World Academy of Science, Engineering and Technology
International Journal of Mathematical and Computational Sciences
Vol:18, No:9, 2024

Application of Legendre Transformation to Portfolio Optimization

Peter Benneth, Tsaroh N. Theophilus, Prince Benjamin

***Abstract*—**This research work aims at studying the application of Legendre Transformation Method (LTM) to Hamilton Jacobi Bellman (HJB) equation which is an example of optimal control problem. We discuss the steps involved in modelling the HJB equation as it relates to mathematical finance by applying the Ito's lemma and maximum principle theorem. By applying the LTM and dual theory, the resultant HJB equation is transformed to a linear Partial Differential Equation (PDE). Also, the Optimal Investment Strategy (OIS) and the optimal value function were obtained under the exponential utility function. Furthermore, some numerical results were also presented with observations that the OIS under exponential utility is directly proportional to the appreciation rate of the risky asset and inversely proportional to the instantaneous volatility, predetermined interest rate, risk averse coefficient. Finally, it was observed that the optimal fund size is an increasing function of the risk free interest rate. This result is consistent with some existing results.

Keywords**—**Legendre transformation method, Optimal investment strategy, Ito's lemma, Hamilton Jacobi Bellman equation, Geometric Brownian motion, financial market.

I. Introduction

THE HJB equation is a standard equation in the field of optimal control theory and explains the optimal feedback control policy for any dynamic system that seeks to optimize a certain objective function. This equation is a nonlinear PDE and is named after William Rowan Hamilton, Carl Gustav Jacob Jacobi, and Richard Bellman. It was first derived by Hamilton in [47], where he used the equation to study the motion of a particle in a conservative field. Bellman derived the HJB equation as a generalization of the dynamic programming principle [48]. The HJB equation describes the value function of an optimal control problem such that the value function gives the optimal value of the objective functions for any given state of the system. This equation can be derived using the optimality principle and can be solved using the LTM.

The LTM is a great tool for solving the HJB equation and have been used to solve a variety of problems in the area of optimal control which include; optimal control of mechanical systems, electrical systems, economic systems and financial system. One example of such a study is solving optimal control problems using the LTM by [1]; the authors used the LTM to solve optimal control problems with constraints on the control variables; they demonstrated the effectiveness of the method through numerical examples, and compare their results with those obtained using other numerical techniques. Several authors such as [2]-[33], all studied optimal control problem arising from investments strategies in financial market. In this paper, we developed a financial portfolio whose risky asset follows the geometric Brownian motion (GBM) and obtained an optimization problem in the form of HJB equation and show how the equation can be solved by applying the LTM and dual theory to obtain the investment strategy.

II. Literature Review

The LTM was first introduced by Bellman in his seminal work on dynamic programming and has since been used extensively in the field of optimal control theory [34]. The transformation involves introducing a new variable called the Legendre variable, which is defined as the exponential of the value function. The LTM is often used to deal with the complicated portfolio selection problems [35]-[37]. One of the advantages of the LTM is its ability to simplify the HJB equation and reduce the number of variables required for its solution. Here are some studies that have applied the LTM in the field of finance:

In [17], they considered a pension scheme with multiple contributors similar to [38] and solve the resultant HJB equation using LTM under exponential and power utility functions. The authors in [39] investigated the stochastic strategies for optimal investment in a defined contribution pension fund (DCPF). They extended the model of [38] from the constant rate of contribution to that with stochastic rate. Furthermore, they applied LTM and dual theory to find closed form solutions for CRRA and CARA utility functions respectively. They obtained a generalized solution for CARA utility function equivalent to the solution with one contributor as in [3] and a different result for CRRA utility function when compared to the one with one contributor. In [15], the authors studied the optimization of wealth investment strategies for DCPF with stochastic salary and extra contributors. They obtained an optimized equation using HJB equation, and then solved the equation using LTM to obtain the explicit solution of the OIS for constant absolute risk aversion (CARA) utility function.

In [40], the authors applied the LTM to solve the optimal constant rebalancing problem in finance; they derived a closed-form solution for the optimal rebalancing frequency by transforming the HJB equation. Also, [41] considered a class of stochastic optimal control problems with a nonlinear cost functional. They used the LTM to transform the HJB equation to a linear PDE that can be solved numerically using finite difference method. In [42], a numerical method for solving the

Peter Bennethis, Tsaroh Theophilus, and Prince Benjamin, graduate student, are with the Department of Mathematics and Statistics, Federal University Otuoke, Bayelsa State, Nigeria (e-mail: peterbenneth@gmail.com, neetsaroh934@gmail.com, princebenjamin130@gmail.com).

singular stochastic control problem using the LTM was presented. They showed that their method can efficiently solve singular stochastic control problems by iteratively solving the HJB equation associated with the problem.

In [43], the LTM was used to study investment and consumption problems and the closed form solutions of the optimal investment and consumption strategies under power and logarithm utility were obtained.

In [44], the authors assumed that the background risks that DC pension fund face include inflation risk and salary volatility risk. Furthermore, they used LTM and dual theory, and obtained a closed-form solution to this optimal problem. They pointed out that the most novel feature of their research was the application of these background risks. Also, they observed that the OIS is an increasing function of the contribution rate, stock risk premium and a decreasing function of stock volatility, which means that the pension fund administrators (PFAs) can buy less stock when the volatility of the stock is higher and vice versa. Since higher volatility implies higher stock risk, the PFAs will buy lesser stock [50]. According to [45], a numerical method was proposed for solving the HJB equation using the LTM.

These studies were also cited to show the application of LTM in solving HJB equations in different fields. The studies include [40] on finance, [41] on stochastic optimal control and [42] on singular stochastic control. The review concludes that LTM presents a promising approach for HJB equations.

III. Methodology and Model Formulations

A. Legendre Transformation and Dual Theory

The Legendre transform is used to convert a non-linear optimization problem into a linear one. By using the LTM, complex equations can be reformulated and its solution can be determined by solving a system of linear equations. The dual theory, on the other hand, is based on the idea that the value of the system can be expressed as the maximum expected reward of a portfolio of non-anticipative assets.

Theorem1. Let $f: R^n \to R$ be a convex function for $z > 0$, defined the Legendre transform $K(z) = \max_m \{ f(m) - zm \}$ where $K(z)$ is the Legendre dual of $f(x)$ [46].

Since $f(x)$ is convex, from theorem 1, we can define the Legendre transform as:

$$h(t,z) = \inf \left\{ \begin{matrix} m \mid V(t,m) \geq \\ zm + \hat{V}(t,m) \end{matrix} \right\} 0 < t < T \tag{1}$$

where $\hat{V}$ is the dual of V and $z > 0$ represents the dual variable of m. Then $h(t,z)$ is the value of m where this optimal value is attained such that h and $\hat{V}$ are very much related and is also called the dual variable of V [49]. Hence, $h(t,z)$ and $\hat{V}$ are related as follows

$$\hat{V}(t,z) = V(t,h) - zh$$

where

$$h(t,z) = m, V_m = z, h = -\hat{V}_z \tag{2}$$

At terminal time, we denote

$$\hat{U}(z) = \sup\{U(m) - zm \mid 0 < m < \infty\}$$

and

$$G(z) = \sup\left\{ m \mid U(m) \geq zm + \hat{U}(z) \right\}$$

As a result

$$G(z) = \left(U^1\right)^{-1}(z) \tag{3}$$

where G is the inverse of the marginal utility.

At terminal time T, we can define

$$h(T,m) = \inf_{m>0} \left\{ m \mid U(m) \geq zm + \hat{V}(t,z) \right\}$$

and

$$\hat{V}(t,z) = \sup_{m>0} \{U(m) - zm\}$$

So that

$$h(T,z) = \left(U^1\right)^{-1}(Z) \tag{4}$$

Next, we differentiate (2) with respect to t and x

$$V_t = \hat{V}_t, V_x = z, V_{mm} = \frac{-1}{\hat{V}_{zz}} \tag{5}$$

B. The Itô's Lemma

Generally, Itô's Lemma has been a valuable tool in mathematical finance, providing a framework for analysing and modelling financial processes that involve randomness. Its applications have led to significant advances in financial theory and practice, and its continued use in ongoing research indicates its continued relevance in the field.

Definition1. Let U_0 be a set of all open subsets in a topological space Ω and H_{U_0} a σ-algebra generated by set U_0. Then H_{U_0} is known as the Borel σ-algebra on Ω and the elements $B \in H_{U_0}$ are known as the Borel sets.

Definition2. $W_H(S,T)$ denotes the class of processes $h(t,\omega) \in$

$\Re$ satisfying:

1. $(t,\omega) \rightarrow h(t,\omega)$ is$B \times \mathcal{F}$-measurable, where B denotes the Borel σ-algebra on $[0,\infty)$;
2. There exists an increasing family of σ-algebras $H(t)$with $t \geq 0$, such that $B_0$is a martingale with respect to $H(t)$and that $h(t)$ is $H(t)$-adapted;
3. $\mathbb{p}\left[\int_S^T h(s,\omega)^2\, ds < \infty\right] = 1.$

Definition3. Let B_t be a one-dimensional Brownian motion on $(\Omega, \mathrm{F}, \mathbb{p})$. A (one-dimensional) Itô process (or stochastic integral) is a stochastic process $X(t)$on $(\Omega, \mathrm{F}, \mathbb{p})$ of the form

$$X(t) = X(0) + \int_S^T \alpha(s,\omega)\, ds + \int_S^T \sigma(s,\omega)\, dB_t \quad (6)$$

where $\sigma \in W_H$ so that

$$\mathbb{p}\left[\int_S^T \alpha(s,\omega)^2\, ds < \infty, \forall t \geq 0\right] = 1.$$

We also assume that u is $H(t)$ −adapted, where $H(t)$ is an increasing family of σ-algebras,

Assuming $X(t)$is an Itô process, the differential form of it is given as

$$dX(t) = \mu dt + \sigma dW(t) \quad (7)$$

where μ is the drift and σ, the standard deviation representing the instantaneous volatility.

Remark1. Let $X(t)$ be an Itô process given by $h(t, X(t)) \in C^2([0,\infty) \times \Re)$. Then $V = h(t, X(t))$ is also an Itô process and

$$dV = dh(t) = \begin{pmatrix} \frac{\partial h}{\partial t}(t, X(t))dt + \frac{\partial h}{\partial x}(t, X(t))dX(t) \\ +\frac{1}{2}\frac{\partial^2 h}{\partial w^2}(t, X(t))(dX(t))^2 \end{pmatrix}, \quad (8)$$

where $(dX(t))^2 = dX(t)dX(t)$ is computed according to the rules: $dtdt = dtdW = dWdt = 0; dWdW = dt.$

Remark2. For Ito processes $U(t)$ and $V(t)$ in $\Re$, Itô's product rule gives

$$d(U(t)V(t)) = U(t)dV(t) + V(t)dU(t) + dU(t)dV(t).$$

C. Wealth Formulation

In this section, we assume that the financial market is made up of a risk free asset (cash deposit) and a risky asset (stock). Let (Ω, F, P) be a complete probability space where Ω is a real space and P is a probability measure, and F is the filtration and denotes the information generated by the Brownian motion. Let S_1 denote the price of the risk free asset, it dynamics is given by

$$\frac{dS_1}{S_1} = Rdt \quad (9)$$

where R is a constant representing the predetermine interest rate of the risk free asset.

We suppose S_2 is the price process of the risky asset such that its dynamic follows the GBM model. Then the stochastic differential equation is given thus as stated [49]

$$\frac{dS_2}{S_2} = \mu dt + \sigma dw(t) \quad (10)$$

where μ is an expected instantaneous rate of return of the risky asset and satisfiers the general conditions $\mu > r_0$. σ is the instantaneous volatility.

The dynamic of the pension wealth is given by

$$dM = M\left(\pi\frac{dS_2}{S_2} + (1-\pi)\frac{dS_1}{S_1}\right) + Cdt \quad (11)$$

where M denotes the wealth and π is the fraction of the wealth invest in the risky asset and $(1-\pi)$, the fraction in risky free asset.

Substituting (9) and (10) into (11) we have

$$dM = \begin{bmatrix} \pi M(\mu dt + \sigma dw) \\ +M(1-\pi)(Rdt) + Cdt \end{bmatrix} \quad (12)$$

Simplifying (12), we have

$$dM = \begin{bmatrix} \left[M(\pi(\mu - R) + R) + C\right]dt \\ +M\sigma\pi dW(t) \end{bmatrix} \quad (13)$$

D. The HJB Equation

In this section, we are interested in maximizing the utility of the plan contributor's terminal relative wealth, our aim is to obtain the optimal value function

$$V(t,m) = \underset{\max}{E}\left(U(M(T))\right) \quad (14)$$

Next we maximize the value function subject to the pension found members wealth. In doing this we make use of the Ito's lemma and maximize principle and the optimal strategy π

$$dV = \frac{\partial v}{\partial t}dt + \frac{\partial v}{\partial M}dM + \frac{1}{2}\frac{\partial^2 v}{\partial M^2}(dM)^2 \quad (15)$$

Substituting (13) into (15), we have

$$dV = \begin{bmatrix} V_t dt \\ +V_m \begin{bmatrix} \left[M\begin{pmatrix} \pi(\mu - R) \\ +R \end{pmatrix} + C\right]dt \\ +M\sigma\pi dW(t) \end{bmatrix} \\ +\frac{1}{2}V_{mm}\left(M^2\sigma^2\pi^2 dt\right) \end{bmatrix} \quad (16)$$

Taking the limit of dV as $dt \to 0$

$$\frac{dV}{dt} = V_t + V_m\left[M\left[\pi\left(\mu - R\right) + R\right] + C\right] + \frac{1}{2}\sigma^2 M^2 \pi^2 V_{mm} = 0$$

$$V_t + \left[[M\left(\pi(\mu - R) + R\right) + C\right]V_m + \frac{1}{2}\left(M^2\sigma^2\pi^2 dt\right)V_{mm} = 0 \quad (17)$$

Differentiate (17) with respect to π

$$\pi = -\frac{(\mu - R)V_m}{\sigma^2 M V_{mm}} \quad (18)$$

Substitute (18) into (17), we have

$$V_t + (MR + C)V_m - \frac{1}{2}\frac{(\mu - R)^2}{\sigma^2}\frac{V_m^{\ 2}}{V_{mm}} = 0 \quad (19)$$

The resultant equation in (19) is the HJB equation for our problem which we intend to solve using LTM under exponential utility to obtain the OIS.

Since the differential equation obtained in (19) (HJB) is a nonlinear PDE, and quite complex to solve by direct method and some known techniques, we then employ the LTM to transform it to a linear PDE by substituting (5) into (18) and (19) where we have

$$\hat{V}_t + \left(MR + C\right)Z - \frac{1}{2}\frac{(\mu - R)^2}{\sigma^2}\left(Z^2 \div \left(-\frac{1}{\hat{V}_{zz}}\right)\right) = 0$$

$$\hat{V}_t + \left(MR + C\right)Z - \frac{1}{2}\frac{(\mu - R)^2}{\sigma^2}\left(Z^2\hat{V}_{zz}\right) = 0 \quad (20)$$

$$\pi = \frac{(\mu - R)\hat{V}_{zz}}{\sigma^2 M} \quad (21)$$

Let

$$V_t = \hat{V}_t,\ V_m = z,\ g(t,z) = M,\ V_m = Z,\ g = -\hat{V}_z \quad (22)$$

Substituting (22) into (20) and (21), we have

$$g_t + (Rg + C) + z(Rg_z) + \frac{1}{2}\frac{(\mu - R)^2}{\sigma^2}Z^2\hat{V}_{zzz}\frac{1}{2}\frac{(\mu - R)^2}{\sigma^2}\hat{V}_{zz}(2z) = 0$$

$$g_t + (Rg + C) + zRg_z - \frac{1}{2}\frac{(\mu - R)^2}{\sigma^2}Z^2 g_{zz} - \frac{(\mu - R)^2}{\sigma^2}Zg_z = 0$$

$$g_t + (Rg + C) + \left(R - \frac{(\mu - R)^2}{\sigma^2}\right)Zg_z - \frac{1}{2}\frac{(\mu - R)^2}{\sigma^2}Z^2 g_{zz} = 0$$

Then solving further

$$g_t - (Rg + C) - \left(R - \frac{(\mu - R)^2}{\sigma^2}\right)zg_z - \frac{1}{2}\left[\frac{(\mu - R)^2}{\sigma^2}\right]z^2 g_{zz} = 0 \quad (23)$$

$$\pi = -\frac{(\mu - R)}{\sigma^2 g}g_z \quad (24)$$

Next, we attempt to solve (23) and substitute it in (24) to obtain the optimal control strategy under exponential utility function.

IV. Main Result

Here, we consider an investor with utility function exhibiting CARA. Since our interest here is to determine the OIS, we use the exponential utility function as follows.

Let's assume an exponential utility

$$U(m) = -\frac{1}{q}e^{-qm},\ q > 0 \quad (25)$$

where m is the wealth of the investor and q is the risk averse coefficient of the investor.

The absolute risk aversion of a decision maker with the utility in (25) is a constant CARA utility. Since $g(T,z) = (U^1)^{-1}(z)$ and the CARA utility function we obtain

$$g(T,z) = -\frac{1}{q}\ln z \quad (26)$$

Next, we conjecture the solution to (23) as follows

$$g(t,z) = -\frac{1}{q}\left[a(t)\ln z + v(t)\right] + b(t) \quad (27)$$

with boundary conditions

$$a(T) = 1,\ \text{v}(T) = 0,\ b(T) = 0 \quad (28)$$

Differentiating (27), we have

$$g_t = -\frac{1}{q}\left[a^1(t)\ln z + v^1(t)\right] + b^1(t) \quad (29)$$

$$g_z = -\frac{1}{q}\left[\frac{a(t)}{z}\right] \quad (30)$$

$$g_{zz} = -\frac{1}{q}\left[\frac{a(t)}{z^2}\right] \quad (31)$$

Substituting (29)-(31) into (23), we have

$$-\frac{1}{q}\left[a^1(t)\ln z+v^1(t)\right]+b^1(t)-R\left[-\frac{1}{q}\left(a(t)\ln z+v(t)\right)+b(t)\right]$$
$$-c-\left(R-\frac{(\mu-R)^2}{\sigma^2}\right)Z\left(-\frac{1}{q}\right)\left(\frac{a(t)}{Z}\right)+\frac{1}{2}\frac{(\mu-R)^2}{\sigma^2}Z^2\left(\frac{a(t)}{qZ^2}\right)=0$$

$$-\frac{1}{q}\left[a^1(t)\ln z+v^1(t)\right]+b^1(t)-R\left[-\frac{1}{q}\left(a(t)\ln z+v(t)\right)+b(t)\right]$$
$$-c-\left(R-\frac{(\mu-R)^2}{\sigma^2}\right)\left(-\frac{1}{q}\right)\left(a(t)\right)+\frac{1}{2}\frac{(\mu-R)^2}{\sigma^2}\left(\frac{a(t)}{q}\right)=0$$

$$\begin{bmatrix}-\frac{a^1(t)\ln z}{q}-\frac{v^1(t)}{q}+b^1(t)+\frac{Ra(t)\ln z}{q}+\frac{Rv}{q}-Rb\\ -c+\frac{Ra(t)}{q}-\frac{(\mu-R)^2a(t)}{q\sigma^2}+\frac{(\mu-R)^2a(t)}{2\sigma^2q}\end{bmatrix}=0$$

$$\begin{bmatrix}-\frac{\ln z}{q}\left(a^1(t)-a(t)R\right)+b^1(t)-Rb-c\\ -\frac{1}{q}\left[v^1-Rv-a(t)R-a(t)\frac{(\mu-R)^2}{\sigma^2}\right]\end{bmatrix}=0 \quad (32)$$

Splitting (32), we have

$$a^1(t)-a(t)R=0 \quad (33)$$

$$b^1(t)-b(t)R=0 \quad (34)$$

$$V^1(t)-Rv(t)-a(t)\left(R-\frac{(\mu-R)^2}{\sigma^2}\right)=0 \quad (35)$$

Solving (33), we have

$$a=k_1e^{Rt} \quad (36)$$

Recalling boundary condition from (28), (36) now becomes

$$a(t)=e^{R(t-T)} \quad (37)$$

Solving (34), we have

$$b(\mathrm{t})=-\frac{c}{R}+k_2e^{Rt} \quad (38)$$

Using the boundary conditions in (28) we have

$$b(t)=\frac{c}{R}\left[e^{R(t-T)}-1\right] \quad (39)$$

From (35)

$$V^1(t)-Rv(t)-a(t)\left(R-\frac{(\mu-R)^2}{\sigma^2}\right)=0$$

Solving (35), we have

$$v(T)=e^{R(t-T)}\left(R-\frac{(\mu-R)^2}{\sigma^2}\right)T+k_3e^{RT} \quad (40)$$

Applying boundary conditions in (28), we have

$$v(t)=e^{R(t-T)}\left(R-\frac{(\mu-R)^2}{\sigma^2}\right)\left[t-T\right] \quad (41)$$

Substituting (37), (39) and (41) into (27), we have

$$g(t,z)=\begin{bmatrix}-\frac{1}{q}\begin{bmatrix}e^{R(t-T)}\ln z\\ +e^{R(t-T)}\left(R-\frac{(\mu-R)^2}{\sigma^2}\right)\left[t-T\right]\end{bmatrix}\\ +\frac{c}{R}\left[e^{R(t-T)}-1\right]\end{bmatrix} \quad (42)$$

Substituting (37) into (30), we have

$$\therefore g_z=-\frac{e^{R(t-T)}}{qz} \quad (43)$$

Substituting (43) into (24), we have

$$\Rightarrow \pi=-\frac{(\mu-R)}{\sigma^2m}\left(-\frac{e^{R(t-T)}}{qz}\right)$$

$$\therefore \pi=\frac{(\mu-R)}{\sigma^2mqz}e^{R(t-T)} \quad (44)$$

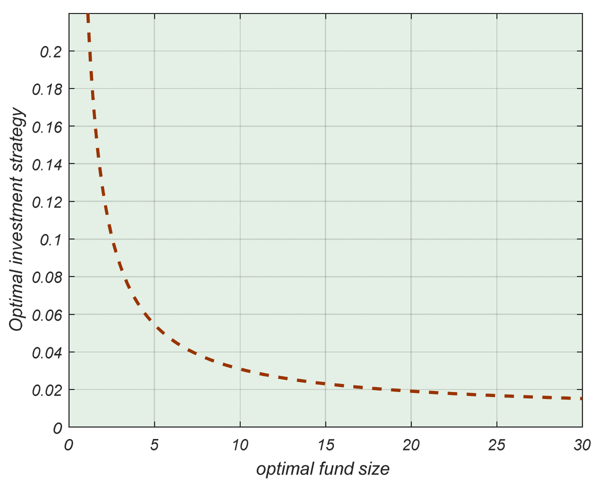

Fig. 1 Evolution of OIS with optimal fund size

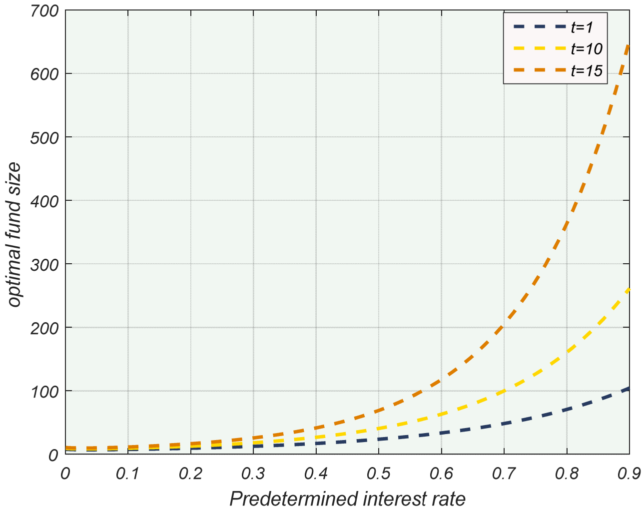

Fig. 2 Evolution of optimal fund size with predetermined interest rate

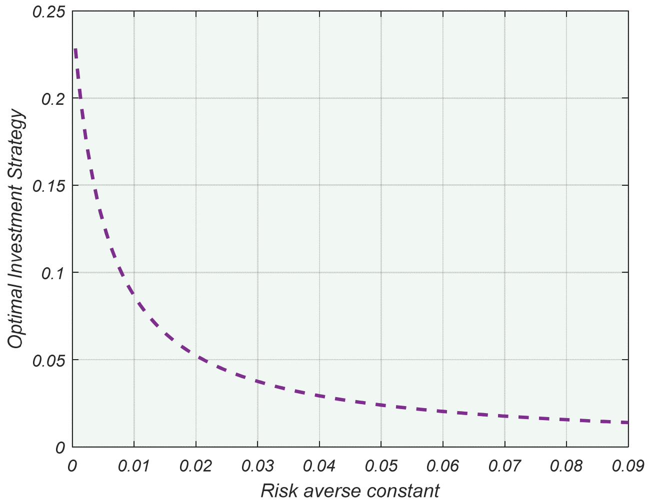

Fig. 3 Evolution of OIS with risk averse constant

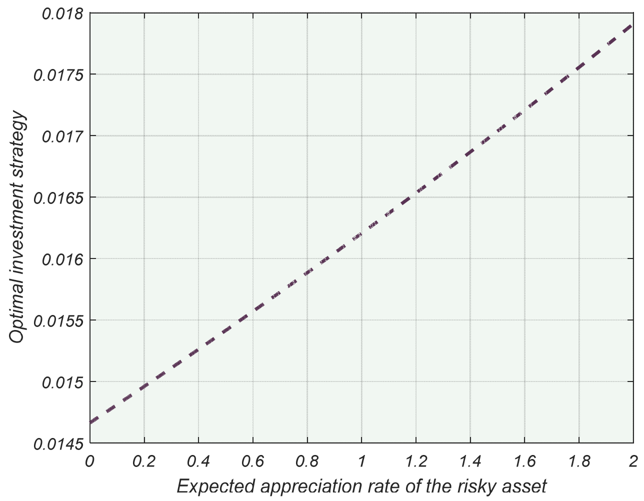

Fig. 4 Evolution of OIS with appreciation rate

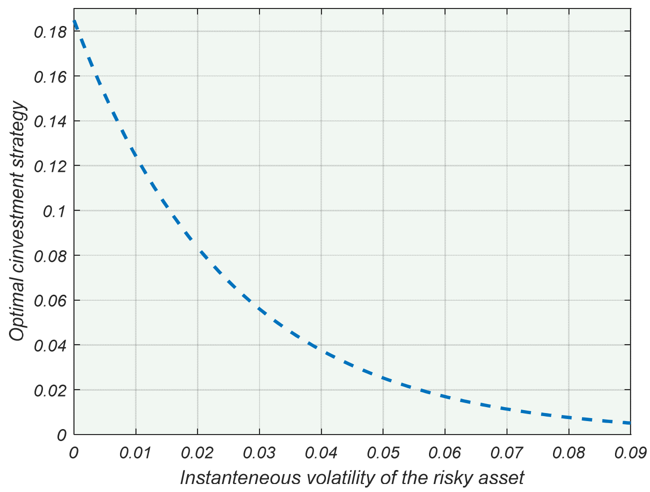

Fig. 5 Evolution of OIS with instantaneous volatility

V. Discussion

In Fig 1, the graph of OIS for the risky asset against the optimal fund size is presented; the graph shows that, OIS for the risky asset decrease as the optimal fund size increases. The implication of this graph is that members with more funds may prefer to invest less in the risky asset as retirement age approaches and vice versa.

In Fig. 2, the graph of the optimal fund size against the predetermined interest rate is presented; the graph shows that the optimal fund size is an increasing function of the predetermined interest rate. The implication of the graph is that members with large funds prefers to invest where there is less risk since they may not want to lose what they have gathered already.

In Fig. 3, the graph of OIS against the risk averse constant is presented; the graph shows that the OIS is inversely proportional to the risk averse coefficient. This implies that the more fearful the investor is to investment in risky asset, the more likelihood that he will reduce the proportion of his wealth to be invested in risky asset.

In Fig. 4, the graph of the OIS against the appreciation rate of the risky asset is presented; the graph shows that the OIS is an increasing function of the appreciation rate of the risky asset. The implication of the graph is that any asset with higher appreciation rate will naturally be appealing and attractive to an investor; hence the investor may be willing to commit more of his resources into such asset with the expectation of more returns and vice versa.

In Fig. 5, the graph of OIS against the instantaneous volatility is presented; the graph shows that the OIS is inversely proportional to the instantaneous volatility of the risky asset. This implies that the higher the instantaneous volatility of the risky asset, the higher the risk involved in the investment in such asset. Hence this may create more fears in the mind of the investor toward investing in the risky asset, furthermore reduce the proportion of the investor's wealth in risky asset.

VI. Conclusion

In this paper, we studied the Legendre transformation and dual theory as a very important method used in handling control problems most especially the HJB equations. This method helps reduce a nonlinear partial differential to a linear PDE which can easily be solved by some available methods. Aside from the application of the Legendre transformation theory, we formulated a portfolio for a given investor with one risk free asset (cash deposit) and one risky asset (stock) and used the Ito's lemma and maximum principle to obtain the HJB equation. With the application of LTM, dual theory and exponential utility function, the optimal value function and the OIS was obtained as seen in (42) and (44). The results obtained in (42) and (44) were simulated using MATLAB programming software and used to study the effect of some sensitive parameters on the OIS. It was observed that the OIS under exponential utility is directly proportional to the appreciation rate of the risky asset and inversely proportional to the instantaneous volatility, predetermined interest rate, risk averse coefficient. Also, we observed that the optimal fund size is an increasing function of the risk free interest rate. This result is consistent with some existing results.

Conflict of Interest

The authors declare that there is no conflict of interests regarding this paper.

Acknowledgment

The authors are very grateful to anonymous referees for their careful reading of the paper and helpful suggestions. This research has not received any sponsorship.

References

[1] Rao, K. S., and Nandy, S. (1992) Solving optimal control problems using the Legendre transformation method," Journal of Optimization Theory and Applications. 72, 1, 1-14.

[2] Gao J. (2008) Stochastic Optimal control of DC Pension funds. Insurance.42(3):1159-1164.

[3] Gao J., (2009). Optimal portfolios for DC pension plan under a CEV model. Insurance *Mathematics and Economics*, 44, 3, 479-490.

[4] Cao Y. & Wan N. (2009). Optimal proportional reinsurance and investment based on Hamilton-Jacobi-Bellman equation, *Insurance: Mathematics and Economics*, 45, 2. 157-162.

[5] Lin X. and Li Y. (2011). Optimal reinsurance and investment for a jump diffusion risk process under the CEV model, *North American Actuarial Journal*, 15, 3, 417-431.

[6] Gu, A., Guo, X., Li, Z. &Zeng, Y. (2012). Optimal control of excess-of-loss reinsurance and investment for insurers under a CEV model. *Insurance: Mathematics and Economics*, 51, 3. 674-684.

[7] Li, Q. and Gu, M. (2013). Optimization Problems of Excess-of-Loss Reinsurance and Investment under the CEV Model. *Mathematical Analysis*, 2013. 1-10.

[8] Egbe, G. A, Awogbemi, C. A. &Osu, B. O. (2013). Portfolio optimization of pension fund contribution in Nigeria. *Mathematical theory and Modelling*, 3, 8, 42-52.

[9] He L. & Liang. Z. (2013). The optimal investment strategy for the DC plan with the return of premiums clauses in a mean-variance framework, *Insurance*, 53, 643-649.

[10] Li D., Rong X. & Zhao H. (2014). Optimal reinsurance-investment problem for maximizing the product of the insurer's and the reinsurer's utilities under a CEV model, *Journal of Computational and Applied Mathematics*, 255, 671-683.

[11] Sheng, D. L. &Rong, X. M. (2014). Optimal time consistent investment strategy for a DC pension with the return of premiums clauses and annuity contracts, *Discrete Dynamics. Natural Science,*2014, 1-13

[12] Li D., Rong X. & Zhao H. (2015). Optimal investment problem for an insurer and a reinsurer, *Journal of Systems Science and Complexity*, 28, 6, 1326-1343.

[13] Ihedioha, S. A. (2015). Optimal Portfolios of an Insurer and a Reinsurer under Proportional Reinsurance and Power Utility Preference, *Open Access Library Journal*, 2, 12, 1-11.

[14] Sheng D. (2016). Explicit Solution of the Optimal Reinsurance-Investment Problem with Promotion Budget, *Journal of Systems Science and Information*, 4, 2, 131-148.

[15] Osu, B. O., Akpanibah, E. E. &Oruh, B. I. (2017). Optimal investment strategies for defined contribution pension fund with multiple contributors via Legendre transform and dual theory. 2, 2, 97-105.

[16] Akpanibah, E. E., Osu, B. O., Njoku, K.N. C. &Akak E. O. (2017). Optimization of Wealth Investment Strategies for a DC Pension Fund with Stochastic Salary and Extra Contributions. *International Journal of Partial Diff. Equations and Application*, 5,1, 33-41.

[17] Osu, B. O., Akpanibah, E. E. &Njoku, K.N.C. (2017). On the Effect of Stochastic Extra Contribution on Optimal Investment Strategies for Stochastic Salary under the Affine Interest Rate Model in a DC Pension Fund, *General Letters in Mathematics*, 2, 3, 138-149.

[18] Njoku, K.N. C, Osu, B. O. Akpanibah, E. E. &Ujumadu, R. N. (2017). Effect of Extra Contribution on Stochastic Optimal Investment Strategies for DC Pension with Stochastic Salary under the Affine Interest Rate Model. *Journal of Mathematical Finance*, 7, 821-833.

[19] Akpanibah, E. E. and Ogheneoro, O. (2018). Optimal Portfolio Selection in a DC Pension with Multiple Contributors and the Impact of Stochastic Additional Voluntary Contribution on the Optimal Investment Strategy. *International journal of mathematical and computational sciences*, 12(1): 14-19.

[20] Wang, Y., Rong, X. & Zhao, H. (2018). Optimal investment strategies for an insurer and a reinsurer with a jump diffusion risk process under the CEV model, *Journal of Computational and Applied Mathematics*, 328, 414-431.

[21] Osu, B.O., Akpanibah, E.E, & Olunkwa, O. (2018). Mean-Variance Optimization of portfolios with returns of premium clauses in DC pension plan with multiple contributors under constant elasticity of variance model. international journal of mathematical and computational sciences pure, 12(5), 85-90

[22] Chunxiang, A., Lai, Y & Shao, Y. (2018). Optimal excess-of-loss reinsurance and investment problem with delay and jump diffusion risk process under the CEV model, *Journal of Computational and Applied Mathematics*, 342, 317-336.

[23] E. E. Akpanibah, B. O. Osu. Optimal Portfolio Selection for a Defined Contribution Pension Fund with Return Clauses of Premium with Predetermined Interest Rate under Mean variance Utility. Asian Journal of Mathematical Sciences. 2(2), (2018), 19 –29.

[24] Deng, C., Bian, W. & Wu, B. (2019). Optimal reinsurance and investment problem with default risk and bounded memory. *International Journal of Control*, 2019, 1-113.

[25] Xiao, H. Ren, T. Bai, Y. & Zhou, Z. (2019). Time-consistent investment reinsurance strategies for the insurer and the reinsurer under the generalized mean-variance criteria. *Mathematics*, 7, 1-25.

[26] Akpanibah, E. E., Osu, B. O., Oruh, B. I. & Obi, C. N. (2019). Strategic optimal portfolio management for a DC pension scheme with return of premium clauses. *Transaction of the Nigerian association of mathematical physics*, 8, 1, 121-130.

[27] Mwanakatwe P. K., Wang X. &Su Y. (2019). Optimal investment and risk control strategies for an insurance fund in stochastic framework, *Journal of Mathematical Finance*, 9, 3, 254-265.

[28] Akpanibah, E. E., Osu, B. O. & Ihedioha, S. A. (2020). On the optimal asset allocation strategy for a defined contribution pension system with refund clause of premium with predetermined interest under Heston's volatility model. *Journal of. Nonlinear Science and Application,* 13, 1, 53-64.

[29] Malik M., Abas S., Sukono M. &Prabowo A. (2020). Optimal reinsurance and investment strategy under CEV model with fractional power utility function, *Engineering Letters*, 28, 4, 1-6.

[30] Zhu J. & Li S. (2020). Time-consistent investment and reinsurance strategies for mean- variance insurers under stochastic interest rate and stochastic volatility, *Mathematics*, 8, 2162-2183.

[31] E. E. Akpanibah, U. O. Ini. Portfolio strategy for an investor with logarithm utility and stochastic interest rate under constant elasticity of variance model, Journal of the Nigerian Society of Physical Sciences, 2

(3) (2020), 186-196.
[32] E. E.Akpanibah, U. O. Ini. An investor's investment plan with stochastic interest rate under the CEV model and the Ornstein-Uhlenbeck process, Journal of the Nigerian Society of Physical Sciences, 3 (3), (2021), 186-196.
[33] Amadi U. I., Ogbogbo, C. P. &Osu, B. O. (2022). Stochastic analysis of stock price changes as markov chain in finite states. *Global journal of pure and applied sciences*, 28, 1, 91-98.
[34] Bertsekas, D. P. (2012), Dynamic programming and optimal control. Athena Scientific 2012, 2(4)
[35] Xiao, J., Hong, Z., And C. (2007). The constant elasticity of variance (CEV) model and the Legendre transform dual solution for annuity contracts. Insurance: Mathematics and Economics, 40(2),302-310.
[36] Gao, J. (2012). Optimal investment and consumption with stochastic interest rate and inflation under power utility. Journal of Optimization Theory and Applications, 154(3), 937-956.
[37] Gao, J. (2013). Optimal consumption and investment with stochastic interest rate and inflation: Legendre transform method. Journal of Industrial and Management Optimization, 9(4), 767-785.
[38] Dawei G and Jingyi Z (2014). "Optimal investment strategies for defined contribution Pension funds with multiple contributors", http://ssrn.com/abstract=2508109
[39] Akpanibah, E. E. &Samaila, S. K. (2017). Stochastic strategies for optimal investment in a defined contribution (DC) pension fund. *International Journal of Applied Science and Mathematical Theory*, 3, 3, 48-55.
[40] Li, X., Zhang, X., & Li, D. (2021). A unified solution of optimal constant rebalancing problem by Legendre transformation. Journal of Systems Science and Complexity, 34(1), 127-148.
[41] Cheng, X., & Liu, H. (2016). On a class of stochastic optimal control problems with nonlinear cost functional. Journal of Industrial and Management Optimization, 12(1), 1-18
[42] Chen, Z., Li, X., & Yan, Y. (2019). A numerical method for solving singular stochastic control problems based on the Legendre transformation. Numerical Algebra, Control and Optimization, 9(1), 67-82.
[43] Chang, H., & Chang, K. (2014). Investment and consumption problem with stochastic interest rate: Legendre transform-dual solution. Journal of Industrial and Management Optimization, 10(1), 127-142.
[44] Bian, L, Li, Z., & Yao, H. (2021) Journal of Industrial & Management Optimization. 17(3), 1383-1410.
[45] Ohtsuka, H., & Takeishi, S. (2017). A numerical method for solving Hamilton-Jacobi-Bellman equations using the Legendre transformation. Applied Mathematics Letters, 69, 53-60.
[46] Jonsson, M. and Sircar, R. (2002). Optimal investment problems and volatility homogenization approximations. *Modern Methods in Scientific Computing and Applications NATO Science Series II*,75:255-281.
[47] Hamilton, W. R. (1834). On a general Method in Dynamics. Philosophical Transactions of the Royal Society, part II, 247-308.
[48] Bellman R. (1957). A Markovian decision process. Journal of Mathematics and Mechanics, 6, 679-684.
[49] Bright O. Osu, Edikan E. Akpanibah and Godswill A. Egbe (2018). Determination of Optimal Investment Strategies for a Defined Contribution pension Fund with Multiple Contributors, Proportional Administrative Cost and Taxation. *MATLAB Journal*, vol 1(1), 40 – 46.
[50] Liu, H. (2013). An Application of Legendre Transform-dual Solutions for DC Pension Funds Optimal Investment Strategy under Background Risk. Industrial Engineering and Engineering Management, https://en.cnki.com.cn/Article_en/CJFDTOTALGLGU201303021.htm