psi_1 \cup C_m$ is an induced graphoidal cover of G and $|\psi| = |\psi_1| + 1 = n + 2$. Hence, $\eta_i(G) \leq n + 2$. Again, for any induced graphoidal cover ψ of G, the n pendant vertices of G and at least one vertex in $U(l;m)$ are external vertices in ψ so that $t \geq n + 1$. Hence, $\eta_i(G) = q - p + t \geq n + 1 + 1 = n + 2$.

When $j = 2$, $deg\ u_0 = 4$ and $deg\ v = 3$ are adjacent to u_0 in C_m. Then $G_1 = G - C_l$ is a unicyclic graph with $deg\ v = 3$ so that $\eta_i(G_1) = n + 1$. Let ψ_1 be a minimum induced graphoidal cover of G_1. Then $\psi = \psi_1 \cup C_l$ is an induced graphoidal cover of G and $|\psi| = |\psi_1| + 1 = n + 2$. Hence, $\eta_i(G) \leq n + 2$. Again, for any induced graphoidal cover ψ of G, the n pendant vertices of G and at least one vertex in $U(l;m)$ are external vertices in ψ so that $t \geq n+1$. Hence, $\eta_i(G) = q - p + t \geq n + 1 + 1 = n + 2$.

When $j = 3$, $deg\ u_0 = 4$ such that u_0 is adjacent to both vertices v and w of degree 3 in C_l and C_m respectively. Then $G - U(l;m)$ gives two trees, say T_1 and T_2, rooted at v and w with $n_1 + 1$ and $n_2 + 1$ pendant vertices respectively such that $n_1 + n_2 = n$. Also, $\eta_i(T_1) = n_1$ and $\eta_i(T_2) = n_2$. Then $\psi = \psi_1 \cup \psi_2 \cup U(l;m)$ is an induced graphoidal cover of G and $|\psi| = |\psi_1| + |\psi_2| + 2 = n_1 + n_2 + 2 = n + 2$. Hence, $\eta_i(G) \leq n + 2$. Again, for any induced graphoidal cover ψ of G, the n pendant vertices of G and at least one vertex on $U(l;m)$ are external vertices in ψ so that $t \geq n + 1$. Hence $\eta_i(G) = q - p + t \geq q - p + n + 1 = n + 2$.

Case(c). When $j = 1$ and $deg\ u_0 \geq 6$. Then $G_1 = G - C_l$ is a unicyclic graph with n pendant vertices so that $\eta_i(G_1) = n$, with ψ_1 as a minimum induced graphoidal cover of G_1. Then $\psi = \psi_1 \cup C_l$ is a minimum induced graphoidal cover of G such that every vertex of degree greater than 1 is an internal vertex of some path in ψ. Hence, $\eta_i(G) = n + 1$.

Similarly, we can prove for $j \geq 2$ and $deg\ u_0 \geq 6$ in one cycle.

When $j = 2$ and $deg\ v \geq 3$ such that u_0 and v are not adjacent. Let T denote this induced $v - u_0$ subgraph section with $n_1 + 2$ pendant vertices so that $\eta_i(T) = n_1 + 1$. Let ψ_1 be a minimum induced graphoidal cover of T. Also, $G_1 = G - T$ is a unicyclic graph with n_2 pendant vertices such that $\eta_i(G_1) = n_2$, where $n_1 + n_2 = n$. Let ψ_2 be a minimum induced graphoidal cover of G_1. Then $\psi = \psi_1 \cup \psi_2$ is a minimum induced graphoidal cover of G and every vertex of degree greater than 1 is an internal vertex of some path in ψ. Hence, $\eta_i(G) = n + 1$.

Similarly, we can prove the result for $j = 3$, $deg\ u_0 \geq 4$ such that u_0 is non adjacent to both v and w of degree 3 in C_l and C_m respectively.

Otherwise, let T be the induced subgraph of G with vertex set $\{u_0, u_1, , \ldots, u_s\}$, where $s < l$ and deg $u_s \geq 3$, on the cycle C_l. Then T has n_1 pendant vertices so that $\eta_i(T) = n_1 - 1$. Let ψ_1 be the minimum induced graphoidal cover of T. Then $G_1 = G - T$ is a unicyclic with $n_2 + 2$ pendant vertices so that $n = n_1 + n_2$ and $\eta_i(G_1) = n_2 + 2$. Let ψ_2 be the minimum induced graphoidal cover of G_1. Then $\psi = \psi_1 \cup \psi_2$, is an induced graphoidal cover of G and every vertex of degree greater than 1 is an internal vertex of some path in ψ. Hence, $\eta_i(G) = n + 1$. ■

Theorem II.8. *Let G be a bicyclic graph with n pendant vertices containing a $D(l,m;i)$ [see, [6]] and j be the number of vertices of degree greater than 2 on cycles in $D(l,m;i)$. Then*

$$\eta_i(G) = \begin{cases} 3 & \text{if } G = D(l,m;i); \\ n+3 & \text{if } j = 3, \text{ exactly one vertex which is} \\ & \text{adjacent to either } u_{l-1} \text{ or } u_{l+i-1} \text{ in} \\ & D(l,m;i) \text{ is of degree 3 and deg } u_{l-1}, \\ & u_{l+i-1} = 3; \text{ or } j = 4, v \text{ and } w \text{ are two} \\ & \text{vertices of degree 3 adjacent to } u_{l-1} \\ & \text{and } u_{l+i-1} \text{respectively and} \\ & deg\ u_{l-1}, u_{l+i-1} = 3; \\ n+2 & \text{if } deg\ u_{l-1} \geq 4,\ deg\ u_{l+i-1} = 3 \\ & \text{and } j \geq 2 \text{ in } C_l; \\ n+1 & otherwise. \end{cases}$$

Proof: G is bicyclic, so G contains at least $C_l = \{u_0, u_1, \ldots, u_{l-1}, u_0\}$, $P_i = \{u_{l-1}, u_l, \ldots, u_{l+i-1}\}$ and $C_m = \{u_{l+i-1}, u_{l+i}, \ldots, u_{l+m+i-2}, u_{l+i-1}\}$ in G.

Case(a). $G = D(l, m; i)$. Then $\psi = \{C_l, P_i, C_m\}$ is an induced graphoidal cover of G with u_{l-1} and u_{l+i-1} as its only external vertices. Hence, $\eta_i(G) = q - p + 2 = 3$.

Case(b). When $j = 3$ and exactly one vertex, say v, which is adjacent to either u_{l-1} or u_{l+i-1} in $D(l, m; i)$ is of degree 3. Suppose v lies in C_l. Then $G_1 = G - C_m$ is a unicyclic graph with $n+1$ pendant vertices so that $\eta_i(G_1) = n+2$. Let ψ_1 be a minimum induced graphoidal cover of G_1. Then $\psi = \psi_1 \cup C_m$ is an induced graphoidal cover of G and $|\psi| = |\psi_1| + 1 = n+3$. Hence, $\eta_i(G) \leq n+3$. Again, for any induced graphoidal cover ψ of G, the n pendant vertices of G and at least two vertices in $D(l, m; i)$ are external vertices in ψ so that $t \geq n + 2$. Hence, $\eta_i(G) = q - p + t \geq n + 2 + 1 = n + 3$.

When $j = 4$, v and w are two vertices of degree 3 adjacent to u_{l-1} and U_{l+i-1} respectively and $deg\ u_{l-1}, u_{l+i-1} = 3$. Let T_1 be the tree rooted at v, then T_1 has $n_1 + 1$ pendant vertices so that $\eta_i(T_1) = n_1$. Let ψ_1 be a minimum induced graphoidal cover of T_1. Similarly, T_2 is also a tree rooted at w, then T_2 has $n_2 + 1$ pendant vertices so that $n = n_1 + n_2$ and $\eta_i(T_2) = n_2$. Let ψ_2 be a minimum induced graphoidal cover of T_2. Then $\psi = \psi_1 \cup \psi_2 \cup D(l, m; i)$ is an induced graphoidal cover of G and $|\psi| = |\psi_1| + |\psi_2| + 3 = n_1 + n_2 + 3 = n + 3$. Hence, $\eta_i(G) \leq n+3$. Again, for any induced graphoidal cover ψ of G, the n pendant vertices of G and at least two vertices on $D(l, m; i)$ are external vertices in ψ so that $t \geq n + 2$. Hence $\eta_i(G) = q - p + t \geq q - p + n + 2 = n + 3$.

Case(c). When $deg\ u_{l-1} \geq 4$, $deg\ u_{l+i-1} = 3$ and $j \geq 2$ in C_l. Here, $G_1 = G - C_m$ is a unicyclic graph with $n + 1$ pendant vertices so that $\eta_i(G_1) = n+1$, with ψ_1 as a minimum induced graphoidal cover of G_1. Then $\psi = \psi_1 \cup C_m$ is an induced graphoidal cover of G and $|\psi| = |\psi_1| + 1 = n + 2$. Hence, $\eta_i(G) \leq n + 2$. Again, for any induced graphoidal cover ψ of G, the n pendant vertices of G and at least one vertex in $D(l, m; i)$ are external vertices in ψ so that $t \geq n+1$. Hence, $\eta_i(G) = q - p + t \geq 1 + n + 1 = n + 2$.

Similarly, we can prove the result for $j = 3$, $deg\ u_{l-1} = deg\ u_{l+i-1} = 3$ and the vertex v of deg 3 is adjacent to neither u_{l-1} nor u_{l+i-1} and $j = 4$, $deg\ v, w = 3$ and at least either v is not adjacent to u_{l-1} or w is not adjacent to u_{l+i-1}.

Case(d). When $j = 2$. Then $G_1 = G - C_l$ is a unicyclic graph with n pendant vertices so that $\eta_i(G_1) = n$, with ψ_1 as a minimum induced graphoidal cover of G_1. Then $\psi = \psi_1 \cup C_l$ is a minimum induced graphoidal cover of G such that every vertex of degree greater than 1 is an internal vertex of some path in ψ. Hence, $\eta_i(G) = n + 1$.

Similarly, it can be proved for $j = 3$ and exactly one vertex, say v, which is adjacent to either u_{l-1} or u_{l+i-1} in $D(l, m; i)$, is of $deg > 3$.

When $j = 3$. let v be the vertex in $D(l, m; i)$ which is non adjacent to either u_{l-1} or u_{l+i-1} in $D(l, m; i)$ is of $deg \geq 3$ and suppose v lie in C_l. Let T denote this induced $v - u_{l-1}$ subgraph section with $n_1 + 2$ pendant vertices so that $\eta_i(T) = n_1 + 1$. Let ψ_1 be a minimum induced graphoidal cover of T. Also, $G_1 = G - T$ is a unicyclic graph with n_2 pendant vertices such that $\eta_i(G_1) = n_2$, where $n_1 + n_2 = n$. Let ψ_2 be a minimum induced graphoidal cover of G_1. Then $\psi = \psi_1 \cup \psi_2$ is a minimum induced graphoidal cover of G and every vertex of degree greater than 1 is an internal vertex of some path in ψ. Hence, $\eta_i(G) = n + 1$.

Otherwise, take an induced tree T containing all vertices in an arc of a cycle in $D(l, m; i)$ such that its end vertices are of $deg \geq 3$. Then T has n_1 pendant vertices so that $\eta_i(T) = n_1 - 1$. Let ψ_1 be a minimum induced graphoidal cover of T. Also, $G_1 = G - T$ is a unicyclic graph with $n_2 + 2$ pendant vertices such that $\eta_i(G_1) = n_2 + 2$, where $n_1 + n_2 = n$. Let ψ_2 be a minimum induced graphoidal cover of G_1. Then $\psi = \psi_1 \cup \psi_2$ is a minimum induced graphoidal cover of G and every vertex of degree greater than 1 is an internal vertex of some path in ψ. Hence, $\eta_i(G) = n + 1$. ■

Theorem II.9. *Let G be a bicyclic graph with n pendant vertices containing a $C_m(i; l)$ [see, [6]] and j be the number of vertices of degree greater than 2 on cycles in $C_m(i; l)$. Then*

$$\eta_i(G) = \begin{cases} 3 & \text{if } G = C_m(i;l) \text{ and } l = 1; \\ 2 & \text{if } G = C_m(i;l) \text{ and } l \geq 2; \\ n+3 & \text{if } j = 2,\ l = 1 \text{ and either } deg\ u_0 \text{ or } \\ & deg\ u_i = 4 \text{ in } C_m(i;l);\ \text{or } j = 2,\ l = 1 \\ & \text{and each adjacent vertices are of } deg\ 4; \\ n+2 & \text{if } j = 2,\ l = 1 \text{ and either } deg\ u_0 \text{ or } \\ & deg\ u_i \geq 4 \text{ in } C_m(i;l);\ \text{or } j = 3,\ l = 1 \\ & deg \text{ of one of } u_i \text{ and } u_0 \text{ is 3 and the other} \\ & \text{is 4 and the third vertex of } deg \geq 3 \text{ is} \\ & \text{adjacent to either } u_0 \text{ or } u_i;\ \text{or } j = 2, \\ & l \geq 2 \text{ and either } deg\ u_0 \text{ or } deg\ u_i = 4 \text{ in} \\ & C_m(i;l);\ \text{or } j = 3,\ l \geq 2 \text{ and third vertex} \\ & \text{of deg 3 is adjacent to either } deg\ u_0 \text{ or} \\ & deg\ u_i = 4 \text{ in } C_m(i;l) \text{ is of degree 3;} \\ n+1 & \text{otherwise.} \end{cases}$$

Proof: G is bicyclic, so it contains at least $C_m = \{u_0, u_1, \ldots, u_i, u_{m-1}, u_0\}$ with $m \geq 4$ the chord $P_l = \{u_0, u_m, u_{m+1}, \ldots, u_{l+m-2}, u_i\}$, $l \geq 1$ and $2 \leq i \leq m - 2$.

Case(a). When $G = C_m(i; l)$ and $l = 1$. Let $P_1 = (u_1, u_0, u_{m-1})$, $P_2 = (u_0, u_m, u_{m+1}, \ldots, u_{l+m-2}, u_i), 2 \leq i \leq m - 2$, and $P_3 = (u_1, u_2, u_i, \ldots, u_{m-1})$ be induced paths in $C_m(i; l)$. Then $\psi = \{P_1, P_2, P_3\}$ is a minimum induced graphoidal cover of G such that any two vertices of deg 2 can be taken as an external vertex. Hence, $\eta_i(G) = q - p + 2 = 3$.

Case(b). When $G = C_m(i; l)$ and $l \geq 2$. Let $P_1 = (u_0, u_1, \ldots, u_{m-1}, u_0)$, $m \geq 4$ and $P_2 = (u_0, u_m, u_{m+1}, \ldots, u_{l+m-2}, u_i)$, $l \geq 2$ be induced paths in $C_m(i; l)$. Then $\psi = \{P_1, P_2\}$ is a minimum induced graphoidal cover of G such that any one vertex in C_m can be taken as an external vertex. Hence, $\eta_i(G) = q - p + 1 = 2$.

Case(c). When $j = 2$, $l = 1$ and either $deg\ u_0$ or $deg\ u_i = 4$ in $C_m(i; l)$. Let u_s, $0 < s < i$, be the vertex in $C_m(i; l)$. Then $P_1 = (u_0, u_s)$, $P_2 = (u_s, u_1, \ldots, u_i)$ be induced paths in $C_m(i; l)$. Let $G_1 = G - \{P_1, P_2\}$ is a unicyclic graph with n pendant vertices so that $\eta_i(G_1) = n+1$. Let ψ_1 be a minimum induced graphoidal cover of G_1. Then $\psi = \psi_1 \cup \{P_1, P_2\}$ is an induced graphoidal cover of G and $|\psi| = |\psi_1|+2 = n+1+2$. Hence, $\eta_i(G) \leq n+3$. Again, for any induced graphoidal cover ψ of G, the n pendant vertices of G and at least two vertices in $C_m(i;l)$ are external vertices in ψ so that $t \geq n+3$. Hence, $\eta_i(G) = q-p+t \geq n+1+2 = n+3$.

Similarly, we can prove for $j = 2,\ l = 1$ and each adjacent vertices are of $deg\ 4$.

Case(d). When $j = 2,\ l = 1$ and either $deg\ u_0$ or $deg\ u_i \geq 4$ in $C_m(i;l)$. Let $u_s,\ 0 < s < i$, be the vertex in $C_m(i;l)$. Then $P_1 = (u_0, u_s)$, $P_2 = (u_s, u_1, \ldots, u_i)$ be induced paths in $C_m(i;l)$. Let $G_1 = G - \{P_1, P_2\}$ is a unicyclic graph with n pendant vertices so that $\eta_i(G_1) = n$. Let ψ_1 be a minimum induced graphoidal cover of G_1. Then $\psi = \psi_1 \cup \{P_1, P_2\}$ is an induced graphoidal cover of G and $|\psi| = |\psi_1|+2 = n+2$. Hence, $\eta_i(G) \leq n+2$. Again, for any induced graphoidal cover ψ of G, the n pendant vertices of G and at least one vertex in $C_m(i;l)$ are external vertices in ψ so that $t \geq n+2$. Hence, $\eta_i(G) = q-p+t \geq n+2 = n+2$.

Similarly, we can prove for $j = 3,\ l = 1$ degree of one of u_i and u_0 is 3 and the other is 4 and the third vertex of $deg \geq 3$ is adjacent to either u_0 or u_i.

When $j = 2,\ l \geq 2$ and either $deg\ u_0$ or $deg\ u_i = 4$ in $C_m(i;l)$. Let $P = (u_0 u_m u_{m+1} \ldots u_{l+m-2} u_i),\ 2 \leq i \leq m-2$, be the chord in $C_m(i;l)$ such that $\eta_i(P) = 1$. Then $G_1 = G - P$ is a unicyclic graph with n pendant vertices so that $\eta_i(G_1) = n+1$. Let ψ_1 be a minimum induced graphoidal cover of G_1. Then $\psi = \psi_1 \cup P$ is an induced graphoidal cover of G and $|\psi| = |\psi_1| + 1 = n+2$. Hence, $\eta_i(G) \leq n+2$. Again, for any induced graphoidal cover ψ of G, the n pendant vertices of G and at least one vertex in $C_m(i;l)$ are external vertices in ψ so that $t \geq n+2$. Hence, $\eta_i(G) = q-p+t \geq n+2 = n+2$.

Similarly, we can prove for $j = 3,\ l \geq 2$ and the third vertex of degree 3 adjacent to either $deg\ u_0$ or $deg\ u_i = 4$ in $C_m(i;l)$.

Case(e). When $j \geq 4,\ l = 1,\ u_s$ and v be the two vertices of $deg \geq 3$ in $C_m(i;1)$ other than $u_0,\ u_i$. Let G_1 be the subgraph of G with vertex set $\{u_0, u_1, \ldots, u_s, u_i\}$, where $0 < s < i$ in $C_m(i;l)$ such that both u_0 and u_i are pendent vertices. Then $T = G_1 - (u_0, u_s)$ is a tree with $n_1 + 1$ pendant vertices so that $\eta_i(T) = n_1$. Let ψ_1 be a minimum induced graphoidal cover of T. Also, $G_2 = G - G_1$ is a unicyclic graph with n_2 pendant vertices so that $n_1 + n_2 = n$ and $\eta_i(G_2) = n_2$. Let ψ_2 be a minimum induced graphoidal cover of G_1. Then $\psi = \psi_1 \cup \psi_2 \cup (u_0, u_s)$ is an induced graphoidal cover of G and every vertex of degree greater than 1 is an internal vertex of some path in ψ. Hence, $\eta_i(G) = n_1 + n_2 + 1 = n+1$.

When $j \geq 2,\ l \geq 2$. Let $P = (u_0 u_m u_{m+1} \ldots u_{l+m-2} u_i), 2 \leq i \leq m-2$, be the chord in $C_m(i;l)$ such that $\eta_i(P) = 1$. Then $G_1 = G - P$ is a unicyclic graph with n pendant vertices so that $\eta_i(G_1) = n$. Let ψ_1 be a minimum induced graphoidal cover of G_1. Then $\psi = \psi_1 \cup P$ is an induced graphoidal cover of G and every vertex of degree greater than 1 is an internal vertex of some path in ψ. Hence, $\eta_i(G) = n+1$. ■

The following result for simple graphoidal covering number also holds for induced graphoidal covering number.

Theorem II.10. *For the wheel $W_p = K_1 + C_{p-1}, p \geq 4$, we have*

$$\eta_i(W_p) = \begin{cases} 4 & \text{if } p = 4,\ 5 \\ 5 & \text{if } p = 6 \\ q-p & \text{if } p \geq 7 \end{cases}$$

Theorem II.11. *For the complete graph $K_p (p \geq 3)$, we have*

$$\eta_i(K_p) = \begin{cases} 1 & \text{if } p = 3 \\ 4 & \text{if } p = 4 \\ q-p & \text{if } p \geq 6 \text{ and } p \text{ is even} \\ q-p+1 & \text{if } p \geq 5 \text{ and } p \text{ is odd.} \end{cases}$$

Corollary II.12. *Let C_p be a cycle with p vertices, then*

$$\eta_i[(C_p)^2] = \begin{cases} q-p & \text{if } p \geq 5 \text{ and } p \text{ is even} \\ q-p+1 & \text{if } p \geq 4 \text{ and } p \text{ is odd.} \end{cases}$$

Corollary II.13. *Let C_p be a cycle with p vertices, then*

$$\eta_i[T(C_p)] = \begin{cases} q-p & \text{if } p \geq 5 \text{ and } p \text{ is even} \\ q-p+1 & \text{if } p \geq 4 \text{ and } p \text{ is odd.} \end{cases}$$

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