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Generalization of Tsallis Entropy from a Q-deformed Arithmetic

J. J. Peña, J. Morales, J. García-Ravelo, J. García-Martínez

***Abstract*—**It is known that by introducing alternative forms of exponential and logarithmic functions, the Tsallis entropy S_q is itself a generalization of Shannon entropy S. In this work, from a deformation through a scaling function applied to the differential operator, it is possible to generate a q-deformed calculus as well as a q-deformed arithmetic, which not only allows generalizing the exponential and logarithmic functions but also any other standard function. The updated q-deformed differential operator leads to an updated integral operator under which the functions are integrated together with a weight function. For each differentiable function, it is possible to identify its q-deformed partner function, which is useful to generalize other algebraic relations proper of the original functions. As an application of this proposal, in this work, a generalization of exponential and logarithmic functions is studied in such a way that their relationship with the thermodynamic functions, particularly the entropy, allows us to have a q-deformed expression of these. As a result, from a particular scaling function applied to the differential operator, a q-deformed arithmetic is obtained, leading to the generalization of the Tsallis entropy.

***Keywords*—**q-calculus, q-deformed arithmetic, entropy, exponential functions, thermodynamic functions.

I. Introduction

THE differential equations are the mathematical representation of many problems in the field of science and engineering. In this sense, there is a wide variety of solution methods for these equations, which range from a simple change of variable to the most elaborate solution methodologies depending on the type of problem we want to solve. In any case, to solve a differential equation, the essential fact is that the coordinate space has been conveniently transformed in order to lead the involved equation into a particular known form and then identify the corresponding solution to the original problem. For example, the canonical transformation method [1] is used to solve the Schrödinger equation (SE). In this method, depending on the potential interaction, a coordinate transformation is proposed along with a similarity transformation in such a way that the original equation can be identified with a differential equation of the mathematical physics. Thus, the solution to the original problem is identified with some orthogonal polynomial or special function, additionally the energy spectrum is also identified. Another powerful method that also uses a transformation of coordinate $x = x(s)$ for solving differential equations is the so-called Nikiforov-Ubanov method [2] which is based on reducing the problem to a generalized equation of hypergeometric type [3].

In this work, we present a proposal in which we deform the coordinate space $x = x_q$ in a particularly simple way using a parametric deformation function $f_q(x)$ (scaling function conditioned to $f_0(x) = 1$) such that the standard differential operator d/dx is transformed to be $f_q(x)d/dx$. This transformation is proposed in such a way that the new q-deformed operator or dilated operator [4] $\widehat{D}_q = f_q(x)d/dx$ satisfies the same differentiation rule as the standard differential operator on the variable x, namely $dx/dx = 1$, then we propose $\widehat{D}_q x_q = 1$. Thus, the new transformation x_q turns out to be a simple change of variable that satisfies the condition $x_0 = x$. This new variable x_q can be used to introduce a new q-deformed functions $F_q(x)$ defined from its original partner $F(x)$ as follows: $F_q(x) = F(x_q)$.

The proposal can be applied to any differentiable function $F(x)$ using an appropriate deformation or scaling function $f_q(x)$, such that the new differential operator $\widehat{D}_q$ acts on $F_q(x)$ in the usual way, namely, $\widehat{D}_q F_q(x) = F_q'(x)$. As an application of this proposal and due to mathematical properties useful in physics, particularly in statistical physics, the standard exponential function $exp\ (x)$ and its corresponding logarithmic function $ln(x)$ are used to propose different q-forms for these functions by using some scaling functions $f_q(x)$, given place to new q-forms of arithmetic (q-arithmetic) as well as a q-deformed calculus.

It is well known that the exponential-type distributions are related to probability densities such as the Boltzmann distribution, describing classical systems via the Boltzmann-Gibbs (BG) entropy. For more complex systems showing asymptotic behavior characterized by stretched exponential [5], power law [6] or log-oscillating [7] that are phenomena typically observed in physical, economics, biological and social systems [8]-[10], several entropic forms have been proposed [11]-[13]. The latter have their origin by replacing the standard logarithmic function appearing in the BG entropy with its generalized version. So, the generalized entropic functions that normally are obtained by maximizing the entropic form under appropriate constraints, take a generalized BG form with a q-

J. J. Peña is with Universidad Autónoma Metropolitana Azc, Av San Pablo 420, col Nueva el Rosario Azc CP 02128 CDMX, México (corresponding author, phone: 52-5553189576; fax: 52-55539576; e-mail: jjpg@azc.uam.mx).
J. Morales is with Universidad Autónoma Metropolitana Azc, (e-mail: jmr@azc.uam.mx).
J. García-Ravelo is with Instituto Politécnico Nacional ESFM, UP. Adolfo López Mateos, Edificio 9, Zacatenco (e-mail: jgravelo@gmail.com).
J. García-Martínez is with Tecnológico de Estudios Superiores de Ixtapaluca, División de Ingeniería Biomédica. (e-mail: bucefalot@gmail.com).
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deformed exponential instead of the standard exponential. The main results of this proposal are focused to generalize entropic functions particularly the BG distribution as well as the Tsallis entropy.

II. q-Deformed Arithmetic

A. q-Calculus

Let us start from the standard differential operator $\frac{d}{dx}$ and its corresponding dilated operator [4] $f_q(x)\frac{d}{dx}$ such that for the particular case:

$$\widehat{D}_q = (1+qx)^{1-k}\frac{d}{dx}, \tag{1}$$

It is immediate to see that for $q \to 0$, $\widehat{D}_0 = \frac{d}{dx}$ is the standard differential operator. Also, we demand that this new operator acts on the new q-deformed variable x_q in similar form as the standard differential operator $\frac{d}{dx}$ acts on the x variable, namely $\frac{d}{dx}x = 1$. So, we propose:

$$\widehat{D}_q x_q = 1 \tag{2}$$

that is to say,

$$(1+qx)^{1-k}\frac{dx_q}{dx} = 1 \tag{3}$$

from which, the new transformed variable x_q is given by:

$$x_q(x) = \int^x \frac{d\tau}{(1+q\tau)^{1-k}} + c(q), \tag{4}$$

$c(q)$ being an integration constant, such that $x_{q\to 0} = x$. From (3), it is easy to have the q-differential operator as:

$$dx_q = \frac{dx}{(1+qx)^{1-k}}. \tag{5}$$

On the other hand, for an arbitrary derivable function $F(x)$ we introduce its partner q-function or q-deformed function $F_q(x)$ defined as:

$$F_q(x) = F(x_q), \tag{6}$$

such that the new q-deformed differential operator acts as:

$$\widehat{D}_q F_q(x) = \frac{dF(x_q)}{dx_q} = F'(x_q) \equiv F'_q(x). \tag{7}$$

So, in the new q-deformed space, the operator $\widehat{D}_q$ has the same role as the standard differential operator $\frac{d}{dx}$ when it acts on any differentiable function $F(x)$, namely $\frac{dF}{dx} = F'(x)$. In fact,

$$\widehat{D}_q F_q(x) = (1+qx)^{1-k}\frac{dF_q(x)}{dx} = (1+qx)^{1-k}\frac{dF(x_q)}{dx} =$$
$$(1+qx)^{1-k}\frac{dx_q}{dx}\frac{dF(x_q)}{dx_q} = (1+qx)^{1-k}\frac{d}{dx}\left(\int^x \frac{d\tau}{(1+q\tau)^{1-k}}\right)\frac{dF(x_q)}{dx_q}$$
$$= F'(x_q) \equiv F'_q(x) \tag{8}$$

where the prime symbol denotes derivative with respect to the argument. For example, if $F(x) = sin(x)$ then its corresponding q-deformed partner $sin_q(x)$ or q-$sin(x)$ function will be:

$$sin_q(x) = sin(x_q) \tag{9}$$

such that,

$$\widehat{D}_q sin_q(x) = (1+qx)^{1-k}\frac{d\,sin_q(x)}{dx} = (1+qx)^{1-k}\frac{dsin(x_q)}{dx}$$
$$= (1+qx)^{1-k}\frac{dx_q}{dx}\frac{dsin(x_q)}{dx_q} = (1+qx)^{1-k}\frac{d}{dx}\left(\int^x \frac{d\tau}{(1+qx)^{1-k}}\right)\frac{d\,sin(x_q)}{dx_q}$$
$$= \frac{d\,sin(x_q)}{dx_q} = cos(x_q) \equiv cos_q(x) \tag{10}$$

In regard to the q-deformed integral operator, for any function $F(x)$, according to (5), it will be defined as follows:

$$_q\int F(x)\,dx_q = \int \frac{F(x)dx}{(1+qx)^{1-k}} \tag{11}$$

whereas for its corresponding q-deformed partner $F_q(x)$, it acts as follows:

$$_q\int F_q(x)\,dx_q = \int F(x_q)\,dx_q \tag{12}$$

For example, as above, if $F(x) = sin(x)$ then,

$$_q\int sin(x)\,dx_q = \int \frac{sin(x)dx}{(1+qx)^{1-k}} \tag{13}$$

and,

$$_q\int sin_q(x)\,dx_q = \int sin(x_q)\,dx_q = -cos(x_q)$$
$$\equiv -cos_q(x) \tag{14}$$

In the special case of the exponential function $F(x) = e^x$, the corresponding q-exponential function $F_q(x)$, according to (6) is given by:

$$F_q(x) = exp_q(x) = e^{x_q(x)}, \tag{15}$$

from which, the q-deformed logarithmic function results in:

$$ln_q(x) = x_q^{-1}(ln(x)) \tag{16}$$

where, we supposed that the transformation $x_q(x)$ has an inverse function. In the next section, we shall deal closer with the explicit form of the q-deformed exponential function and the q-deformed logarithmic function.

B. q-Deformed Exponential and Logarithmic Functions

Due to the different interesting properties that the exponential and logarithmic functions have, we will now apply the approach given above to these functions, finding useful interesting results. Namely, from the q-exponential and q-logarithmic functions emerge a new q-deformed algebra or q-deformed arithmetic which will have applications in the field of statistical physics.

We consider the explicit form of (4) which, after realizing the integral with $k = \alpha$, we have:

$$x_{q\alpha} = \frac{(1+qx)^{\alpha}-1}{q\alpha}, \tag{17}$$

where we have taken $c(q) = \frac{-1}{q\alpha}$. At this point, it worth noting that,

$$\lim_{q\to 0} x_{q\alpha} = x\,,\ \ for\ any\ \alpha \tag{18}$$

Hence, from (15), the q-deformed exponential function is:

$$exp_{q\alpha}(x) = e^{\frac{(1+qx)^{\alpha}-1}{q\alpha}} \tag{19}$$

and the q-deformed logarithmic function is:

$$ln_{q\alpha}(x) = \frac{1}{q}\left(\left(1 + q\alpha\, ln(x)\right)^{\frac{1}{\alpha}} - 1\right). \tag{20}$$

From (17) and (19), it is straightforward to show that for any α,

$$\lim_{q\to 0} exp_{q\alpha}(x) = e^{x} \tag{21}$$

and,

$$\lim_{q\to 0} ln_{q\alpha}(x) = ln(x)\,, \tag{22}$$

where the L'Hôpital's rule has been used.

C. q-Deformed Arithmetic

The q-deformed function given above will be conditioned to preserving the standard structure of the algebraic properties of the original functions, namely $e^{x}e^{y} = e^{x+y}$, $\frac{e^{x}}{e^{y}} = e^{x-y}$, $ln(x) + ln(y) = ln(xy)$ and $ln(x) - ln(y) = ln(x/y)$, thus emerging a new q-deformed arithmetic. Specifically,

$$exp_{q\alpha}(x)exp_{q\alpha}(y) \equiv exp_{q\alpha}(x \oplus_{q\alpha} y) \tag{23}$$

$$\frac{exp_{q\alpha}(x)}{exp_{q\alpha}(y)} \equiv exp_{q\alpha}(x \ominus_{q\alpha} y) \tag{24}$$

$$ln_{q\alpha}(x) + ln_{q\alpha}(y) \equiv ln_{q\alpha}(x \otimes_{q\alpha} y) \tag{25}$$

$$ln_{q\alpha}(x) - ln_{q\alpha}(y) \equiv ln_{q\alpha}(x \oslash_{q\alpha} y) \tag{26}$$

where $x \oplus_{q\alpha} y$, $x \ominus_{q\alpha} y$, $x \otimes_{q\alpha} y$ and $x \oslash_{q\alpha} y$ are new algebraic relations that will determine a new q-deformed arithmetic. Explicitly,

$$exp_{q\alpha}(x)exp_{q\alpha}(y) = exp\left(\frac{(1+qx)^{\alpha}-1}{q\alpha}\right) exp\left(\frac{(1+qy)^{\alpha}-1}{q\alpha}\right)$$
$$= exp\left(\frac{(1+qx)^{\alpha}-1+(1+qy)^{\alpha}-1}{q\alpha}\right) = exp\left(\frac{[(1+qx)^{\alpha}+(1+qy)^{\alpha}-1]-1}{q\alpha}\right). \tag{27}$$

So, if the q-addition $\oplus_{q\alpha}$ is defined as:

$$x \oplus_{q\alpha} y = \frac{[\,(1+qx)^{\alpha} + (1+qy)^{\alpha} - 1]^{\frac{1}{\alpha}} - 1}{q}, \tag{28}$$

Then,

$$\left(1 + q\left(x \oplus_{q\alpha} y\right)\right)^{\alpha} = [\,(1+qx)^{\alpha} + \,(1+qy)^{\alpha} - 1] \tag{29}$$

from which, after substituting it in (27), it takes the form:

$$exp_{q\alpha}(x)exp_{q\alpha}(y) = exp\left(\frac{\left(1+q\left(x\oplus_{q\alpha}y\right)\right)^{\alpha}-1}{q\alpha}\right). \tag{30}$$

Finally, by comparing it with (19), (30) reduces to:

$$exp_{q\alpha}(x)exp_{q\alpha}(y) = exp_{q\alpha}\left(x \oplus_{q\alpha} y\right). \tag{31}$$

Likewise,

$$\frac{exp_{q\alpha}(x)}{exp_{q\alpha}(y)} = \frac{exp\left(\frac{(1+qx)^{\alpha}-1}{q\alpha}\right)}{exp\left(\frac{(1+qy)^{\alpha}-1}{q\alpha}\right)} = exp\left(\frac{(1+qx)^{\alpha}-1}{q\alpha} - \frac{(1+qy)^{\alpha}-1}{q\alpha}\right) =$$
$$exp\left(\frac{[(1+qx)^{\alpha}-(1+qy)^{\alpha}+1]-1}{q\alpha}\right). \tag{32}$$

So, when the q-deformed subtraction $x \ominus_{q\alpha} y$ is defined in the form:

$$x \ominus_{q\alpha} y = \frac{[\,(1+qx)^{\alpha} - (1+qy)^{\alpha}+1]^{\frac{1}{\alpha}} - 1}{q}, \tag{33}$$

such that,

$$[\,(1+qx)^{\alpha} - \,(1+qy)^{\alpha} + 1] = \left(1 + q\left(x \ominus_{q\alpha} y\right)\right)^{\alpha}, \tag{34}$$

then (32) writes down as:

$$\frac{exp_{q\alpha}(x)}{exp_{q\alpha}(y)} = exp_{q\alpha}\left(x \ominus_{q\alpha} y\right) \tag{35}$$

Regarding the q-deformed product $\otimes_{q\alpha}$ and division $\oslash_{q\alpha}$ these are deduced as follows:

$$ln_{q\alpha}(x) + ln_{q\alpha}(y) = \frac{(1+q\alpha ln(x))^{\frac{1}{\alpha}} - 1}{q} + \frac{(1+q\alpha ln(y))^{\frac{1}{\alpha}} - 1}{q} =$$
$$\frac{\left[(1+q\alpha ln(x))^{\frac{1}{\alpha}}+(1+q\alpha ln(y))^{\frac{1}{\alpha}}-1\right]-1}{q} \tag{36}$$

such that, according with the structure of (20), if:

$$x \otimes_{q\alpha} y = exp\left(\frac{\left[(1+q\alpha ln(x))^{\frac{1}{\alpha}}+(1+q\alpha ln(y))^{\frac{1}{\alpha}}-1\right]^{\alpha}-1}{q\alpha}\right) \quad (37)$$

is introduced as the q-deformed product, then:

$$ln_{q\alpha}(x) + ln_{q\alpha}(y) = ln_{q\alpha}(x \otimes_{q\alpha} y). \quad (38)$$

Similarly,

$$ln_{q\alpha}(x) - ln_{q\alpha}(y) = \frac{(1+q\alpha ln(x))^{\frac{1}{\alpha}}-1}{q} - \frac{(1+q\alpha ln(y))^{\frac{1}{\alpha}}-1}{q} = \frac{\left[(1+q\alpha ln(x))^{\frac{1}{\alpha}}-(1+q\alpha ln(y))^{\frac{1}{\alpha}}+1\right]-1}{q} \quad (39)$$

In this case, if the q-deformed division $x \oslash_{q\alpha} y$ is defined as:

$$x \oslash_{q\alpha} y = exp\left(\frac{\left[(1+q\alpha ln(x))^{\frac{1}{\alpha}}-(1+q\alpha ln(y))^{\frac{1}{\alpha}}+1\right]^{\alpha}-1}{q\alpha}\right), \quad (40)$$

it leads to,

$$ln_{q\alpha}(x) - ln_{q\alpha}(y) = ln_{q\alpha}(x \oslash_{q\alpha} y). \quad (41)$$

It is worth mentioning that the limit case $q \to 0$ leads to the standard arithmetic: $x \oplus_{0\alpha} y = x + y$, $x \ominus_{0\alpha} y = x - y$, $x \otimes_{0\alpha} y = xy$ and $x \oslash_{0\alpha} y = \frac{x}{y}$, for any α.

III. Tsallis Entropy and Its Generalization

One of the most important entropic functions in statistical physics is the so-called Entropy S, which can be defined using different concepts, depending on the application area we are dealing with. It constitutes the cornerstone in statistical physics and statistical mechanics due its applications in physics and engineering. In general, it has many applications in different areas of knowledge [14]. For example, in the information theory the entropy measures how much uncertainty can exist in any data source [15]. In the economic theory, the entropy is useful when an economic system has a great disorder as a result of a deficit in regulations, which would lead to the economy to the chaos [16]. In the field of psychology, it deals with social systems in which human beings have differences and similarities at the same time, differences that must be preserved to increase possibilities of optimal development [17]. In computing, entropy is the measure of the uncertainty that is present in a series of messages, of which only one will ultimately be received. It is a type of information measure that is used to eliminate or reduce uncertainty [18]. In the area of linguistics, the concept of entropy is related with the information that is organized and selected to reach the optimal analysis that has greater depth in a communication process [19].

A. Tsallis Entropy

The functional form of the Boltzman-Gibbs (BG) entropy for a discrete set of probabilities $\{p_i\}$ is given by:

$$S^{BG} = -k\sum_{i=1}^{\Omega} p_i ln(p_i) \quad (42)$$

with $\sum_{i=1}^{\Omega} p_i = 1$, Ω= number of states of the system and k is the Boltzmann constant.

The case of equal probabilities $p_i = \frac{1}{\Omega}$, $i = 1, 2, 3, \ldots\ldots \Omega$ leads to Boltzmann formula:

$$S^{BG} = -k\sum_{i=1}^{\Omega} p_i ln(p_i) = -k\sum_{i=1}^{\Omega}\frac{1}{\Omega} ln\left(\frac{1}{\Omega}\right) = -k\frac{1}{\Omega} ln\left(\frac{1}{\Omega}\right)\sum_{i=1}^{\Omega} 1 = -k\frac{1}{\Omega} ln\left(\frac{1}{\Omega}\right)\Omega$$

$$S^{BG} = kln(\Omega). \quad (43)$$

In his famous proposal about the generalization of the entropy, Tsallis introduces [11] the relation for the q-entropy as:

$$S_q^T = k\frac{1-\sum_{i=1}^{\Omega} p_i^q}{q-1} \quad (44)$$

which, for equal probabilities $\left(p_i = \frac{1}{\Omega}\right)$ reduces to:

$$S_q^T = k\frac{\Omega^{1-q}-1}{1-q} = kln_q(\Omega), \quad (45)$$

where,

$$ln_q(\Omega) = \frac{\Omega^{1-q}-1}{1-q} \quad (46)$$

is defined as the q-deformed logarithmic function whose corresponding inverse function is the q-deformed exponential [20].

$$exp_q(\Omega) = (1 + (1-q)\Omega)^{\frac{1}{1-q}}. \quad (47)$$

As q goes to one, both q-deformed functions go to their corresponding standard ones. Namely,

$$\lim_{q\to 1} exp_q(\Omega) = e^{\Omega} \quad (48)$$

$$\lim_{q\to 1} ln_q(\Omega) = ln(\Omega), \quad (49)$$

which leads to $S_1^T = S^{BG}$.

B. Generalized Tsallis Entropy

If instead of using the standard logarithm $ln(p_i)$ and $ln(\Omega)$ in the definitions of standard Boltzman-Gibbs (BG) entropy given in (42) and (43), we use the generalized q-deformed logarithmic function $ln_{q\alpha}(p_i)$ given in (20), we get:

$$S_{q\alpha}^{BG} = -k\sum_{i=1}^{\Omega} p_i ln_{q\alpha}(p_i) = -k\sum_{i=1}^{\Omega} p_i \frac{(1+q\alpha\, ln(p_i))^{\frac{1}{\alpha}}-1}{q}$$

$$= k\frac{\sum_{i=1}^{\Omega} p_i - \sum_{i=1}^{\Omega} p_i\left(1+q\alpha\, ln(p_i)\right)^{\frac{1}{\alpha}}}{q} \tag{50}$$

If the fact that $\sum_{i=1}^{\Omega} p_i = 1$ is used and,

$$h_{q\alpha}(p_i) = \left(1 + q\alpha\, ln(p_i)\right)^{\frac{1}{\alpha}} \tag{51}$$

Then,

$$S_{q\alpha}^{BG} = k\frac{1-\sum_{i=1}^{\Omega} p_i\, h_{q\alpha}(p_i)}{q} \tag{52}$$

It is worth noting that the above result generalizes the Tsallis entropy given in (44). Namely, the limit case $\alpha \to 0$ leads to:

$$\lim_{\alpha\to 0} S_{q\alpha}^{BG} = S_{q0}^{BG} = S_q^T \tag{53}$$

with,

$$S_{q0}^{BG} = k\frac{1-\sum_{i=1}^{\Omega} p_i\, h_{q0}(p_i)}{q} \tag{54}$$

To show that, it is enough to calculate the $\lim_{\alpha\to 0} h_{q\alpha}(p_i) = h_{q0}(p_i)$, which is equivalent to calculate:

$$\lim_{\alpha\to 0} ln\left(h_{q\alpha}(p_i)\right) = ln\left(h_{q0}(p_i)\right) \tag{55}$$

Explicitly,

$$\lim_{\alpha\to 0} ln\left(h_{q\alpha}(p_i)\right) = ln\left(h_{q0}(p_i)\right) = \lim_{\alpha\to 0} ln\left(1 + q\alpha\, ln(p_i)\right)^{\frac{1}{\alpha}} = \lim_{\alpha\to 0} \frac{ln\left(1+q\alpha\, ln(p_i)\right)}{\alpha} = ln(p_i)^q \tag{56}$$

Then $ln\left(h_{q0}(p_i)\right) = ln(p_i)^q$ or $h_{q0}(p_i) = p_i^q$ such that (54) writes down as:

$$S_{q0}^{BG} = k\frac{1-\sum_{i=1}^{\Omega} p_i^{1+q}}{q} \tag{57}$$

which, with $q \to q-1$ is the Tsallis entropy given in (44) obtained as a limit case of (57). Namely,

$$S_{(q-1)0}^{BG} = S_q^T\,. \tag{58}$$

IV. Applications

As an application of the results given above, we are going to show the non-extensive property of the Tsallis entropy using the formulae obtained before. To start with, unlike limits given in (21) and (22), in this case the next limits hold:

$$\lim_{\alpha\to 0} exp_{q\alpha}(\Omega) = exp_{q0}(\Omega) = (1+q\Omega)^{\frac{1}{q}} \tag{59}$$

and,

$$\lim_{\alpha\to 0} ln_{q\alpha}(\Omega) = ln_{q0}(\Omega) = \frac{\Omega^q - 1}{q}, \tag{60}$$

from which, with $q \to 1-q$ result in those given from the Tsalli's proposal of (47) and (46) respectively:

$$exp_{1-q0}(\Omega) \equiv exp_q(\Omega) = (1+(1-q)\Omega)^{\frac{1}{1-q}} \tag{61}$$

$$ln_{1-q0}(\Omega) \equiv ln_q(\Omega) = \frac{\Omega^{1-q}-1}{1-q}\,. \tag{62}$$

Next, we consider two independent systems A and B with Ω_A and Ω_B being total number of allowed states of each system, that is $\{1, 2, 3, \ldots.\ \Omega_A\}$ and $\{1, 2, 3, \ldots.\ \Omega_B\}$. Then, from (45):

$$S_q^T(\Omega_A) = k\frac{\Omega_A^{1-q}-1}{1-q} \tag{63}$$

and,

$$S_q^T(\Omega_B) = k\frac{\Omega_B^{1-q}-1}{1-q}, \tag{64}$$

then,

$$S_q^T(\Omega_A) + S_q^T(\Omega_B) = k\frac{\Omega_A^{1-q}-1+\Omega_B^{1-q}-1}{1-q} = k\frac{\left[\Omega_A^{1-q}+\Omega_B^{1-q}-1\right]-1}{1-q} = k\frac{\left[\left(\Omega_A^{1-q}+\Omega_B^{1-q}-1\right)^{\frac{1}{1-q}}\right]^{1-q}-1}{1-q} \tag{65}$$

Similarly, to the q-product defined in (37), if:

$$\Omega_A \otimes_q \Omega_B = \left(\Omega_A^q + \Omega_B^q - 1\right)^{\frac{1}{q}} \tag{66}$$

then,

$$S_q^T(\Omega_A) + S_q^T(\Omega_B) = k\frac{\left[\Omega_A \otimes_{1-q} \Omega_B\right]^{1-q}-1}{1-q} \tag{67}$$

which, according of the format of (63) and (64):

$$S_q^T(\Omega_A) + S_q^T(\Omega_B) = S_q^T\left(\Omega_A \otimes_{1-q} \Omega_B\right) \tag{68}$$

On the other hand, from (45):

$$S_q^T(\Omega_A) S_q^T(\Omega_B) = k^2\frac{\left(\Omega_A^{1-q}-1\right)\left(\Omega_B^{1-q}-1\right)}{(1-q)^2} \tag{69}$$

such that,

$$\frac{1-q}{k} S_q^T(\Omega_A) S_q^T(\Omega_B) = k\frac{(\Omega_A\Omega_B)^{1-q}-\Omega_A^{1-q}-\Omega_B^{1-q}+1}{1-q} = k\frac{(\Omega_A\Omega_B)^{1-q}-1}{1-q} - k\frac{\Omega_A^{1-q}-1}{1-q} - k\frac{\Omega_B^{1-q}-1}{1-q} = S_q^T(\Omega_A\Omega_B) - S_q^T(\Omega_A) - S_q^T(\Omega_B) \tag{70}$$

where,

$$S_q^T(\Omega_A\Omega_B) = k\frac{(\Omega_A\Omega_B)^{1-q}-1}{1-q} \tag{71}$$

By arranging terms, it yields:

$$S_q^T(\Omega_A\Omega_B) = S_q^T(\Omega_A) + S_q^T(\Omega_B) + \frac{(1-q)}{k} S_q^T(\Omega_A) S_q^T(\Omega_B) \tag{72}$$

from which, the non-extensive property of Tsallis entropy is exhibited by the last term of the previous relation. In addition, if $\{(1,1),\ (1,2),\ \dots(\Omega_A, \Omega_B)\}$ is the ensembles of configurational possibilities associated to $A \cup B$, (72) can be written as [21]:

$$\frac{S_q^T(\Omega_{A\cup B})}{k} = \frac{S_q^T(\Omega_A)}{k} + \frac{S_q^T(\Omega_B)}{k} + (1-q)\frac{S_q^T(\Omega_A)}{k}\frac{S_q^T(\Omega_B)}{k} \quad (73)$$

such that,

$$\frac{S_q^T(\Omega_{A\cup B})}{k} = \frac{S_q^T(\Omega_A)}{k} \oplus_{1-q} \frac{S_q^T(\Omega_B)}{k} \quad (74)$$

which, unlike (73), shows the additivity of the Tsallis entropy in the frame of the q-deformed arithmetic as given in (7) in [11]. The q-addition $\oplus_{1-q}$ is a particular case of that given in (28) with $q \to (1-q)$. Namely,

$$x \oplus_q y \equiv \lim_{\alpha\to 0} x \oplus_{q\alpha} y = x + y + qxy \quad (75)$$

As an observation, due to the properties of the q-deformed arithmetic, (74) cannot be written as:

$$S_q^T(\Omega_{A\cup B}) = S_q^T(\Omega_A) \oplus_{1-q}\ S_q^T(\Omega_B) \quad (76)$$

However, when $q \to 1$ the additivity of the BG entropy is obtained as well. That is $S_1^T(\Omega_{A\cup B}) = S^{BG}(\Omega_{A\cup B})$, explicitly:

$$S^{BG}(\Omega_{A\cup B}) = S^{BG}(\Omega_A) + S^{BG}(\Omega_B) \quad (77)$$

V. Conclusions

In the framework of q-calculus, the proposal of a q-deformed differential operator led to the definition of q-deformed associated functions and a new arithmetic (q-arithmetic) which allowed us to find partner forms for arbitrary differentiable functions, particularly the exponential and logarithmic functions. These q-deformed partner functions are directly involved with entropy functions which are useful in many areas of the science. By maintaining the already known algebraic properties of their corresponding standard functions, it was possible to find new arithmetic structures from which the generalization of the entropic functions was achieved, particularly the Tsallis entropy. The proposal is general in the sense that the differential operator can be deformed by proposing other deformation (dilation) functions thus finding a new q-deformed calculus as well as its corresponding arithmetic structures that would lead to new applications in thermodynamics and statistical physics.

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