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Approximation to the Hardy Operator on Topological Measure Spaces

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Abstract—We consider a Hardy type operator generated by a family of open subsets of a Hausdorff topological space. The family is indexed with non-negative real numbers and is totally ordered. For this operator, we obtain two-sided bounds of its norm, a compactness criterion and bounds for its approximation numbers. Previously bounds for its approximation numbers have been established only in the one-dimensional case, while we do not impose any restrictions on the dimension of the Hausdorff space. The bounds for the norm and conditions for compactness have been found earlier but our approach is different in that we use domain partitions for all problems under consideration.

Keywords—Approximation numbers, boundedness and compactness, multidimensional Hardy operator, Hausdorff topological space.

I. Introduction

THE one-dimensional Hardy inequality

$$\left[\int_0^\infty u(x)\left(\int_0^x f\right)^q d\mu\left(x\right)\right]^{1/q} \le C\left(\int_0^\infty f^p v d\nu\right)^{1/p}$$

has been studied in detail and complete characterizations of its validity for all non-negative functions f have been obtained in terms of pairs of weights u, v and measures μ, ν for all pairs of exponents p, q, see [4]-[7], [9] for the history and extensive references.

In the one-dimensional case most researchers have used tools of one-dimensional calculus, such as integration by parts [16]. The lack of such tools has been the main obstacle on the way to multidimensional results. Some results have been established under simplifying assumptions (for $p \le q$ [3], by using spherical coordinates [14], [1] or the polar decomposition [12], [13] or assuming that the weights are products of functions of one variable [17], [11]).

Sinnamon [15] and Mynbaev [10] have obtained very general results in the multidimensional case, without simplifying assumptions of the type just mentioned. Our statements on the boundedness and compactness of the Hardy operator are less general than in these two papers (the compactness condition has been obtained by Sinnamon). However, the partitions method applied here has the advantage that, unlike [15] and [10], it does not use advanced properties of measures beyond σ-additivity and it opens the door to many generalizations of the results that are known in the one-dimensional case, whether they are based on partitions or not. The bounds for approximation numbers we give in the multidimensional case are new, see [2], [8].

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II. Main Results

We consider integration over expanding subsets $\Omega(t)$ of an arbitrary open set Ω in a Hausdorff topological space X with σ-additive Borel measures μ, ν. In the classical case one can notice that the subdomain $\Omega\left(t\right) = \left(0, t\right)$ of $\Omega = \left(0, \infty\right)$ has $\omega\left(t\right) = t$ as the boundary in the relative topology and that $\Omega\left(t\right) = \left\{s \in \Omega : \omega\left(s\right) < \omega\left(t\right)\right\}$. Our conditions on the family $\left\{\Omega\left(t\right)\right\}$ are based on this observation. Specifically, they are stated as follows:

Assumption a) $\{\Omega(t) : t \ge 0\}$ is a one-parametric family of open subsets of Ω which satisfy monotonicity: for $t_1 < t_2,\ \Omega(t_1)$ is a proper subset of $\Omega(t_2)$.

b) $\Omega(t)$ start at the empty set and eventually cover almost all Ω: $\Omega(0) = \cap_{t>0}\Omega(t) = \emptyset,\ \nu\left(\Omega\backslash \cup_{t>0}\Omega(t)\right) = 0$.

c) Further, denote $\omega(t) = \overline{\Omega(t)} \cap \overline{(\Omega\backslash\Omega(t))}$ the boundary of $\Omega(t)$ in the relative topology. We require the boundaries to be disjoint and cover almost all Ω: $\omega(t_1) \cap \omega(t_2) = \emptyset,\ t_1 \ne t_2,\ \nu(\Omega\backslash \cup_{t>0}\omega(t)) = 0$.

d) Passing to a different parametrization, if necessary, we can assume that $\nu\left(\Omega\backslash \cup_{t\le N}\omega(t)\right) > 0$ for any $N < \infty$.

e) Finally, we assume that boundaries are thin in the sense that $\nu(\omega(t)) = 0$ for all $t > 0$.

Operator T definition

The main implication is that for ν-almost each $x \in \Omega$ there exists a unique $\tau(x) > 0$ such that $x \in \omega(\tau(x))$, which allows us to define

$$Tf(x) = \int_{\Omega(\tau(x))} f d\nu,\ x \in \Omega,$$

for any non-negative $\mathfrak{M}$-measurable f. Thus the Hardy type inequality is

$$\left(\int_\Omega |Tf|^q u d\mu\right)^{1/q} \le C\left(\int_\Omega |f|^p v d\nu\right)^{1/p}.$$

$L^p_{vd\nu}(\Omega)$ denotes the space with the norm $\|f\|_{L^p_{vd\nu}(\Omega)} = \left(\int_\Omega |f|^p v d\nu\right)^{1/p}$ where v is a weight function. $\|T\| = \|T\|_{L^p_{vd\nu}(\Omega)\to L^q_{ud\mu}(\Omega)}$ is the norm of a linear operator T acting from $L^p_{vd\nu}(\Omega)$ to $L^q_{ud\mu}(\Omega)$ and $C = \|T\|$ is the best constant in

$$\left[\int_\Omega\left|\int_{\Omega(\tau(x))} f d\nu\right|^q u(x) d\mu(x)\right]^{1/q} \le C\left(\int_\Omega |f|^p\, v d\nu\right)^{1/p}.$$

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Put

$$\Psi(t)=\left(\int_{\Omega\setminus\Omega(t)}ud\mu\right)^{1/q}\left(\int_{\Omega(t)}v^{-p'/p}d\nu\right)^{1/p'}.$$

The notation $A\approx B$ means that the ratio A/B is bounded from above and below by some absolute constants.

Theorem 1. *If $1<p\le q<\infty$, then $C\approx A$ where $A=\sup_{t>0}\Psi(t)$.*

Let $0<q<p$, $1<p<\infty$ and put $1/r=1/q-1/p$,

$$\Phi(y)=\left(\int_{\Omega\setminus\Omega(\tau(y))}ud\mu\right)^{1/p}\left(\int_{\Omega(\tau(y))}v^{-p'/p}d\nu\right)^{1/p'}.$$

Theorem 2. *If $1<p<\infty$ and $0<q<p$, then $C\approx B$, where $B=\left(\int_\Omega \Phi^r ud\mu\right)^{1/r}$.*

Proof steps

- Discretize the problem using binary partitions.
- To discrete sums apply analogs of one-dimensional tools (like integration by parts and an integral of a full derivative).
- Estimate resulting discrete sums by integrals.

Here is an example of the property applied to discrete sums.

Lemma 1. *Let $a\ge 1$. We have*

$$\sum_{j\ge k}\left(\sum_{i\ge j+1}V_i\right)^{a-1}V_{j+1}\ge\frac{1}{a}\left(\sum_{i\ge k+1}V_i\right)^{a}$$

for any non-negative numbers V_i such that the left side is finite.

Result for the adjoint operator

Analogs of Theorems 1 and 2 hold for the adjoint operator T^*. Denote

$$\begin{aligned}\Psi^*(t)&=\left(\int_{\Omega(t)}ud\mu\right)^{1/q}\left(\int_{\Omega\setminus\Omega(t)}v^{-p'/p}d\nu\right)^{1/p'},\\ \Phi^*(t)&=\left(\int_{\Omega(\tau(y))}ud\mu\right)^{1/p}\left(\int_{\Omega\setminus\Omega(\tau(y))}v^{-p'/p}d\nu\right)^{1/p'}\end{aligned}$$

and consider the inequality

$$\left[\int_\Omega u(x)\left(\int_{\Omega\setminus\Omega(\tau(x))}fd\nu\right)^q d\mu(x)\right]^{1/q}\le C^*\left(\int_\Omega f^p vd\nu\right)^{1/p}.$$

Theorem 3. *Let the sets $\{\Omega(t)\}$, weight functions u,v and measures μ,ν satisfy the same conditions as before.*

1) If $1<p\le q<\infty$ then $C^*\approx A^*$ where $A^*=\sup_{t>0}\Psi^*(t)$.

2) If $0<q<p$, $1<p<\infty$ then $C^*\approx B^*$ where $B^*=\left(\int_\Omega(\Phi^*)^r ud\mu\right)^{1/r}$, $1/r=1/q-1/p$.

The next subject is compactness of T. The notation allows one to trace similarity with [15]. Denote

$$\begin{aligned}a(x)&=\int_{\Omega\setminus\Omega(x)}ud\mu,\\ b(x)&=\int_{\Omega(x)}v^{-p'/p}d\nu,\ 0<x<\infty,\\ l_i&=\limsup_{x\to i}a(x)^{1/q}b(x)^{1/p'},\ \text{for } i=0,\infty,\\ l&=\max\{l_0,l_\infty\}.\end{aligned}$$

Lemma 2. *Suppose that $a(x)<\infty$, $b(x)<\infty$ on $(0,\infty)$. If $l>\varepsilon>0$ then there exists a sequence $\{g_n\}$ such that $\|g_n\|_{L^p_{vd\nu}(\Omega)}=1$, $\|Tg_n-Tg_m\|_{L^q_{ud\mu}(\Omega)}>\varepsilon$.*

Theorem 4. *a) If $1<p\le q<\infty$, then T is compact if and only if $A<\infty$ and $l=0$. b) If $1<q<p$ and T is bounded, then T is compact.*

Our next task is to obtain bounds for approximation numbers (a-numbers) of the operator T. Let X,Y be two Banach spaces. For a bounded linear operator $T:X\to Y$ its n-th a-number, $n\in N$, is defined by

$$a_n(T)=\inf\{\|T-P\|\}$$

where the $\inf$ is taken over all bounded linear operators of $rankP<n$.

For $[a,b]\subseteq[0,\infty)$ we initially consider the question of how well the operator $\chi_{[a,b]}T$ is approximated by averages. To this end, successively define

$$\begin{aligned}\mu_u(\Omega[a,b])&=\int_{\Omega[a,b]}ud\mu,\\ \bar{T}f&=\frac{1}{\mu_u(\Omega[a,b])}\int_{\Omega[a,b]}(Tf)ud\mu, \qquad (1)\\ T_{[a,b]}f(x)&=\chi_{\Omega[a,b]}(x)\left(Tf(x)-\bar{T}_{[a,b]}f\right).\end{aligned}$$

Theorem 5 *Choose the point c so that*

$$\mu_u(\Omega[a,c])=\mu_u(\Omega[c,b])=\frac{1}{2}\mu_u(\Omega[a,b]).$$

a) Let $1<p\le q<\infty$ and denote

$$\begin{aligned}A^*[a,c]&=\sup_{a<\tau(x)<c}\left(\int_{\Omega[a,\tau(x)]}ud\mu\right)^{1/q}\\ &\quad\left(\int_{\Omega[\tau(x),c]}v^{-p'/p}d\nu\right)^{1/p'},\\ A[c,b]&=\sup_{c<\tau(x)<b}\left(\int_{\Omega[\tau(x),b]}ud\mu\right)^{1/q}\\ &\quad\left(\int_{\Omega[c,\tau(x)]}v^{-p'/p}d\nu\right)^{1/p'}.\end{aligned}$$

Then $\left\|T_{[a,b]}\right\|_{L^p_{vd\nu}(\Omega)\to L^q_{ud\mu}(\Omega)}\asymp\max\{A^*[a,c],A[c,b]\}$.

b) Let $1 < q < p < \infty$ *and put*

$$B^*[a,c] = \left[\int_{\Omega[a,c]}\left(\int_{\Omega[a,\tau(x)]} ud\mu\right)^{r/p}\left(\int_{\Omega[\tau(x),c]} v^{-p'/p}d\nu\right)^{r/p'} u(x)\,d\mu(x)\right]^{1/r},$$

$$B[c,b] = \left[\int_{\Omega[c,b]}\left(\int_{\Omega[\tau(x),b]} ud\mu\right)^{r/p}\left(\int_{\Omega[c,\tau(x)]} v^{-p'/p}d\nu\right)^{r/p'} u(x)\,d\mu(x)\right]^{1/r}.$$

Then $\left\|T_{[a,b]}\right\|_{L^p_{vd\nu}(\Omega)\to L^q_{ud\mu}(\Omega)} \asymp \max\{B^*[a,c], B[c,b]\}$.

Denote

$$\mathbf{A}[a,b] = \max\{A^*[a,c], A[c,b]\},\ 0 \le a < b \le \infty.$$

Obviously, for any $0 < x < \infty$ we have

$$\mathbf{A}[a,b] \to 0 \text{ if } a,b \to x;\ \mathbf{A}[a,b] > 0 \text{ if } a < b.$$

Everywhere below we assume that T is compact.

Lemma 3. *Let* $1 < p \le q < \infty$ *and* $0 < \varepsilon < \max\Psi$. *There exist points* $0 = t_0 < t_1 < ... < t_N < t_{N+1} = \infty$ *such that with the notation* $\Delta_k = [t_k, t_{k+1})$, $k = 0, ..., N$ *one has*

$$\begin{aligned} \sup_{t\in\Delta_0}\Psi(t) &= \varepsilon,\ \max_{k=1,...,N-2}\mathbf{A}(\Delta_k) = \varepsilon, \\ \mathbf{A}(\Delta_{N-1}) &\le \varepsilon,\ \sup_{t\in\Delta_N}\Psi(t) = \varepsilon. \end{aligned} \tag{2}$$

With $\Omega_k = \Omega(\Delta_k)$ put for $k = 1, ..., N-1$

$$\begin{aligned} T_k f(x) &= \int_{\Omega(\tau(x))\setminus\Omega(t_k)} fd\nu,\ \mu_u(\Omega_k) = \int_{\Omega_k} ud\mu, \\ \bar{T}_k f &= \frac{1}{\mu_u(\Omega_k)}\int_{\Omega_k}(Tf)ud\mu, \end{aligned}$$

$$\begin{aligned} P_k f(x) &= \chi_{\Omega_k}(x)\left\{Tf(x) - \left[T_k f(x) - \bar{T}_k f\right]\right\} \quad (3) \\ &= \chi_{\Omega_k}(x)\left\{\int_{\Omega(t_k)} fd\nu + \bar{T}_k f\right\}, \end{aligned}$$

$$P_0 f = 0,\ P_N f(x) = \chi_{\Omega_N}(x)\int_{\Omega(t_N)} fd\nu.$$

Each of P_k is one-dimensional, so $P = \sum_{k=1}^N$ has rank $rankP \le N$.

We use the approach developed in [2].

Theorem 6. *Let* $1 < p \le q < \infty$ *and suppose the covering* $\{\Omega_k : k = 0, ..., N\}$ *satisfies* (2). *Then*

$$c_1\varepsilon(N-2)^{1/q-1/p} \le a_{N-1}(T),\ a_{N+1}(T) \le c_2\varepsilon. \tag{4}$$

Remark. Obviously, when $p = q$, (4) gives a same-order two-sided bound for a-numbers. Besides, the upper bound on a-numbers gives an upper bound for the Gelfand, Kolmogorov and entropy numbers because the a-numbers are the largest among s-numbers of linear operators.

To consider the case $1 < q < p < \infty$ we assume that $\|T\| < \infty$ and therefore $B < \infty$ by Theorem 2. Denote

$$\begin{aligned} \Phi^*_{[a,b]}(x) &= \left(\int_{\Omega[a,\tau(x)]} ud\mu\right)^{1/p}\left(\int_{\Omega[\tau(x),b]} v^{-p'/p}d\nu\right)^{1/p'}, \\ \Phi_{[a,b]}(x) &= \left(\int_{\Omega[\tau(x),b]} ud\mu\right)^{1/p}\left(\int_{\Omega[a,\tau(x)]} v^{-p'/p}d\nu\right)^{1/p'} \\ \mathbf{B}[a,b] &= \left[\int_{\Omega[a,b]}\left(\Phi^*_{[a,c]}\chi_{[a,c]} + \Phi_{[c,b]}\chi_{[c,b]}\right)^r ud\mu\right]^{1/r} \end{aligned}$$

where $c = c(a,b)$ is the constant from Theorem 5.

Bound from above Let $0 < \varepsilon < B$. Select t', t'' to satisfy

$$\begin{aligned} \left(\int_{\Omega(t')}\Phi^r ud\mu\right)^{1/r} &= \varepsilon, \quad (5) \\ \left(\int_{\Omega[t'',\infty]}\Phi^r ud\mu\right)^{1/r} &= \varepsilon. \end{aligned}$$

Let $\{\Delta_k : k = 1, ..., N\}$ be a uniform (and finite) partition of $[t', t'']$ into segments Δ_k of length m. From the bound

$$\sum_{k=1}^N \mathbf{B}(\Delta_k)^r \le \max_k \sup_{\tau(x)\in\Delta_k}\left(\Phi^*_{\Delta_k}(x) + \Phi_{\Delta_k}(x)\right)\int_{\Omega[t',t'']} ud\mu$$

we see that m can be chosen so that

$$\left(\sum_{k=1}^N \mathbf{B}(\Delta_k)^r\right)^{1/r} = \varepsilon. \tag{6}$$

With definitions (3) and putting $\Omega_k = \Omega(\Delta_k)$ we have for each k by Theorem 1

$$\left(\int_{\Omega_k}|Tf - P_k f|^q ud\mu\right)^{1/q} \le c\mathbf{B}(\Delta_k)\left(\int_{\Omega_k}|f|^p vd\nu\right)^{1/p}.$$

By Theorem 2 (5) implies

$$\begin{aligned} \left(\int_{\Omega(t')}|Tf|^q ud\mu\right)^{1/q} &\le c\varepsilon\left(\int_{\Omega(t')}|f|^p vd\nu\right)^{1/p}, \\ \left(\int_{\Omega[t'',\infty]}|Tf|^q ud\mu\right)^{1/q} &\le c\varepsilon\left(\int_{\Omega[t'',\infty]}|f|^p vd\nu\right)^{1/p}. \end{aligned}$$

We use definitions of $P_0, ..., P_N$ from Theorem 5. Put $P = \sum_{k=1}^N P_k$. The last three estimates and (6) give

$$\left(\int_\Omega |Tf - Pf|^q ud\mu\right)^{1/q} \le c\varepsilon\left(\int_\Omega |f|^p vd\nu\right)^{1/p}.$$

Since $rankP \le N$ this proves that $a_{N+1}(T) \le c\varepsilon$. Thus, with the partition defined above we have

Theorem 7. *Suppose* $1 < q < p < \infty$, T *is bounded and* $0 < \varepsilon < B$. *Then* $a_{N+1}(T) \le c\varepsilon$.

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Bound from below. Let t', t'' be chosen as in (5) and put $t_0 = 0$, $t_1 = t'$. On the n-th step, if $\sup_{t>t_n} \mathbf{B}(t_n, t) \geq \varepsilon$ then we put $t_{n+1} = \min\{t > t_n : \mathbf{B}(t_n, t) = \varepsilon\}$. If $\sup_{t>t_n} \mathbf{B}(t_n, t) < \varepsilon$ we put $t_{n+1} = \infty$. This process stops in a finite number of steps. Suppose that it does not and that $t_n \to t \leq \infty$. Then

$$\max\left\{\Phi^*_{[a,b]}(x), \Phi_{[a,b]}(x)\right\}$$
$$\leq \left(\int_{\Omega[t',\infty]} u d\mu\right)^{1/p} \left(\int_{\Omega(t'')} v^{-p'/p} d\nu\right)^{1/p'} = c$$

for $t' \leq a < b \leq t''$.

Hence, for each k, $\varepsilon^r = \mathbf{B}(\Delta_k)^r \leq (2c)^r \int_{\Omega(\Delta_k)} u d\mu$, $\sum_k \int_{\Omega(\Delta_k)} u d\mu = \infty$, which contradicts compactness.

Denoting N the total number of segments, for an arbitrary bounded linear operator $P : L^p_{vd\nu} \to L^q_{ud\mu}$, $rank P < N - 1$, we have the following statement:

Theorem 8 *Suppose* $1 < q < p < \infty$, T *is bounded and* $0 < \varepsilon < B$. *Then* $a_{N-1}(T) \geq c\varepsilon$.

III. Conclusion

This research shows that domain partitions can be productively used in the current setup to obtain generalizations of many one-dimensional results.

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