World Academy of Science, Engineering and Technology
International Journal of Mathematical and Computational Sciences
Vol:5, No:8, 2011

# Hutchinson-Barnsley Operator in Fuzzy Metric Spaces

R. Uthayakumar[a] and D. Easwaramoorthy[b,*]



*Abstract*—The purpose of this paper is to present the fuzzy contraction properties of the Hutchinson-Barnsley operator on the fuzzy hyperspace with respect to the Hausdorff fuzzy metrics. Also we discuss about the relationships between the Hausdorff fuzzy metrics on the fuzzy hyperspaces. Our theorems generalize and extend some recent results related with Hutchinson-Barnsley operator in the metric spaces.

*Keywords*—Fractals, Iterated Function System, Hutchinson-Barnsley Operator, Fuzzy Metric Space, Hausdorff Fuzzy Metric.

## I. Introduction

Fuzzy set theory was introduced by Zadeh in 1965 [1]. Many authors have introduced and discussed several notions of fuzzy metric space in different ways [2], [3], [4] and also proved fixed point theorems with interesting consequent results in the fuzzy metric spaces [5].

The Fractal Analysis was introduced by Mandelbrot in 1975 [6] and popularized by various mathematicians [7], [8], [9], [10], [11]. Sets with non-integral Hausdorff dimension, which exceeds its topological dimension, are called Fractals by Mandelbrot [6]. Hutchinson [7] and Barnsley [8] initiated and developed the Hutchinson-Barnsley theory (HB theory) in order to define and construct the fractal as a compact invariant subset of a complete metric space generated by the Iterated Function System (IFS) of contractions. That is, Hutchinson introduced an operator on hyperspace of nonempty compact sets called as Hutchinson-Barnsley operator (HB operator) to define a fractal set as a unique fixed point by using the Banach Contraction Theorem in the metric spaces. Here we introduce the concepts and properties of HB operator in the fuzzy metric spaces.

In this paper, we present the fuzzy contraction properties of the HB operator on the fuzzy hyperspace of non-empty compact sets with respect to the Hausdorff fuzzy metrics. Also we discuss about the relationships between the Hausdorff fuzzy metrics on the fuzzy hyperspaces. Here our theorems generalize and extend some recent results related with Hutchinson-Barnsley operator in the metric spaces.

Department of Mathematics,
The Gandhigram Rural Institute – Deemed University,
Gandhigram – 624 302, Dindigul, Tamil Nadu, India.
E-mail: [a]uthayagri@gmail.com and [b]easandk@gmail.com
Tel.: +91 451 2452371; Fax: +91 451 2453071.
*Corresponding Author.

## II. Preliminaries

### A. *HB Operator in Metric Space*

In this section, we recall the Hutchinson-Barnsley theory (HB theory) to define HB operator in the metric space.

**Definition II.1** ([8], [9]). *Let $(X, d)$ be a metric space and $\mathscr{K}_o(X)$ be the collection of all non-empty compact subsets of $X$.*

*Define, $d(x, B) := \inf_{y\in B} d(x, y)$ and $d(A, B) := \sup_{x\in A} d(x, B)$ for all $x \in X$ and $A, B \in \mathscr{K}_o(X)$. The Hausdorff metric or Hausdorff distance $(H_d)$ is a function $H_d : \mathscr{K}_o(X) \times \mathscr{K}_o(X) \longrightarrow \mathbb{R}$ defined by*

$$H_d\Big(A, B\Big) = \max\Big\{d(A, B), d(B, A)\Big\}.$$

*Then $H_d$ is a metric on the hyperspace of compact sets $\mathscr{K}_o(X)$ and hence $(\mathscr{K}_o(X), H_d)$ is called a Hausdorff metric space.*

**Definition II.2** ([8], [9]). *Let $(X, d)$ be a metric space. We note that, $\Big(\mathscr{K}_o(\mathscr{K}_o(X)), \mathscr{H}_{H_d}\Big)$ is also a metric space, where $\mathscr{K}_o(\mathscr{K}_o(X))$ is the hyperspace of all non-empty compact subsets of $(\mathscr{K}_o(X), H_d)$ and $\mathscr{H}_{H_d}$ is the Hausdorff metric on $\mathscr{K}_o(\mathscr{K}_o(X))$ implied by the Hausdorff metric $H_d$ on $\mathscr{K}_o(X)$.*

*That is, for all $\mathscr{A}, \mathscr{B} \in \mathscr{K}_o(\mathscr{K}_o(X))$,*

$$\mathscr{H}_{H_d}\Big(\mathscr{A}, \mathscr{B}\Big) = \max\Big\{H_d(\mathscr{A}, \mathscr{B}), H_d(\mathscr{B}, \mathscr{A})\Big\},$$

*where $H_d(\mathscr{A}, \mathscr{B}) := \sup_{A\in\mathscr{A}} H_d(A, \mathscr{B})$ and $H_d(A, \mathscr{B}) := \inf_{B\in\mathscr{B}} H_d(A, B)$ for all $A \in \mathscr{K}_o(X)$ and $\mathscr{A}, \mathscr{B} \in \mathscr{K}_o(\mathscr{K}_o(X))$.*

**Definition II.3** ([7], [8]). *Let $(X, d)$ be a metric space and $f_n : X \longrightarrow X$, $n = 1, 2, 3, ..., N_o$ $(N_o \in \mathbb{N})$ be $N_o$ - contraction mappings with the corresponding contractivity ratios $k_n$, $n = 1, 2, 3, ..., N_o$. The system $\{X; f_n,\ n = 1, 2, 3, ..., N_o\}$ is called an Iterated Function System (IFS) or Hyperbolic Iterated Function System with the ratio $k = \max_{n=1}^{N_o} k_n$.*

*Then the Hutchinson-Barnsley operator (HB operator) of the IFS is a function $F : \mathscr{K}_o(X) \longrightarrow \mathscr{K}_o(X)$ defined by*

$$F(B) = \bigcup_{n=1}^{N_o} f_n(B), \quad \text{for} \quad \text{all} \quad B \in \mathscr{K}_o(X).$$

**Theorem II.1** ([7], [8]). *Let $(X, d)$ be a metric space. Let $\{X; f_n,\ n = 1, 2, 3, ..., N_o; N_o \in \mathbb{N}\}$ be an IFS. Then, the $HB$ operator $(F)$ is a contraction mapping on $(\mathscr{K}_o(X), H_d)$.*

**Theorem II.2** (HB Theorem [7], [8]). *Let $(X, d)$ be a complete metric space and $\{X; f_n,\ n = 1, 2, 3, ..., N_o; N_o \in \mathbb{N}\}$ be an IFS. Then, there exists only one compact invariant set $A_\infty \in \mathscr{K}_o(X)$ of the $HB$ operator $(F)$ or, equivalently, $F$ has a unique fixed point namely $A_\infty \in \mathscr{K}_o(X)$.*

**Definition II.4** ([8]). *The fixed point $A_\infty \in \mathscr{K}_o(X)$ of the $HB$ operator $F$ described in the Theorem II.2 is called the Attractor (Fractal) of the IFS. Sometimes $A_\infty \in \mathscr{K}_o(X)$ is called as Fractal generated by the IFS and so called as IFS Fractal.*

### B. *Fuzzy Metric Space*

In [1], Zadeh defined a fuzzy set on $X$ as a function $f : X \longrightarrow [0, 1]$. Here we state the required concepts of fuzzy metric spaces as follows:

**Definition II.5** ([12]). *A binary operation $* : [0, 1] \times [0, 1] \longrightarrow [0, 1]$ is a continuous t-norm, if $([0, 1], *)$ is a topological monoid with unit 1 such that $a * b \le c * d$ whenever $a \le c$, $b \le d$ and $a, b, c, d \in [0, 1]$.*

**Definition II.6** (Kramosil and Michalek [2]). *The 3-tuple $(X, M, *)$ is said to be a fuzzy metric space if $X$ is an arbitrary set, $*$ is a continuous t-norm and $M$ is a fuzzy set on $X \times X \times [0, \infty)$ satisfying the following conditions:*

1) $M(x, y, 0) = 0$,
2) $M(x, y, t) = 1$ *for all* $t > 0$ *if and only if* $x = y$,
3) $M(x, y, t) = M(y, x, t)$,
4) $M(x, y, t) * M(y, z, s) \le M(x, z, t + s)$,
5) $M(x, y, \cdot) : [0, \infty) \longrightarrow [0, 1]$ *is left continuous,*

$x, y, z \in X$ *and* $t, s > 0$.

In order to introduce a Hausdorff topology on the fuzzy metric space, George and Veeramani [3] modified the above definition and gave the following.

**Definition II.7** (George and Veeramani [3]). *The 3-tuple $(X, M, *)$ is said to be a fuzzy metric space if $X$ is an arbitrary set, $*$ is a continuous t-norm and $M$ is a fuzzy set on $X \times X \times (0, \infty)$ satisfying the following conditions:*

1) $M(x, y, t) > 0$,
2) $M(x, y, t) = 1$ *if and only if* $x = y$,
3) $M(x, y, t) = M(y, x, t)$,
4) $M(x, y, t) * M(y, z, s) \le M(x, z, t + s)$,
5) $M(x, y, \cdot) : (0, \infty) \longrightarrow [0, 1]$ *is continuous,*

$x, y, z \in X$ *and* $t, s > 0$.

The function $M(x, y, t)$ represents the degree of nearness between $x$ and $y$ with respect to $t$. We identify $x = y$ with $M(x, y, t) = 1$, for $t > 0$ and $M(x, y, t) = 0$ as $t \longrightarrow \infty$. From now onwards, a fuzzy metric space refers a fuzzy metric space of George and Veeramani [3], unless mentioned specifically.

**Definition II.8** ([3]). *Let $(X, d)$ be a metric space. Define $a * b = a \cdot b$, the usual multiplication for all $a, b \in [0, 1]$, and let $M_d$ be the function defined on $X \times X \times (0, \infty)$ by*

$$M_d(x, y, t) = \frac{t}{t + d(x, y)},$$

*for all $x, y \in X$ and $t > 0$. Then $(X, M_d, *)$ is a fuzzy metric space called standard fuzzy metric space, and $M_d$ is called as the standard fuzzy metric induced by the metric $d$.*

**Definition II.9** ([3]). *Let $(X, M, *)$ be a fuzzy metric space. The open ball $B(x, r, t)$ for $t > 0$ with centre $x \in X$ and radius $r, 0 < r < 1$, is defined as*

$$B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}.$$

*Define*

$$\tau_M = \Big\{A \subset X : x \in A \iff \text{ there exists } t > 0 \text{ and } r,\ 0 < r < 1, \text{ such that } B(x, r, t) \subset A\Big\}.$$

*Then $\tau_M$ is a topology on $X$ induced by a fuzzy metric $M$.*

The topologies induced by the metric and the corresponding standard fuzzy metric are the same. That is, if $(X, d)$ is a metric space, then the topology $\tau_d$ induced by the metric $d$ coincides with the topology $\tau_{M_d}$ induced by the standard fuzzy metric $M_d$.

**Definition II.10** ([13], [14]). *A fuzzy B-contraction (Sehgal contraction) on a fuzzy metric space $(X, M, *)$ is a self-mapping $f$ on $X$ for which*

$$M(f(x), f(y), kt) \ge M(x, y, t),$$

*for all $x, y \in X$ and $t > 0$, where $k$ is a fixed constant in $(0, 1)$.*

### C. *Hausdorff Fuzzy Metric Space*

In [15], Rodriguez-Lopez and Romaguera defined the Hausdorff metric on fuzzy hyperspace $\mathscr{K}_o(X)$ and constructed the Hausdorff fuzzy metric space. Besides that the necessary results of the Hausdorff fuzzy metric on fuzzy hyperspaces are proved in [15].

**Definition II.11** ([15]). *Let $(X, M, *)$ be a fuzzy metric space and $\tau_M$ be the topology induced by the fuzzy metric $M$. We shall denote by $\mathscr{K}_o(X)$, the set of all non-empty compact subsets of $(X, \tau_M)$.*

*Define, $M(x, B, t) := \sup_{y \in B} M(x, y, t)$ and $M(A, B, t) := \inf_{x \in A} M(x, B, t)$ for all $x \in X$ and $A, B \in \mathscr{K}_o(X)$. The Hausdorff fuzzy metric $(H_M)$ is a function $H_M : \mathscr{K}_o(X) \times \mathscr{K}_o(X) \times (0, \infty) \longrightarrow [0, 1]$ defined by*

$$H_M(A, B, t) = \min\Big\{M(A, B, t), M(B, A, t)\Big\}.$$

*Then $H_M$ is a fuzzy metric on the fuzzy hyperspace of compact sets, $\mathscr{K}_o(X)$, and hence $(\mathscr{K}_o(X), H_M, *)$ is called a Hausdorff fuzzy metric space.*

**Proposition II.1** ([15]). *Let $(X, d)$ be a metric space. Then, the Hausdorff fuzzy metric $(H_{M_d})$ of the standard fuzzy metric $(M_d)$ coincides with the standard fuzzy metric $(M_{H_d})$ of the Hausdorff metric $(H_d)$ on $\mathscr{K}_o(X)$, i.e., $H_{M_d}(A, B, t) = M_{H_d}(A, B, t)$ for all $A, B \in \mathscr{K}_o(X)$ and $t > 0$.*

**Definition II.12.** *Let $(X, M, *)$ be a fuzzy metric space. We observe that, $\Big(\mathscr{K}_o(\mathscr{K}_o(X)), \mathscr{H}_{H_M}, *\Big)$ is also a fuzzy metric space, where $\mathscr{K}_o(\mathscr{K}_o(X))$ is the fuzzy hyperspace of all non-empty compact subsets of $(\mathscr{K}_o(X), H_M, *)$ and $\mathscr{H}_{H_M}$ is the Hausdorff fuzzy metric on $\mathscr{K}_o(\mathscr{K}_o(X))$ implied by the Hausdorff fuzzy metric $H_M$ on $\mathscr{K}_o(X)$. That is, for all $\mathscr{A}, \mathscr{B} \in \mathscr{K}_o(\mathscr{K}_o(X))$,*

$$\mathscr{H}_{H_M}\Big(\mathscr{A}, \mathscr{B}\Big) = \min\Big\{H_M(\mathscr{A}, \mathscr{B}), H_M(\mathscr{B}, \mathscr{A})\Big\},$$

*where $H_M(\mathscr{A}, \mathscr{B}) := \inf_{A\in\mathscr{A}} H_M(A, \mathscr{B})$ and $H_M(A, \mathscr{B}) := \sup_{B\in\mathscr{B}} H_M(A, B)$ for all $A \in \mathscr{K}_o(X)$ and $\mathscr{A}, \mathscr{B} \in \mathscr{K}_o(\mathscr{K}_o(X))$.*

## III. Fuzzy HB Operator

In this section, we define the Fuzzy IFS and Fuzzy HB operator in the fuzzy metric spaces.

**Definition III.1.** *Let $(X, M, *)$ be a fuzzy metric space and $f_n : X \longrightarrow X$, $n = 1, 2, 3, ..., N_o$ ($N_o \in \mathbb{N}$) be $N_o$ - fuzzy B-contraction mappings. Then the system $\{X; f_n,\ n = 1, 2, 3, ..., N_o\}$ is called a Fuzzy Iterated Function System (FIFS) of fuzzy B-contractions on the fuzzy metric space $(X, M, *)$.*

**Definition III.2.** *Let $(X, M, *)$ be a fuzzy metric space. Let $\{X; f_n,\ n = 1, 2, 3, ..., N_o; N_o \in \mathbb{N}\}$ be a FIFS of fuzzy B-contractions. Then the Fuzzy Hutchinson-Barnsley operator ($FHB$ operator) of the FIFS is a function $F : \mathscr{K}_o(X) \longrightarrow \mathscr{K}_o(X)$ defined by*

$$F(B) = \bigcup_{n=1}^{N_o} f_n(B), \quad \text{for} \quad \text{all} \quad B \in \mathscr{K}_o(X).$$

**Definition III.3.** *Let $(X, M, *)$ be a complete fuzzy metric space. Let $\{X; f_n,\ n = 1, 2, 3, ..., N_o; N_o \in \mathbb{N}\}$ be a FIFS of fuzzy B-contractions and $F$ be the $FHB$ operator of the FIFS. We say that the set $A_\infty \in \mathscr{K}_o(X)$ is Fuzzy Attractor (Fuzzy Fractal) of the given FIFS, if $A_\infty$ is a unique fixed point of the $FHB$ operator $F$. Usually such $A_\infty \in \mathscr{K}_o(X)$ is also called as Fractal generated by the FIFS of fuzzy B-contractions and so called as FIFS Fractal of fuzzy B-contractions.*

## IV. Fuzzy Contractivity of HB Operator

Now we prove the interesting results about the contraction properties of operators with respect to the Hausdorff Fuzzy Metric on $\mathscr{K}_o(X)$.

**Theorem IV.1.** *Let $(X, d)$ be a metric space. Let $f : X \longrightarrow X$ be a contraction function, with a contractivity ratio $k$. Then,*

$$H_{M_d}\Big(f(A), f(B), t\Big) \geq H_{M_d}\Big(A, B, t\Big)$$

*for all $A, B \in \mathscr{K}_o(X)$ and $t > 0$.*

*Proof:*

Fix $t > 0$ and let $A, B \in \mathscr{K}_o(X)$. Since $f$ is contraction on $(X, d)$ with the contractivity ratio $k \in (0, 1)$ and by Theorem II.1 for the case $N = 1$, we have

$$H_d(f(A), f(B)) \leq kH_d(A, B).$$

Since $t > 0$ and $k \in (0, 1)$,

$$\frac{kt}{kt + H_d(f(A), f(B))} \geq \frac{kt}{kt + kH_d(A, B)} = \frac{t}{t + H_d(A, B)}$$

By using the Proposition II.1, we have

$$\begin{aligned} H_{M_d}(f(A), f(B), kt) &= M_{H_d}(f(A), f(B), kt) \\ &= \frac{kt}{kt + H_d(f(A), f(B))} \\ &\geq \frac{t}{t + H_d(A, B)} \\ &= M_{H_d}(A, B, t) \\ &= H_{M_d}(A, B, t). \end{aligned}$$

■

The above Theorem IV.1 shows that $f$ is a fuzzy B-contraction function on $\mathscr{K}_o(X)$ with respect to the Hausdorff fuzzy metric $H_{M_d}$ implied by the standard fuzzy metric $M_d$, if $f$ is contraction on a metric space $(X, d)$. The following theorem is somewhat generalization of the Theorem IV.1.

**Theorem IV.2.** *Let $(X, M, *)$ be a fuzzy metric space. Let $(\mathscr{K}_o(X), H_M, *)$ be the corresponding Hausdorff fuzzy metric space. Suppose $f : X \longrightarrow X$ is a fuzzy B-Contraction function on $(X, M, *)$. Then for $k \in (0, 1)$,*

$$H_M\Big(f(A), f(B), kt\Big) \geq H_M\Big(A, B, t\Big)$$

*for all $A, B \in \mathscr{K}_o(X)$ and $t > 0$.*

*Proof:*

Fix $t > 0$. Let $A, B \in \mathscr{K}_o(X)$.
For given $k \in (0, 1)$, we get

$$\begin{aligned} M(f(x), f(y), kt) &\geq M(x, y, t), \quad \forall x, y \in X \\ M(f(x), f(y), kt) &\geq M(x, y, t), \quad \forall x \in A \;\&\; y \in B \\ \sup_{y\in B} M(f(x), f(y), kt) &\geq \sup_{y\in B} M(x, y, t), \quad \forall x \in A \\ M(f(x), f(B), kt) &\geq M(x, B, t), \quad \forall x \in A \\ \inf_{x\in A} M(f(x), f(B), kt) &\geq \inf_{x\in A} M(x, B, t) \\ M(f(A), f(B), kt) &\geq M(A, B, t) \end{aligned}$$

Similarly, $M(f(B), f(A), kt) \geq M(B, A, t)$.

Hence
$\min\Big\{M\Big(f(A), f(B), kt\Big), M\Big(f(B), f(A), kt\Big)\Big\}$
$\geq \min\Big\{M(A, B, t), M(B, A, t)\Big\}$.
*i.e.*, $H_M\Big(f(A), f(B), kt\Big) \geq H_M\Big(A, B, t\Big)$.
This completes the proof. ■

The above Theorem IV.2 shows that $f$ is a fuzzy B-contraction function on $\mathscr{K}_o(X)$ with respect to the Hausdorff fuzzy metric $H_M$, if $f$ is fuzzy B-contraction on a fuzzy metric space $(X, M, *)$.

**Lemma IV.1.** *Let $(X, M, *)$ be a fuzzy metric space. If $B, C \subset X$ such that $B \subset C$, then $M(x, B, t) \leq M(x, C, t)$ for all $x \in X$ and $t > 0$.*

*Proof:*
Fix $t > 0$. Let $x \in X$ and $B, C \subset X$ such that $B \subset C$.
Then, $M(x, B, t) = \sup_{b \in B} M(x, b, t)$
$\leq \sup_{b \in C} M(x, b, t) = M(x, C, t)$. ∎

**Lemma IV.2.** *Let $(X, M, *)$ be a fuzzy metric space. If $B, C \subset X$ such that $B \subset C$, then $M(A, B, t) \leq M(A, C, t)$ for all $A \subset X$ and $t > 0$.*

*Proof:*
Fix $t > 0$. Let $A, B, C \subset X$ such that $B \subset C$.
By the Lemma IV.1, we have

$$\begin{aligned} M(A, B, t) &= \inf_{a \in A} M(a, B, t) \\ M(A, B, t) &\leq M(a, B, t), \forall a \in A \\ M(A, B, t) &\leq M(a, C, t), \forall a \in A \\ M(A, B, t) &\leq \inf_{a \in A} M(a, C, t) \\ M(A, B, t) &\leq M(A, C, t). \end{aligned}$$

∎

**Lemma IV.3.** *Let $(X, M, *)$ be a fuzzy metric space. If $A, B, C \subset X$, then $M\Big(A \cup B, C, t\Big) = \min\Big\{M(A, C, t), M(B, C, t)\Big\}$ for all $t > 0$.*

*Proof:*
Fix $t > 0$. Let $A, B, C \subset X$.
Then,

$$\begin{aligned} M\Big(A \cup B, C, t\Big) &= \inf_{x \in A \cup B} M(x, C, t) \\ &= \min\Big\{\inf_{x \in A} M(x, C, t), \inf_{x \in B} M(x, C, t)\Big\} \\ &= \min\Big\{M(A, C, t), M(B, C, t)\Big\}. \end{aligned}$$

∎

**Lemma IV.4.** *Let $(X, M, *)$ be a fuzzy metric space. Let $(\mathscr{K}_o(X), H_M, *)$ be the corresponding Hausdorff fuzzy metric space. If $A, B, C, D \in \mathscr{K}_o(X)$, then*

$$H_M\Big(A \cup B, C \cup D, t\Big) \geq \min\Big\{H_M(A, C, t), H_M(B, D, t)\Big\},$$

$\forall t > 0$.

*Proof:*
Fix $t > 0$. Let $A, B, C, D \in \mathscr{K}_o(X)$.
By using the Lemmas IV.2 and IV.3, we get

$$\begin{aligned} &M\Big(A \cup B, C \cup D, t\Big) \\ &= \min\Big\{M(A, C \cup D, t), M(B, C \cup D, t)\Big\} \\ &\geq \min\Big\{M(A, C, t), M(B, D, t)\Big\} \\ &\geq \min\Big\{H_M(A, C, t), H_M(B, D, t)\Big\}. \end{aligned}$$

Similarly, $M\Big(C \cup D, A \cup B, t\Big)$
$\geq \min\Big\{H_M(A, C, t), H_M(B, D, t)\Big\}$.
Hence
$\min\Big\{M(A \cup B, C \cup D, t), M(C \cup D, A \cup B, t)\Big\}$
$\geq \min\Big\{H_M(A, C, t), H_M(B, D, t)\Big\}$.
This completes the proof. ∎

The following theorem is a generalized version of the Theorem IV.2.

**Theorem IV.3.** *Let $(X, M, *)$ be a fuzzy metric space. Let $(\mathscr{K}_o(X), H_M, *)$ be the corresponding Hausdorff fuzzy metric space. Suppose $f_n : X \longrightarrow X, n = 1, 2, ..., N_o; N_o \in \mathbb{N}$, is a fuzzy B-Contraction function on $(X, M, *)$. Then the HB operator is also fuzzy B-Contraction on $(\mathscr{K}_o(X), H_M, *)$.*

*Proof:*
Fix $t > 0$. Let $A, B \in \mathscr{K}_o(X)$.
By using the Lemma IV.4 and the Theorem IV.2 for a given $k \in (0, 1)$, we get

$$\begin{aligned} H_M\Big(F(A), F(B), kt\Big) &= H_M\left(\bigcup_{n=1}^{N_o} f_n(A), \bigcup_{n=1}^{N_o} f_n(B), kt\right) \\ &\geq \min_{n=1}^{N} H_M\Big(f_n(A), f_n(B), kt\Big) \\ &\geq H_M(A, B, t). \end{aligned}$$

This completes the proof. ∎

From the above Theorem IV.3, we conclude that the HB operator $F$ is a fuzzy B-contraction function on $\mathscr{K}_o(X)$ with respect to the Hausdorff fuzzy metric $H_M$, if $f_n$ is fuzzy B-contraction on a fuzzy metric space $(X, M, *)$ for each $n \in \{1, 2, 3, ..., N_o\}$.

## V. Hausdorff Fuzzy Metrics on $\mathscr{K}_o(X)$ and $\mathscr{K}_o(\mathscr{K}_o(X))$

Now, we discuss about the relationships between the fuzzy hyperspaces $\mathscr{K}_o(X)$ and $\mathscr{K}_o(\mathscr{K}_o(X))$ and the Hausdorff fuzzy metrics $H_M$ and $\mathscr{H}_{H_M}$.

**Theorem V.1.** *Let $(X, M, *)$ be a fuzzy metric space. Let $\mathscr{A}, \mathscr{B} \in \mathscr{K}_o(\mathscr{K}_o(X))$ be such that*

$$\{a \in A : A \in \mathscr{A}\}, \{b \in B : B \in \mathscr{B}\} \in \mathscr{K}_o(X).$$

*Then*

$$H_M\Big(\{a \in A : A \in \mathscr{A}\}, \{b \in B : B \in \mathscr{B}\}, t\Big) \geq \mathscr{H}_{H_M}(\mathscr{A}, \mathscr{B}, t)$$

*for all $t > 0$.*

*Proof:*
Fix $t > 0$. Firstly, we note that

$$\begin{aligned}
&M\Big(B, \{a \in A : A \in \mathscr{A}\}, t\Big) \\
&\quad = \inf_{b \in B} M\Big(b, \{a \in A : A \in \mathscr{A}\}, t\Big) \\
&\quad = \inf_{b \in B} \sup_{\{a \in A : A \in \mathscr{A}\}} M(b, a, t) \\
&\quad = \inf_{b \in B} \sup_{A \in \mathscr{A}} \sup_{a \in A} M(b, a, t) \\
&\quad \geq \sup_{A \in \mathscr{A}} \inf_{b \in B} \sup_{a \in A} M(b, a, t) \\
&\quad = \sup_{A \in \mathscr{A}} M(B, A, t).
\end{aligned}$$

It follows that

$$\begin{aligned}
&M\Big(\{b \in B : B \in \mathscr{B}\}, \{a \in A : A \in \mathscr{A}\}, t\Big) \\
&\quad = \inf_{\{b \in B : B \in \mathscr{B}\}} M\Big(b, \{a \in A : A \in \mathscr{A}\}, t\Big) \\
&\quad = \inf_{B \in \mathscr{B}} \inf_{b \in B} M\Big(b, \{a \in A : A \in \mathscr{A}\}, t\Big) \\
&\quad = \inf_{B \in \mathscr{B}} M\Big(B, \{a \in A : A \in \mathscr{A}\}, t\Big) \\
&\quad \geq \inf_{B \in \mathscr{B}} \sup_{A \in \mathscr{A}} M(B, A, t).
\end{aligned}$$

Similarly, $M\Big(\{a \in A : A \in \mathscr{A}\}, \{b \in B : B \in \mathscr{B}\}, t\Big)$ $\geq \inf_{A \in \mathscr{A}} \sup_{B \in \mathscr{B}} M(A, B, t)$.

Hence

$$\begin{aligned}
&H_M\Big(\{a \in A : A \in \mathscr{A}\}, \{b \in B : B \in \mathscr{B}\}, t\Big) \\
&\quad = \min\Big\{M\Big(\{a \in A : A \in \mathscr{A}\}, \{b \in B : B \in \mathscr{B}\}, t\Big), \\
&\quad \geq \min\Big\{\inf_{A \in \mathscr{A}} \sup_{B \in \mathscr{B}} M(A, B, t), \inf_{B \in \mathscr{B}} \sup_{A \in \mathscr{A}} M(B, A, t)\Big\}.
\end{aligned}$$

But

$$\begin{aligned}
&\mathscr{H}_{H_M}(\mathscr{A}, \mathscr{B}, t) \\
&\quad = \min\Big\{H_M(\mathscr{A}, \mathscr{B}, t), H_M(\mathscr{B}, \mathscr{A}, t)\Big\} \\
&\quad = \min\Big\{\inf_{A \in \mathscr{A}} \sup_{B \in \mathscr{B}} H_M(A, B, t), \inf_{B \in \mathscr{B}} \sup_{A \in \mathscr{A}} H_M(B, A, t)\Big\} \\
&\quad = \min\Big\{\inf_{A \in \mathscr{A}} \sup_{B \in \mathscr{B}} \min\Big\{M(A, B, t), M(B, A, t)\Big\}, \\
&\qquad\quad \inf_{B \in \mathscr{B}} \sup_{A \in \mathscr{A}} \min\Big\{M(B, A, t), M(A, B, t)\Big\}\Big\} \\
&\quad \leq \min\Big\{\inf_{A \in \mathscr{A}} \sup_{B \in \mathscr{B}} M(A, B, t), \inf_{B \in \mathscr{B}} \sup_{A \in \mathscr{A}} M(B, A, t)\Big\}.
\end{aligned}$$

The above two inequalities concludes the proof. ∎

The above Theorem V.1 declares that $H_M$ is a 'stronger' Hausdorff fuzzy metric than $\mathscr{H}_{H_M}$.

## VI. Conclusion

This paper presented the fuzzy contraction properties of the Hutchinson-Barnsley operator on the fuzzy hyperspace of with respect to the Hausdorff fuzzy metrics. Further, we discussed about the relationships between the Hausdorff fuzzy metrics $H_M$ and $\mathscr{H}_{H_M}$ on the fuzzy hyperspaces $\mathscr{K}_o(X)$ and $\mathscr{K}_o(\mathscr{K}_o(X))$ respectively. This paper will leads our direction to improve the Hutchinson-Barnsley Theory in the sense of fuzzy B-contractions in order to define a fractal set in the fuzzy metric spaces as a unique fixed point of the Fuzzy HB operator.

## Acknowledgment

The research work has been supported by University Grants Commission (*UGC - MRP & SAP*), New Delhi, India.

## References

[1] L.A. Zadeh, Fuzzy Sets, *Information and Control*, 8 (1965) 338–353.
[2] I. Kramosil and J. Michalek, Fuzzy metrics and statistical metric Spaces, *Kybernetika*, 11(5) (1975) 336–344.
[3] A. George and P. Veeramani, On some results in fuzzy metric spaces, *Fuzzy Sets and Systems*, 64 (1994) 395–399.
[4] A. George and P. Veeramani, On some results of analysis for fuzzy metric spaces, *Fuzzy Sets and Systems*, 90 (1997) 365–368.
[5] V. Gregori and A. Sapena, On fixed-point theorems in fuzzy metric spaces, *Fuzzy Sets and Systems*, 125 (2002) 245–252.
[6] B.B. Mandelbrot, *The Fractal Geometry of Nature*, W.H. Freeman and Company, New York, 1983.
[7] J.E. Hutchinson, Fractals and self similarity, *Indiana University Mathematics Journal*, 30 (1981) 713–747.
[8] M. Barnsley, *Fractals Everywhere*, 2nd ed. Academic Press, USA, 1993.
[9] M. Barnsley, *SuperFractals*, Cambridge University Press, New York, 2006.
[10] G. Edgar, *Measure, Topology, and Fractal Geometry*, 2nd ed. Springer, New York, 2008.
[11] K. Falconer, *Fractal Geometry: Mathematical Foundations and Applications*, 2nd ed. John Wiley & Sons Ltd., England, 2003.
[12] B. Schweizer and A. Sklar, Statistical metric spaces, *Pacific Journal of Mathematics*, 10 (1960) 313–334.
[13] V.M. Sehgal and A.T. Bharucha-Reid, Fixed points of contraction mappings on probabilistic metric spaces, *Math. Systems Theory*, 6 (1972) 97–102.
[14] R. Farnoosh, A. Aghajani and P. Azhdari, Contraction theorems in fuzzy metric space, *Chaos, Solitons and Fractals*, 41 (2009) 854–858.
[15] J. Rodriguez-Lopez and S. Romaguera, The Hausdorff fuzzy metric on compact sets, *Fuzzy Sets and Systems*, 147 (2004) 273–283.

**R. Uthayakumar** was born in Dindigul, India; in 1967. He received the Ph.D. Degree in Mathematics from The Gandhigram Rural Institute - Deemed University, Gandhigram, India; in 2000. He is currently an Associate Professor in Mathematics in The Gandhigram Rural Institute - Deemed University, Gandhigram, India. His major research interests include Mathematical Modeling, Fractal Analysis, Fuzzy Spaces and Inventory Models. He is an author and coauthor of more than 60 scientific journal articles and 30 papers published in proceedings of conferences.

**D. Easwaramoorthy** was born in Rasipuram, Namakkal, Tamil Nadu, India; in 1986. He received the M.Sc. Degree in Mathematics from the Bharathidasan University, Tiruchirappalli, India; in 2008. Currently, he is a Research Scholar in the Department of Mathematics, The Gandhigram Rural Institute - Deemed University, Gandhigram, India. His research interests include Mathematical Modeling and Fractal Analysis in Fuzzy Spaces.