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Closed-Form Solution of Second Order Linear Ordinary Differential Equations

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Abstract—A transformational method is employed to obtain closed-form integral solutions for nonhomogeneous second order linear ordinary differential equations in terms of a particular solution of the corresponding homogeneous part. To find the particular solution of the homogeneous part, the equation is first transformed into a simple Riccati equation from which the general solution of the nonhomogeneous second order linear differential equation, in the form of a closed integral equation, is inferred. The method is applied to the solution of Schrödinger equation for hydrogen-like atoms. A generic nonhomogeneous second order linear differential equation has also been solved to further exemplify the methodology.

Keywords—Closed form, Second order ordinary differential equations, explicit, linear equations, differential equations.

I. Introduction

MANY nonhomogeneous second order linear ordinary differential equations are solved in terms of the infinite series, e.g., Legendre, Bessel, hypergeometric, hermit polynomials [1], [2]. If possible, closed form solutions are always preferable. This article presents a method for solving in closed form a particular variety of non-homogenous second order linear differential equations for which a particular solution is available and for which the ensuing integrals can be solved in closed form.

The order of derivation is as follows: In Section II, we will derive a solution for nonhomogeneous second order linear differential equations in terms of the general solution of their corresponding homogeneous part. The resulting solution will be an integral equation. In Section III, the homogenous equation will be transformed into an equivalent simple Riccati equation form of

$$z^2 + \dot{z} = \mathcal{R} \tag{1}$$

From this point on (1) is referred to as the simple Riccati equation. The results from Section III provide us with the necessary method to solve the general second order linear differential equations. In Section IV, the method will be applied to solve the Schrödinger equation hydrogen-like atoms. The subject of Section V is the solution of a nonhomogeneous second order differential equation with transcendental type functions.

II. Nonhomogeneous Second Order Linear Differential Equations

Equation (2) represents a nonhomogeneous second order linear differential equation:

$$\ddot{y}(x) + p(x)\dot{y}(x) + q(x)y(x) = s(x) \tag{2}$$

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Let u and v be twice differentiable functions and such that $u(x)$ be the solution of the homogeneous part of (2). Then u satisfies (3):

$$\ddot{u}(x) + p(x)\dot{u}(x) + q(x)u(x) = 0 \tag{3}$$

Additionally, let us define y such that $y(x) = v(x)u(x)$. If y is now substituted into (2), we will obtain (4) as:

$$\begin{aligned}\ddot{u}(x)v(x) + \ddot{v}(x)u(x) + 2\dot{u}(x)\dot{v}(x) + p(x)\dot{u}(x)v(x) \\ + p(x)\dot{v}(x)u(x) + q(x)u(x)v(x) = s(x)\end{aligned} \tag{4}$$

Equation (4) can be rearranged as (5):

$$\begin{aligned}[\ddot{u}(x) + p(x)\dot{u}(x) + q(x)u(x)]v(x) \\ + [u(x)\ddot{v}(x) + (2\dot{u}(x) + p(x)u(x))\,\dot{v}(x)] = s(x)\end{aligned} \tag{5}$$

Since $u(x)$ is the solution of the homogeneous part of (2), (5) simplifies as (6):

$$u(x)\ddot{v(x)} + (2\dot{u}(x) + p(x)u(x))\,\dot{v}(x) = s(x) \tag{6}$$

Now if any particular solution, $u_p(x)$ of the (6) is known, (6) can be rearranged as (7):

$$\ddot{v}(x) + \left[\frac{2\dot{u}_p(x)}{u_p(x)} + p(x)\right]\dot{v}(x) = \frac{s(x)}{u_p(x)} \tag{7}$$

A change of variable of $\vartheta(x) = \dot{v}(x)$ will modify this equation as:

$$\dot{\vartheta}(x) + \left[\frac{2\dot{u}_p(x)}{u_p(x)} + p(x)\right]\vartheta(x) = \frac{s(x)}{u_p(x)} \tag{8}$$

The solution of (8) is given by (9):

$$\vartheta = \frac{\int e^{\int p(x)dx}u_p(x)s(x)dx + c}{u_p(x)^2 e^{\int p(x)dx}} \tag{9}$$

From this, we obtain the solution $v(x)$ as (10):

$$v(x) = \int \frac{e^{\int p(x)dx}u(x)s(x)dx + C}{u(x)^2 e^{\int p(x)dx}}dx + C_1 \tag{10}$$

The particular solution of (2) is given by (11):

$$\begin{aligned}y_p(x) = u_p(x)v(x) = \\ u_p(x)\left[\int \frac{\int e^{\int p(x)dx}u_p(x)s(x)dx + c}{u_p(x)^2 e^{\int p(x)dx}}dx + c_1\right]\end{aligned} \tag{11}$$

Since for $s(x) = 0$, $y(x)$ and $v(x)$ have to be reduced to $u(x)$ and 1 respectively, therefore the values for the constants of integrations will be: $c = 0$ and $c_1 = 1$. As a result, the general solution of any nonhomogeneous second order linear differential equation in terms of the general solution of its corresponding homogeneous part, $u(x)$ will be as in (12):

$$y(x) = u(x) + y_p(x) \tag{12}$$

This can be written as (13):

$$y(x) = u(x) + u_p(x) \int \frac{e^{\int p(x)dx} u_p(x) s(x) dx}{u_p(x)^2 e^{\int p(x)dx}} dx \tag{13}$$

III. Solution of the Homogeneous Part

It is well known that the variable change of $\frac{\dot{u}(x)}{u(x)} = \omega(x)$ transforms the homogenous second order differential (3) into (14):

$$\omega(x)^2 \; + p(x)\omega(x) + q(x) \; + \dot{\omega(x)} = 0 \tag{14}$$

This equation is known as the Riccati equation [1], [4]. The Riccati equation reduces to the simple Riccati equation as in (15):

$$z^2(x) + \dot{z}(x) = \mathcal{R}(x) \tag{15}$$

Theorem 1: All Riccati equations and consequently homogenous second order linear differential equations can be reduced to the simple Riccati equation.

Proof: Equation (14) can be factored partially as follows:

$$\left(\omega(x) + \frac{P(x)}{2} - \frac{\sqrt{p(x)^2 - 4q(x)}}{2}\right) \times \left(\omega(x) + \frac{P(x)}{2} + \frac{\sqrt{p(x)^2 - 4q(x)}}{2}\right) + \dot{\omega}(x) = 0 \tag{16}$$

The variable $z(x) = \omega(x) + \frac{p(x)}{2}$ transforms (16) into (17):

$$z^2(x) + \dot{z}(x) = \mathcal{R}(x), \quad \mathcal{R}(x) = \frac{p(x)^2}{4} + \frac{\dot{p}(x)}{2} - q(x) \tag{17}$$

Now, if we substitute $\omega(x) = \frac{\dot{u}(x)}{u(x)}$ into relation $z(x) = \omega(x) + \frac{p(x)}{2}$, we will obtain (18):

$$\frac{du(x)}{u(x)} = \left(z(x) - \frac{p(x)}{2}\right) dx \tag{18}$$

The solution of (18), in terms of the general solution of the simple Riccati equation is given by (19):

$$u(x) = u_0 e^{-\int \frac{p(x)}{2} dx} e^{\int z(x) dx}, \quad u_0, \text{a constant} \tag{19}$$

■

Equations (3), (17) and (19) indicate that all second order linear differential equations with the same $\mathcal{R}(x)$ will have the same $z(x)$, and therefore their final solution is different only in the term $e^{-\int \frac{p(x)}{2} dx}$.

If $z_1(x)$ is a particular solution of the simple Riccati equation, then its general solution will be given by (20);

$$z(x) = z_1(x) + \frac{1}{\Omega(x)} \tag{20}$$

where;

$$\Omega\ (x) = \frac{\left[\int e^{-2\int z_1\ (x)dx} dx + c\right]}{e^{-2\int z_1\ (x)\ \mathrm{dx}}} \tag{21}$$

Therefore, (20) can be written as (22):

$$z(x) = z_1(x) + \frac{e^{-2\int z_1\ (x)\ dx}}{\left[\int e^{-2\int z_1\ (x)\ \mathrm{dx}} dx + c\right]} \tag{22}$$

We may write (22) as (23):

$$z(x) = z_1(x) + \frac{d}{dx} \ln \left[\int e^{-2\int z_1\ (x)\ \mathrm{dx}} dx + c\right] \tag{23}$$

This completes the solution of (19) in terms of a particular solution of the Riccati equation which is given by (24):

$$u(x) = u_0 e^{-\int \frac{p(x)}{2} dx} e^{\int \left[z_1(x) + \frac{d}{dx} \ln\left[\int e^{-2\int z_1\ (x)\ \mathrm{dx}} dx + c\right]\right] dx} \tag{24}$$

If we integrate the second term on the right-hand side of the above equation the general solution for $u(x)$ will be as in (25):

$$u(x) = u_0 e^{-\int \frac{p(x)}{2} dx} e^{\int z_1(x) dx} \left[\int e^{-2\int z_1(x) dx} dx + c\right] \tag{25}$$

Finally, by substituting (25) into (13), the general solution of the nonhomogeneous second order differential equation in terms of a particular solution of its corresponding Simple Riccati equation will be;

$$y(x) = u_0 e^{-\int \frac{p(x)}{2} dx} e^{\int z_1(x) dx} \left[\int e^{-2\int z_1(x) dx} dx + c\right] + u_p(x) \int \frac{\int e^{\int p(x)dx} u_p(x) s(x) dx}{u_p(x)^2 e^{\int p(x)dx}}] \tag{26}$$

IV. Example: Apply to Schrödinger Equation

Schrödinger equation in spherical coordinate system for hydrogen-like atoms is as follows [1], [5], [6]:

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \psi(r,\theta,\varphi)}{\partial r}\right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial \psi(r,\theta,\varphi)}{\partial \theta}\right) + \frac{1}{r^2 (\sin\theta)^2} \frac{\partial^2 \psi(r,\theta,\phi)}{\partial^2 \phi} + \frac{2\mu}{\hbar^2} \left(E + \frac{z_a\ e^2}{4\pi\epsilon_0\ r}\right) = 0 \tag{27}$$

By separation of variables $\psi(r,\theta,\phi)$ as be expressed as $\psi(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi)$, and be resolved into three equations as follows:

$$0 = \ddot{\Phi}(\phi) + B\Phi(\phi) \tag{28}$$

$$0 = \frac{\sin\theta}{\Theta(\theta)} \frac{d}{d\theta} \left(\sin\theta \frac{d\Theta(\theta}{\partial\theta}\right) + A\sin^2\theta - B \tag{29}$$

$$0 = \frac{d^2R(r)}{dr^2} + \frac{2}{r}\frac{dR(r)}{d(r)} + \left[\frac{2\mu}{\hbar^2}\left(E + \frac{z_a\ e^2}{4\pi\epsilon_0\ r}\right) - \frac{A}{r^2}\right] R(r) = 0 \tag{30}$$

A and B are constants. The solutions are obtained as follows:

A. Azimuthal Component

Equation (28) is a homogeneous differential equation with constant coefficients. According to (1), the simple Riccati equation related to (28) is (31):

$$z(\phi)^2 + \dot{z}(\phi) = \mathcal{R}(\phi) \tag{31}$$

Since $p(\phi)$ and $q(\phi)$ in (28) are 0 and $-B$, respectively, then according to (17), $\mathcal{R}(\phi) = -B$, and

$$z(\phi)^2 + \dot{z}(\phi) = -B \tag{32}$$

A particular solution of this equation is;

$$z_1(\phi) = \sqrt{-B}\tanh(\sqrt{-B}\phi) \tag{33}$$

The general solution of (28) is obtained by substituting (33) into (25). It is given by (34):

$$\Phi(\phi) = \Phi_0 e^{\int \sqrt{-B}\tanh(\sqrt{-B}\phi)d\phi} \times \left[\int e^{-2\sqrt{-B}\tanh(\sqrt{-B}\phi)d\phi}\, d\phi + c\right] \tag{34}$$

This equation simplifies as (35):

$$\Phi(\phi) = c_1 e^{-\sqrt{-B}\phi} + c_2 e^{\sqrt{-B}\phi}, \quad c_1, c_2, \text{ constants} \tag{35}$$

Since ϕ is the azimuthal angle, Φ must exhibit a periodic character, i.e., $B > 0$. If we let $B = m^2, m = 0, \pm 1, \cdots$, then the final solution will be given by (36):

$$\Phi_m(\phi) = c_1\cos(m\phi) + c_2\sin(m\phi) \tag{36}$$

B. Polar Component

Since $B = m^2$, we can express (29) in terms of $x = \cos\,\theta$ to obtain (37):

$$\left(1 - x^2\right)\frac{d^2\Theta(x)}{dx^2} - 2x\frac{d\Theta(x)}{dx} + \left(A - \frac{m^2}{1 - x^2}\right)\Theta(x) = 0 \tag{37}$$

To solve this equation by our method, first we write it in the following form:

$$\frac{d^2\Theta(x)}{dx^2} - \frac{2x}{1 - x^2}\frac{d\Theta(x)}{d} + \left(\frac{A}{1 - x^2} - \frac{m^2}{\left(1 - x^2\right)^2}\right)\Theta(x) = 0 \tag{38}$$

Here, $p(x) = -\frac{2x}{1-x^2}$ and $q(x) = \left(\frac{A}{1-x^2} - \frac{m^2}{(1-x^2)^2}\right)$. With these, the simple Riccati equation of (29) will be as follows:

$$z(x)^2 + \dot{z}(x) = \frac{\left(m^2 - 1\right) - A(1 - x^2)}{\left(1 - x^2\right)^2} \tag{39}$$

If we choose $z_1(x) = \frac{\alpha x}{(1-x^2)}$ for a particular solution of (39), we will have the following identity to satisfy:

$$x^2\left(\alpha^2 + \alpha - A\right) + \left(\alpha - m^2 + 1 + A\right) \equiv 0 \tag{40}$$

For this to be satisfied, it must be that $\alpha = m - 1, \quad and\ A = m(m-1)$. These will give the particular solution for the Riccati equation as in (41):

$$z_1(x) = \frac{(m - 1)x}{(1 - x^2)} \tag{41}$$

Again, if we substitute z_1 into (25) we will get the general solution for (38) as (42):

$$\Theta(x) = \Theta_0 e^{-\int \frac{-2x}{2(1-x^2)}dx} e^{\int \frac{(m-1)x}{(1-x^2)}dx} \times \left[\int e^{-2\int \frac{(m-1)x}{(1-x^2)}dx} dx + c\right] \tag{42}$$

Now, if we replace x by $\cos(\theta)$ and compute the integrals in this equations the final solution of (29) will be as follows:

$$\Theta(x) = \Theta_1 \sin\theta^{-m} + \Theta_2 \sin\theta^{-m}\int \sin\theta^{2m-1}d\theta, \quad \Theta_1 \text{and } \Theta_2 \text{ constants} \tag{43}$$

C. Radial Component

The radial component of the Schrödinger equation in terms of the eigenvalue m is (44):

$$\frac{d^2R(r)}{dr^2} + \frac{2}{r}\frac{dR(r)}{dr} + \left[\frac{2\mu}{\hbar^2}\left(E + \frac{z_a e^2}{4\pi\epsilon_0\ r}\right) - \frac{m(m-1)}{r^2}\right]R(r) = 0 \tag{44}$$

This equation can be written in the simple Riccati form as follows:

$$z(r)^2 + \dot{z}(r) = \frac{m(m-1))}{r^2} - \frac{\mu z_a e^2}{2\pi\epsilon_0 h^2}\frac{1}{r} - \frac{2\mu E}{\hbar^2} \tag{45}$$

As before, we choose $z_1 = \frac{\alpha}{r} + \beta$ for a particular solution and from the resulting identity we deduce that

$$\alpha = m\ , \quad \beta = -\frac{\mu z_a e^2}{4\pi\epsilon_0 h^2}\frac{1}{m}, E = -\frac{\mu z_a^2 e^4}{32\pi^2\epsilon_0 h^2}\left(\frac{1}{m^2}\right)$$

Thus, a particular solution of simple Riccati equation will be (46).

$$z_1 = \frac{m}{r} - \frac{\mu z_a e^2}{4\pi\epsilon_0 h^2}\frac{1}{m} \tag{46}$$

From this the general solution of (30) will be (47):

$$R(r) = R_0 e^{-\int \frac{1}{r}dr} e^{\int\left(\frac{m}{r} - \frac{\mu z_a e^2}{4\pi\epsilon_0 h^2}\frac{1}{m}\right)dr} \left[\int e^{-2\int\left(\frac{m}{r} - \frac{\mu z_a e^2}{4\pi\epsilon_0 h^2}\frac{1}{m}\right)dr} dr + c\right] \tag{47}$$

This equation simplifies to (48):

$$R_m(r) = R_0 r^{m-1} e^{-\frac{\mu z_a e^2}{4\pi\epsilon_0 h^2}\frac{1}{m}r} + R_1 r^{m-1} e^{-\frac{mu z_a e^2}{4\pi\epsilon_0 h^2}\frac{r}{m}}\int r^{-2m} e^{-\frac{\mu z_a e^2}{2\pi\epsilon_0 h^2}\frac{r}{m}} dr \tag{48}$$

for constants R_0 and R_1.

Finally, the wave function is obtained as in (49):

$$\psi(r, \theta, \phi) = \sum_m R_m(r)\Theta_m(\theta)\Phi_m(\phi) \tag{49}$$

V. Example: Generic Second Order

We consider the following equation:

$$\ddot{y}(x) + \tanh(x)\ \dot{y}(x) + \big(\frac{1}{2} - \frac{1}{4}\tanh^2(x)\big)y(x) = \frac{1}{\sqrt{\cosh(x)}} \quad (50)$$

The corresponding homogeneous component of the above equation is (51):

$$\ddot{u}(x) + \tanh(x)\dot{u}(x) + (\frac{1}{2} - \frac{1}{4}\tanh^2(x))u(x) = 0 \quad (51)$$

The corresponding Riccati equation according to (17) is as follows:

$$z^2(x) + \dot{z}(x) = 0 \quad (52)$$

A particular solution for this equation is

$$z_1(x) = \frac{1}{x} \quad (53)$$

According to (26) the solution of all second order linear differential equations with this Riccati equation is as follows:

$$y(x) = u_0 e^{-\int \frac{p(x)}{2}dx} e^{\int \frac{1}{x}dx}\left[\int e^{-2\int \frac{1}{x}dx}dx + c\right] + u_p(x)\int \frac{\int e^{\int p(x)dx} u_p(x)s(x)dx}{u_p(x)^2 e^{\int p(x)dx}}dx \quad (54)$$

Thus, according to (54) the general solution for (50) will be as follows:

$$u(x) = u_0 e^{-\int \frac{\tanh(x)}{2}dx} e^{\int (\frac{1}{x}dx}\left[\int e^{-2\int \frac{1}{x}dx}dx + c\right] + u_p(x)\int \frac{\int e^{\int \tanh x dx} u_p(x)\frac{1}{\sqrt{\cosh(x)}}dx}{u_p(x)^2 e^{\int \tanh x dx}}dx \quad (55)$$

The first term in the right hand sight of (55) is the general solution of the homogeneous part and simplifies to (56):

$$u(x) = \frac{A}{\sqrt{\cosh(x)}} + \frac{Bx}{\sqrt{\cosh(x)}} \quad (56)$$

for constants A and B.

If we substitute $u_p(x) = \frac{1}{\sqrt{\cosh(x)}}$ as a particular solution of the above equation in the (52) will be as follows:

$$y(x) = \left[\frac{A}{\sqrt{\cosh(x)}} + \frac{Bx}{\sqrt{\cosh(x)}}\right] + \frac{1}{\sqrt{\cosh(x)}}\int \frac{\int e^{\int \tanh(x)dx}\frac{1}{\sqrt{\cosh(x)}}\frac{1}{\sqrt{\cosh(x)}}dx}{(\frac{1}{\sqrt{\cosh(x)}})^2 e^{\int \tanh(x)dx}}dx \quad (57)$$

The final solution of (50) is (58):

$$y(x) = \frac{A}{\sqrt{\cosh(x)}} + \frac{Bx}{\sqrt{\cosh(x)}} + \frac{x^2}{2\sqrt{\cosh(x)}} \quad (58)$$

VI. Discussion

In Section II we found a particular solution for a particular nonhomogeneous second order linear differential equation to obtain the general solution in a closed form. The method introduced here applies to those second order linear ordinary differential equations for which a particular solution is available and the particular solution is such that the ensuing integrals can be evaluated. If this is a possibility, then a closed form solution is possible for those differential equations which satisfy these conditions. This method allows some of the important equations in physics and engineering such as hypergeometric [2], [3], harmonic oscillator [7] and Bessel equation of order one half [2], [3], that for decades has been solved by the method of series solution be solved in an explicit form.

As stated in the corollary of Section II and as shown in Section V, one more fundamental advantage of this method compared to the other methods, is that, knowing the solution of any single homogenous linear second order differential equation, the solution of innumerable linear homogenous and non-homogenous second order differential equations with the same $\mathcal{R}(x)$ function will be available already.

VII. Conclusion

The analytical solution of differential equations, if possible, is always desired. The advantage of having a closed form solution is that the analysis of the aspects of the subject of the differential equation becomes explicit and direct because the analysis will reduce to simple algebra even if the form is more entangled. What this presentation offers is a transformation of the second order linear ordinary nonhomogeneous differential equation to a form which is readily available for direct algebraic evaluation to the level of elementary calculus. The difficulty is the necessity of having access to a particular solution to implement the method. But the necessity of a particular solution is shared by almost all methods of resolution of differential equations.

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