Forming the Differential-Algebraic Model of Radial Power Systems for Simulation of both Transient and Steady-State Conditions

Saleh A. Al-Jufout

Abstract—This paper presents a procedure of forming the mathematical model of radial electric power systems for simulation of both transient and steady-state conditions. The research idea has been based on nodal voltages technique and on differentiation of Kirchhoff's current law (KCL) applied to each non-reference node of the radial system, the result of which the nodal voltages has been calculated by solving a system of algebraic equations. Currents of the electric power system components have been determined by solving their respective differential equations. Transforming the three-phase coordinate system into Cartesian coordinate system in the model decreased the overall number of equations by one third. The use of Cartesian coordinate system does not ignore the DC component during transient conditions, but restricts the model's implementation for symmetrical modes of operation only. An example of the input data for a four-bus radial electric power system has been calculated.

Keywords—Mathematical Modelling, Radial Power System, Steady-State, Transients

I. INTRODUCTION

SIMULATION languages and development in simulation methodologies have made mathematical modelling one of the most used tools in scientific research and system analysis.

Study of the interaction of a complex system, or of a part within a complex system can be achieved by simulation. The experience obtained in designing a mathematical model may be of great value toward suggesting improvement in the system under study and by changing the inputs and observing the resulting output; valuable knowledge may be obtained in which variables are most important and how variables interact [1], [2].

Modelling of electric power systems is an area of increasing interest in the transmission, distribution and control systems. Mathematical modelling and simulation of the electric power system is necessary for both planning and operation and depends on appropriate mathematical models. In the operation sector, models are typically more complex than those used for planning. Operational models support analysis of incoming data as well as simulation of expected and unexpected operational conditions [3]-[5].

Designing, maintenance and verification of mathematical models are significant activities for most operators. Distribution companies also face new mathematical modelling tasks with the increasing deployment of distribution management systems.

S. A. Al-Jufout is with the Electrical Engineering Department,, Tafila Technical University, Tafila 66110, Jordan (e-mail: drjufout@ttu.edu.jo).

While their capabilities vary from system to system, all require mathematical models of the sub-transmission and networks.

Mathematical model for maintenance solutions developed for energy management systems are rarely suitable. Unlike most main transmission systems, a typical distribution system is subject to continuous expansion and restructuring. Thus, it is worth reviewing the traditional modelling approaches to find developments. New techniques should enable engineers to meet challenges they now face.

The current paper aims to develop a procedure for mathematical modelling of the radial power system and that can be used for analysis of both transient and steady-state conditions. In section two, the procedure of formation of the model is explained using a four-bus study system. Section three shows an example of input-data preparation.

II. MATHEMATICAL MODEL FORMATION

In Fig. 1, the one-line diagram of a simple 4-bus radial electric power system is shown and considered, in this paper, for forming the differential-algebraic model.

Transforming the voltages of the power supply system from three-phase coordinate system into Cartesian coordinate system is obtained as follows:

$$\begin{bmatrix} v_{1x} \\ v_{1y} \end{bmatrix} = \frac{2}{3} \begin{bmatrix} \cos(0) & \cos(\frac{2\pi}{3}) & \cos(-\frac{2\pi}{3}) \\ \sin(0) & \sin(\frac{2\pi}{3}) & \sin(-\frac{2\pi}{3}) \end{bmatrix} \begin{bmatrix} v_{1a} \\ v_{1b} \\ v_{1c} \end{bmatrix} \quad (1)$$

thus,

$$v_{1x} = v_{1a} \quad (2)$$

$$v_{1y} = \frac{v_{1b} - v_{1c}}{\sqrt{3}} \quad (3)$$

where

v_{1x}, v_{1y} - the voltages at the infinite bus (node 1) represented in rectangular system of coordinates;

v_{1a}, v_{1b}, v_{1c} - the phase voltages at the infinite bus.

When Cartesian coordinate system is used, the number of the differential and algebraic equations is decreased by one third and the mathematical model is restricted for analysis of the symmetrical modes of operations only. However, the DC component is not ignored.

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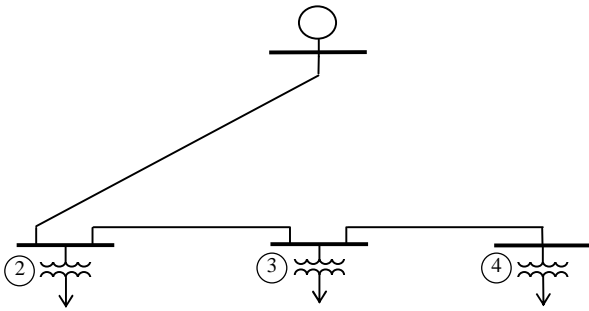International Journal of Mathematical and Computational Sciences
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Fig. 1 The one-line diagram of the study 4-bus radial power system

In this paper, transmission lines are assumed to be short and represented by their lumped series impedances. Thus, the derivatives of the currents flowing through the transmission lines, with positive direction according to their double-subscript notation in x axis, are:

$$\frac{di_{12x}}{dt} = \frac{v_{1x} - v_{2x}}{L_{12}} - \frac{R_{12}}{L_{12}} i_{12x} \qquad (4)$$

$$\frac{di_{23x}}{dt} = \frac{v_{2x} - v_{3x}}{L_{23}} - \frac{R_{23}}{L_{23}} i_{23x} \qquad (5)$$

$$\frac{di_{34x}}{dt} = \frac{v_{3x} - v_{4x}}{L_{34}} - \frac{R_{34}}{L_{34}} i_{34x} \qquad (6)$$

where

$i_{12x}, i_{23x}, i_{34x}$ - the currents flowing through the transmission lines connected between buses 1-2, 2-3 and 3-4 in x axis, respectively;

v_{2x}, v_{3x}, v_{4x} - the nodal voltages at buses 2, 3 and 4 represented in x axis, respectively;

R_{12}, R_{23}, R_{34} - the resistances of the transmission lines connected between buses 1-2, 2-3 and 3-4, respectively;

L_{12}, L_{23}, L_{34} - the inductances of the transmission lines connected between buses 1-2, 2-3 and 3-4, respectively.

If the transmission lines are considered to be medium-length lines then the shunt capacitors should be taken into account [6]. Transformers are also represented by resistive-inductive branches to which the load impedance is connected in series. The load parameters, in this example, are considered constant and thus the derivatives of the load currents flowing through the transformers, in x axis, can be expressed as:

$$\frac{di_{2x}}{dt} = \frac{v_{2x}}{L_{T2} + L_{L2}} - \frac{R_{T2} + R_{L2}}{L_{T2} + L_{L2}} i_{2x} \qquad (7)$$

$$\frac{di_{3x}}{dt} = \frac{v_{3x}}{L_{T3} + L_{L3}} - \frac{R_{T3} + R_{L3}}{L_{T3} + L_{L3}} i_{3x} \qquad (8)$$

$$\frac{di_{4x}}{dt} = \frac{v_{4x}}{L_{T4} + L_{L4}} - \frac{R_{T4} + R_{L4}}{L_{T4} + L_{L4}} i_{4x} \qquad (9)$$

where

i_{2x}, i_{3x}, i_{4x} - the load currents flowing through the transformers connected to buses 2, 3 and 4 in x axis, respectively;

R_{T2}, R_{T3}, R_{T4} - the resistances of the transformers connected to buses 2, 3 and 4, respectively;

R_{L2}, R_{L3}, R_{L4} - the resistances of the loads connected to buses 2, 3 and 4, respectively;

L_{T2}, L_{T3}, L_{T4} - the inductances of the transformers connected to buses 2, 3 and 4, respectively;

L_{L2}, L_{L3}, L_{L4} - the inductances of the loads connected to buses 2, 3 and 4, respectively.

Applying KCL to the non-reference nodes 2, 3 and 4 respectively yields:

$$i_{12x} - i_{23x} - i_{2x} = 0 \qquad (10)$$

$$i_{23x} - i_{34x} - i_{3x} = 0 \qquad (11)$$

$$i_{34x} - i_{4x} = 0 \qquad (12)$$

Differentiating (10)-(12), substituting (4)-(9) in them and rearranging with respect to the unknown voltages yield:

$$\left(\frac{1}{L_{12}} + \frac{1}{L_{23}} + \frac{1}{L_{T2} + L_{L2}} \right) v_{2x} - \frac{1}{L_{23}} v_{3x} = \frac{1}{L_{12}} v_{1x} - \frac{R_{12}}{L_{12}} i_{12x} + \frac{R_{23}}{L_{23}} i_{23x} + \frac{R_{T2} + R_{L2}}{L_{T2} + L_{L2}} i_{2x} \qquad (13)$$

$$\left(\frac{1}{L_{23}} + \frac{1}{L_{34}} + \frac{1}{L_{T3} + L_{L3}} \right) v_{3x} - \frac{1}{L_{23}} v_{2x} - \frac{1}{L_{34}} v_{4x} = -\frac{R_{23}}{L_{23}} i_{23x} + \frac{R_{34}}{L_{34}} i_{34x} + \frac{R_{T3} + R_{L3}}{L_{T3} + L_{L3}} i_{3x} \qquad (14)$$

$$\left(\frac{1}{L_{34}} + \frac{1}{L_{T4} + L_{L4}} \right) v_{4x} - \frac{1}{L_{34}} v_{3x} = -\frac{R_{34}}{L_{34}} i_{34x} + \frac{R_{T4} + R_{L4}}{L_{T4} + L_{L4}} i_{4x} \qquad (15)$$

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In matrix form:

$$V_x = B^{-1}I_x \qquad (16)$$

where

$$V_x = \begin{bmatrix} v_{2x} \\ v_{3x} \\ v_{4x} \end{bmatrix}$$

$$I_x = \begin{bmatrix} \dfrac{1}{L_{12}}v_{1x} - \dfrac{R_{12}}{L_{12}}i_{12x} + \dfrac{R_{23}}{L_{23}}i_{23x} + \dfrac{R_{T2}+R_{L2}}{L_{T2}+L_{L2}}i_{2x} \\[2mm] -\dfrac{R_{23}}{L_{23}}i_{23x} + \dfrac{R_{34}}{L_{34}}i_{34x} + \dfrac{R_{T3}+R_{L3}}{L_{T3}+L_{L3}}i_{3x} \\[2mm] -\dfrac{R_{34}}{L_{34}}i_{34x} + \dfrac{R_{T4}+R_{L4}}{L_{T4}+L_{L4}}i_{4x} \end{bmatrix}$$

$$B = \begin{bmatrix} \dfrac{1}{L_{12}} + \dfrac{1}{L_{23}} + \dfrac{1}{L_{T2}+L_{L2}} & -\dfrac{1}{L_{23}} & 0 \\[3mm] -\dfrac{1}{L_{23}} & \dfrac{1}{L_{23}} + \dfrac{1}{L_{34}} + \dfrac{1}{L_{T3}+L_{L3}} & -\dfrac{1}{L_{34}} \\[3mm] 0 & -\dfrac{1}{L_{34}} & \dfrac{1}{L_{34}} + \dfrac{1}{L_{T4}+L_{L4}} \end{bmatrix}$$

The nodal voltages, in x axis, can be calculated by solving (16) with zero-initial values of the currents. The determined nodal voltages are used then to find the next iteration of the currents in (4)-(9).

The above algorithm should be written for y axis as well, thus, the system of algebraic equations for the study system, in matrix form, is as follows:

$$\begin{bmatrix} V_x \\ V_y \end{bmatrix} = \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix}^{-1} \begin{bmatrix} I_x \\ I_y \end{bmatrix} \qquad (17)$$

where

$$I_y = \begin{bmatrix} \dfrac{1}{L_{12}}v_{1y} - \dfrac{R_{12}}{L_{12}}i_{12y} + \dfrac{R_{23}}{L_{23}}i_{23y} + \dfrac{R_{T2}+R_{L2}}{L_{T2}+L_{L2}}i_{2y} \\[2mm] -\dfrac{R_{23}}{L_{23}}i_{23y} + \dfrac{R_{34}}{L_{34}}i_{34y} + \dfrac{R_{T3}+R_{L3}}{L_{T3}+L_{L3}}i_{3y} \\[2mm] -\dfrac{R_{34}}{L_{34}}i_{34y} + \dfrac{R_{T4}+R_{L4}}{L_{T4}+L_{L4}}i_{4y} \end{bmatrix}$$

$$V_y = \begin{bmatrix} v_{2y} \\ v_{3y} \\ v_{4y} \end{bmatrix}$$

and where

i_{2y}, i_{3y}, i_{4y} - the load currents flowing through the transformers connected to buses 2, 3 and 4 in y axis, respectively;

$i_{12y}, i_{23y}, i_{34y}$ - the currents flowing through the transmission lines connected between buses 1-2, 2-3 and 3-4 in y axis, respectively;

v_{2y}, v_{3y}, v_{4y} - the nodal voltages at buses 2, 3 and 4 represented in y axis, respectively.

The system of differential equations for y axis is as follows:

$$\frac{di_{12y}}{dt} = \frac{v_{1y} - v_{2y}}{L_{12}} - \frac{R_{12}}{L_{12}}i_{12y} \qquad (18)$$

$$\frac{di_{23y}}{dt} = \frac{v_{2y} - v_{3y}}{L_{23}} - \frac{R_{23}}{L_{23}}i_{23y} \qquad (19)$$

$$\frac{di_{34y}}{dt} = \frac{v_{3y} - v_{4y}}{L_{34}} - \frac{R_{34}}{L_{34}}i_{34y} \qquad (20)$$

$$\frac{di_{2y}}{dt} = \frac{v_{2y}}{L_{T2}+L_{L2}} - \frac{R_{T2}+R_{L2}}{L_{T2}+L_{L2}}i_{2y} \qquad (21)$$

$$\frac{di_{3y}}{dt} = \frac{v_{3y}}{L_{T3}+L_{L3}} - \frac{R_{T3}+R_{L3}}{L_{T3}+L_{L3}}i_{3y} \qquad (22)$$

$$\frac{di_{4y}}{dt} = \frac{v_{4y}}{L_{T4}+L_{L4}} - \frac{R_{T4}+R_{L4}}{L_{T4}+L_{L4}}i_{4y} \qquad (23)$$

The calculated nodal voltages, currents of the transmission lines and loads are transformed then from Cartesian coordinate system into the three-phase coordinate system. For example, the three-phase voltage at bus 2 can be found using the transformation matrix as follows:

$$\begin{bmatrix} v_{2a} \\ v_{2b} \\ v_{2c} \end{bmatrix} = \begin{bmatrix} \cos(0) & \sin(0) \\ \cos(\dfrac{2\pi}{3}) & \sin(\dfrac{2\pi}{3}) \\ \cos(-\dfrac{2\pi}{3}) & \sin(-\dfrac{2\pi}{3}) \end{bmatrix} \begin{bmatrix} v_{2x} \\ v_{2y} \end{bmatrix} \qquad (24)$$

thus,

$$v_{2a} = v_{2x} \qquad (25)$$

$$v_{2b} = -\frac{1}{2}(v_{2x} - \sqrt{3}\,v_{2y}) \qquad (26)$$

$$v_{2c} = -\frac{1}{2}(v_{2x} + \sqrt{3}\,v_{2y}) \qquad (27)$$

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where

v_{2a}, v_{2b}, v_{2c} - the phase voltages at bus 2.

Fault can be simulated by a fault switch at the fault location [7], [8]. This switch allows taking into account the effect of the stored energy in the load on the fault current and increases the accuracy of the simulation results.

III. INPUT DATA PREPARATION

Assume that the three-phase transformers of Fig.1 are identical and rated 25 MVA, 138/20 kV with leakage reactance of 11% and resistance of 2%. The loads are assumed to be constant-impedance load. In this article, it is modeled by constant impedance that is calculated as follows [8]:

$$Z_{load} = \frac{V_L^2}{S_{load}} \angle \cos^{-1} \varphi \qquad (28)$$

where

S_{load}, $\cos\varphi$ - the given apparent power and the power factor of the given load, respectively.

Table I shows the parameters of the loads connected to the buses of the study system. These parameters are indicated in both actual values and per unit (pu).

Assuming a base power of 25 MVA, the base values for the distribution part of the power system are as follows:

$$V_{base} = \frac{\sqrt{2} V_L}{\sqrt{3}} = 16.329 \text{ kV} \qquad (29)$$

$$I_{base} = \sqrt{2} \frac{S_{base}}{\sqrt{3} V_L} = 1020.62 \text{ A} \qquad (30)$$

$$Z_{base} = \frac{V_{base}}{I_{base}} = 16 \ \Omega \qquad (31)$$

TABLE I
PARAMETERS OF THE LOADS

Parameter	Bus to which load is connected		
	2	3	4
Power, MVA	8	12	10
Power factor	0.8	0.85	0.9
Line current, A	230.94	346.41	288.675
Resistance, Ω	40	28.333	36
Inductive reactance, Ω	30	17.559	17.436
Resistance, pu	2.5	1.771	2.25
Inductive reactance, pu	1.875	1.097	1.089

Table II shows the transmission line parameters, where the conductors of the transmission line are assumed to be Rock with resistance of 0.1603 Ω/mi and inductive reactance of 0.8277 Ω/mi. The base values for transmission part of the power system are:

$$V_{base} = \frac{\sqrt{2} V_L}{\sqrt{3}} = 107.778 \text{ kV} \qquad (32)$$

$$I_{base} = \sqrt{2} \frac{S_{base}}{\sqrt{3} V_L} = 154.64 \text{ A} \qquad (33)$$

$$Z_{base} = \frac{V_{base}}{I_{base}} = 696.961 \ \Omega \qquad (34)$$

TABLE II
PARAMETERS OF THE TRANSMISSION LINES

Parameter	Transmission line		
	1-2	2-3	3-4
Length, mi	45	48	50
Resistance, Ω	7.214	7.694	8.016
Inductive reactance, Ω	37.247	39.730	41.386
Resistance, pu	0.010	0.011	0.012
Inductive reactance, pu	0.053	0.057	0.060

To overcome the voltage drop on the transmission lines, the power supply voltage, in this example, should be more than 1.0 pu.

IV. CONCLUSION

The procedure of forming the differential-algebraic model of the radial electric power system is developed. This procedure is based on nodal voltages technique and on differentiation of Kirchhoff's current law applied to each non-reference node. It is, thus, composed of two systems of equations: differential for currents flowing through the power system components and algebraic for nodal voltages. It can be used for simulation of both transients and steady-state conditions.

The use of the rectangular Cartesian coordinate system decreases the overall number of the equations by one third. This restricts the use of the developed model for analysis of symmetrical modes of operation only. It can be used for protection and stability studies.

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