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Tsunami Modelling using the Well-Balanced Scheme

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***Abstract*—**A well balanced numerical scheme based on stationary waves for shallow water flows with arbitrary topography has been introduced by Thanh et al. [18]. The scheme was constructed so that it maintains equilibrium states and tests indicate that it is stable and fast. Applying the well-balanced scheme for the one-dimensional shallow water equations, we study the early shock waves propagation towards the Phuket coast in Southern Thailand during a hypothetical tsunami. The initial tsunami wave is generated in the deep ocean with the strength that of Indonesian tsunami of 2004.

***Keywords*—**Tsunami study, shallow water, conservation law, well-balanced scheme, topography.

Subject classification: 86 A 05, 86 A 17.

I. INTRODUCTION

A tsunami is a large and often destructive ocean wave(s) resulting mainly from sudden vertical movement of the ocean floor. When an undersea earthquake or other major disturbance causes a section of ocean floor to rise or sink abruptly, the mass of water above the affected area is also suddenly displaced and large waves are formed. A wave is defined as a shallow water wave when the ratio between the water depth and its wave length gets very small (< 0.05). A tsunami can have very long wavelengths and because of this, it is a shallow-water wave.

We still vividly remember the tragedy on Sunday, 26th December, 2004 when the great Sumatra earthquake occurred at 7:59 a.m. local time, about 160 km from North-Sumatra, Indonesia (Fig. 1). The Indian Ocean undersea earthquake (Mw: 9.3) which was located off the west coast of Northern Sumatra, Indonesia, triggered widespread tsunamis causing unprecedented loss of life and damage to property in the coastal areas of the Indian ocean rim countries. The west coast of Southern Thailand faces Sumatra Island where there is a permanent tsunami source zone. Phuket Island on the west coast of Southern Thailand lies between latitudes 7.5° and 8° N, and longitudes 98° and 98.4° E. It is located approximately 500 km east from the tsunami source and was severely affected by the tsunami. As the Indian Ocean has several seismic sources, recurrence of the tsunami on the scale of the 2004 event can be anticipated in the future.

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Recently, a numerical scheme based on stationary waves for shallow water flows has been developed by Thanh et al. [18]. The scheme was constructed so that it maintains equilibrium states and gives appropriate approximations of the exact solutions. Moreover the scheme was found to be stable and fast. Related studies can be found in Marchesin and Paes-Leme [17], Isaacson and Temple [11], [12], Goatin and LeFloch [7], LeFloch and Thanh [15], [16], Andrianov and Warnecke [1] , [2], etc.

It has been shown that for nonconservative hyperbolic systems, or hyperbolic systems with source terms, traditional discretisations of the right-hand side do not give satisfactory results. In particular, oscillations may appear or the errors may grow when the mesh size is reduced. This was observed in the scalar case by Botchorishvili et al. [6], and in the case of systems for the modeling of flows in a nozzle with variable cross-section by Kröner and Thanh [14]. Well-balanced scheme can overcome this problem. In the case of scalar conservation laws, well-balanced schemes have been constructed by Greenberg and Leroux [9], Greenberg et al. [10], Bouchut [4], Gosse [8], Botchorishvili et al. [5], Botchorishvili and Pironneau [6], Audusse et al. [3] and Jin and Wen [13].

In this paper, the numerical scheme that was developed by Thanh et al. [18] has been used to compute a hypothetical tsunami propagation towards Phuket Island.

II. GOVERNING EQUATIONS AND THE WELL-BALANCED SCHEME

A. 1-D Shallow Water Equations

Let us consider the one-dimensional shallow water equations:

$$\begin{aligned} &\partial_t h + \partial_x(hu)=0, \\ &\partial_t (hu) + \partial_x(h(u^2+gh/2))=-gh\partial_x a, \end{aligned} \tag{1}$$

where, h is the height of the water from the bottom to the surface, u is the velocity, g is the gravity constant, and a, is the height of the bottom from a given level.

To see how the tsunami appears and propagate a short time after the earthquake which caused the tsunami happened, it is sufficient to use the one dimensional shallow water equation. This is because the quake which caused the tsunami sends a huge momentum to the ocean water along a ray from the source to a relatively near position we are interested in. The use of a 1-D model is however not appropriate to model the propagation of a tsunami to a relatively distant position i.e. a long time after it has occurred.

Given the real data of the "bed function" $a=a(x)$, we can apply the well-balanced scheme constructed in Section II. B to study the propagation of the early shock waves to a relatively close position. We use the depths along a ray from the source to the Island of Phuket.

B. The Well-Balanced Scheme

Given a uniform time step Δt and a spatial mesh size Δx, setting $x_j=j\Delta x$, $j \in Z$, and $t_n=n\Delta t$, $n \in N$, we denote by U_j^n in what follows as the approximation of the values $U(x_j,t_n)$ of the exact solution $U = (h, hu)$ of (1).

Let us take any standard finite difference scheme for gas dynamics equations with the numerical flux g^c. The classical scheme is of the form

$$U_j^{n+1}=U_j^n-\lambda(g^C(U_j^n, U_{j+1}^n) - g^C(U_{j-1}^n, U_j^n)) + \frac{\lambda}{2}(0,-gh_j^n(a_{j+1}-a_{j-1}))^T, \tag{2}$$

where $a_j := a(x_j)$ and $\lambda=\frac{\Delta t}{\Delta x}$. The *modified Lax-Friedrichs scheme* is of the form (2) with the Lax-Friedrichs numerical flux:

$$g^c(U,V):= \frac{1}{2}(f(U)+f(V)) - \frac{1}{2\lambda}(V-U),$$
$$U:=(h,hu), \quad f(U):=(hu, h(u^2+gh/2)). \tag{3}$$

The constant λ is also required to satisfy the so-called *CFL stability condition*

$$\lambda \max_U \{|u|+\sqrt{gh}\} \le 1. \tag{4}$$

The *well-balanced scheme* is defined by

$$U_j^{n+1}=U_j^n-\lambda(g^N(U_j^n, U_{j+1,-}^n) - g^N(U_{j-1,+}^n, U_j^n)), \tag{5}$$

where $g^N(U,V)$ can be any standard numerical flux for shallow water equations without topography, and $U_{j-1,+}^n, U_{j+1,-}^n$ are given below. We take the Lax-Friedrichs numerical flux as: $g^N(U,V) = g^C(U,V)$

In the scheme (5), the states

$$U_{j+1,-}^n=\left(h,hu\right)_{j+1,-}^n, \text{ and } U_{j-1,+}^n=\left(h,hu\right)_{j-1,+}^n,$$

are defined by

$$h_{j+1}^n u_{j+1}^n = h_{j+1,-}^n u_{j+1,-}^n$$
$$\left(u_{j+1}^n\right)^{2/2} + g\left(h_{j+1}^n + a_{j+1}\right) = \left(u_{j+1,-}^n\right)^{2/2} + g\left(h_{j+1,-}^n + a_{j+1,-}\right) \tag{6}$$

and

$$h_{j-1}^n u_{j-1}^n = h_{j-1,+}^n u_{j-1,+}^n$$
$$\frac{1}{2}\left(u_{j-1}^n\right)^2 + g\left(h_{j-1}^n + a_{j-1}\right) = \frac{1}{2}\left(u_{j-1,+}^n\right)^2 + g\left(h_{j-1,+}^n + a_{j-1,+}\right) \tag{7}$$

III. Boundary and Initial Conditions

The main generating force of a tsunami triggered by an earthquake is the uplift or subsidence of the sea-floor which accompanies the earthquake. The generation mechanism of the 26 December 2004 tsunami, for example, was mainly due a static sea bed deformation caused by an abrupt slip at the India/Burma plate interface. It was estimated that the uplift and the subsidence zone is between 92º E to 97º E and 2º N to 10º N with a maximum uplift of 507 cm at the west and maximum subsidence of 474 cm at the east so that the uplift to subsidence is approximately from west to east.

For our hypothetical tsunami, we assume that ocean displacement is the same as sea surface displacement due to the incompressibility of the ocean. The initial tsunami wave is generated in the deep ocean (open sea) with the strength that of Indonesian tsunami of 2004 in the form of sea level fall and rise. Other than this localized region the initial values of sea level are taken as zero; in addition the initial values of u are also taken as zero everywhere. The coastal belt of Phuket (land boundary) is the closed boundary where the component of the velocity is taken as zero.

We use cubic splines with 50 knots to interpolate the bottom along a ray from the fault line to Phuket coast.

IV. Computations of Wave Propagations in a Tsunami

The tsunami simulations were performed by numerically solving the non-linear 1-D shallow water equations using the well balanced scheme. We set the grid size of the computation as approximately 10 km. The ocean depth data (h) for the model are collected from the Admiralty bathymetric charts.

The unit in the horizontal axis in the Figs. 2 - 6 of ocean bottom and water height is 100 km (i.e. 1 unit =100 km) and the unit in the vertical axis is 1 km, while the units in velocity are 100 km/h (vertical axis) and 100 km (horizontal axis). Figs. 2 – 6 show the distribution of water height and water velocity with ocean bottom at different instants of time. Figs. 2 – 4 show the early stages of the tsunami. We can see from the subplot of water velocity where the velocity of water attains a maximum level at some positions x (t)at the time t (about 350 km/h in Fig. 2, 400km/h in Fig. 3, 430 km/h in Fig. 4 and 470 km/h in Fig. 5). From the subplot for water velocity in Fig. 6, we can see that after approximately 50 minutes, the first shock waves of the tsunami are propagating towards the coast of Phuket with a speed approximately 500 km/h with maximum water height 2.8 m. Thus the well-balanced scheme is capable of propagating a tsunami towards Phuket Island.

V. Conclusion and Recommendation

In this paper we have applied on the well-balanced scheme to model the propagation of a hypothetical tsunami towards the Phuket coast. The one-dimensional well - balanced scheme is shown to be capable of propagating a tsunami from a source in the Indian Ocean towards Phuket.

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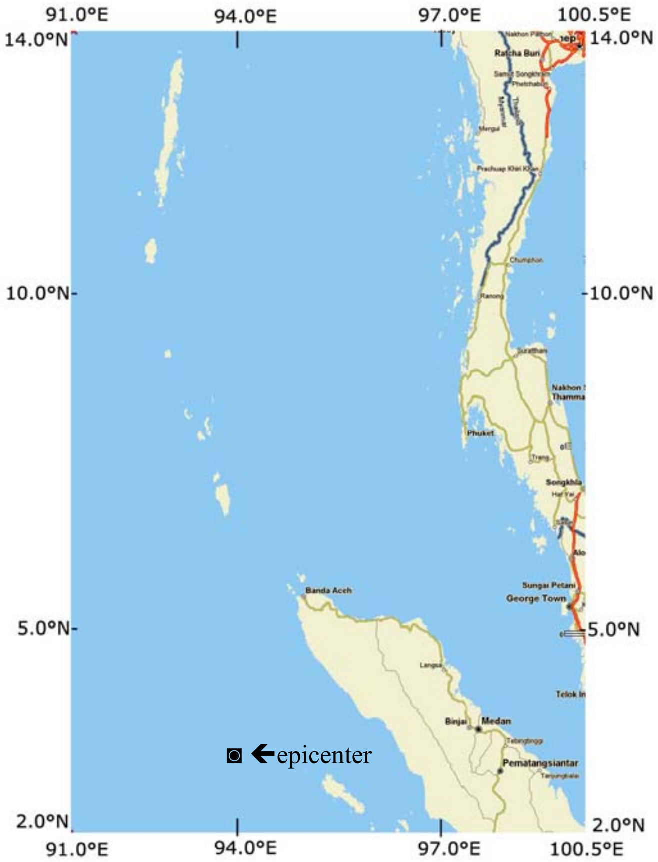

Fig. 1 Tsunami source zone (west of north Sumatra) including west coast of Thailand, Peninsular Malaysia

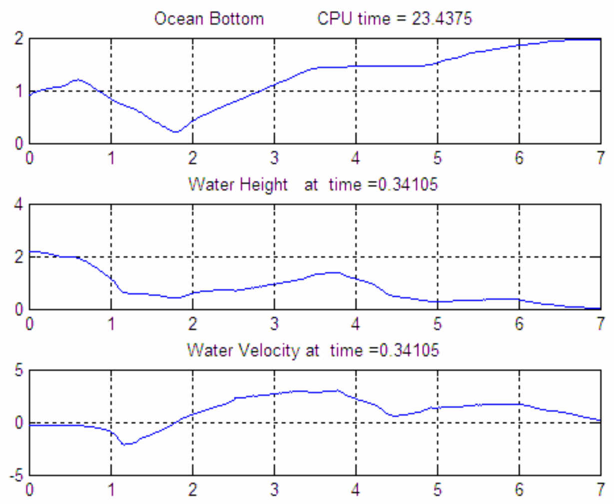

Fig. 2 Water level and velocity at 20 minutes

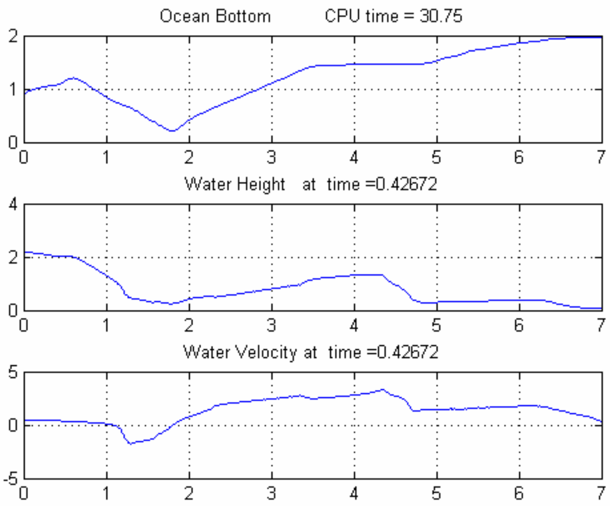

Fig. 3 Water level and velocity at 25 minutes

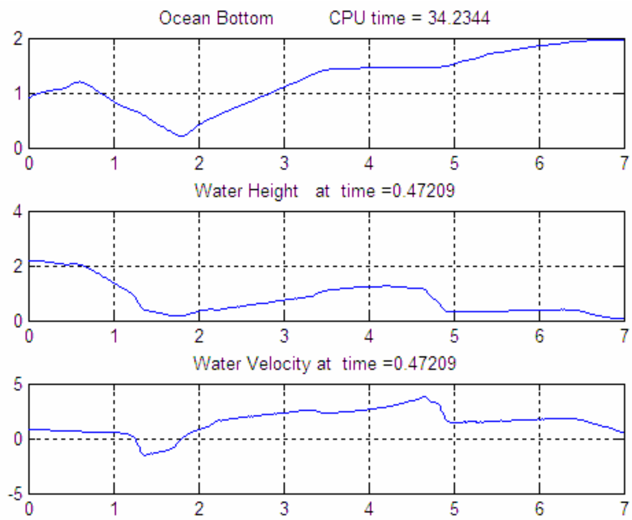

Fig. 4 Water level and velocity at 30 minutes

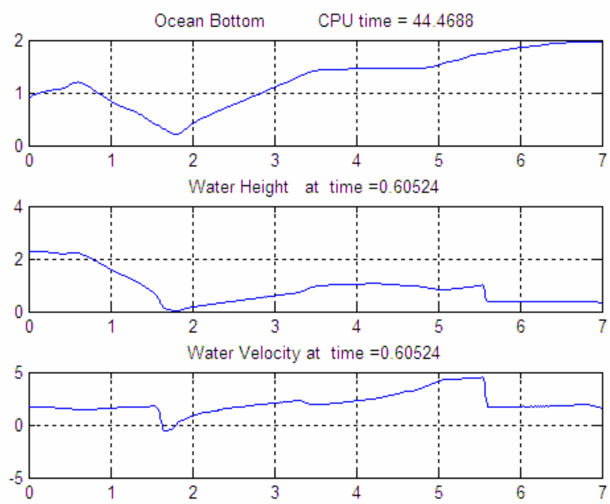

Fig. 5 Water level and velocity at 35 minutes

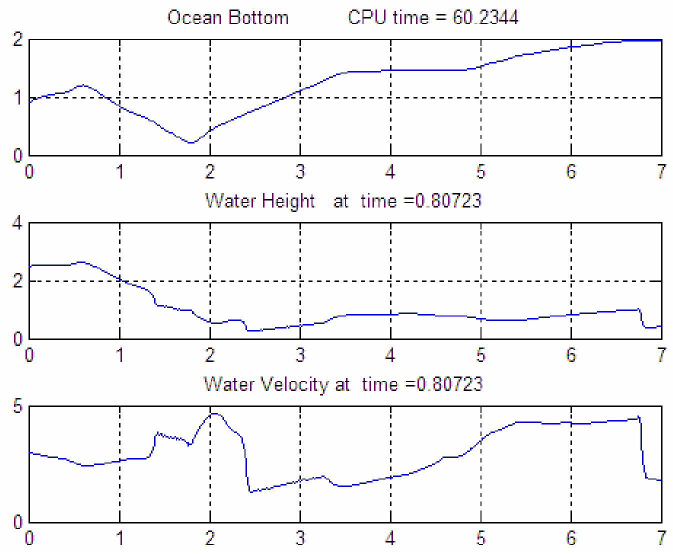

Fig. 6 Water level and velocity at 50 minutes

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