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The Analogue of a Property of Pisot Numbers in Fields of Formal Power Series

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Abstract—This study delves into the intriguing properties of Pisot and Salem numbers within the framework of formal Laurent series over finite fields, a domain where these numbers' spectral characteristics, $\Lambda_m(\beta)$ and $l^m(\beta)$, have yet to be fully explored. Utilizing a methodological approach that combines algebraic number theory with the analysis of power series, we extend the foundational work of Erdos, Joo, and Komornik to this setting. Our research uncovers bounds for $l^m(\beta)$, revealing how these depend on the degree of the minimal polynomial of β and thus offering a characterization of Pisot and Salem formal power series. The findings significantly contribute to our understanding of these numbers, highlighting their distribution and properties in the context of formal power series. This investigation not only bridges number theory with formal power series analysis but also sets the stage for further interdisciplinary research in these areas.

Keywords—Pisot numbers, Salem numbers, Formal power series, Minimal polynomial degree.

I. Introduction

PISOT and Salem numbers occupy a distinctive place in number theory, known for their unique algebraic and spectral properties. These numbers are integral to various mathematical disciplines, including algebraic number theory and dynamical systems, due to their intriguing characteristics and the complex problems they present. The study of Pisot and Salem numbers has evolved over the years, leading to a rich body of knowledge that explores their applications and theoretical significance. Formal power series and Laurent series over finite fields provide a fertile ground for advancing this exploration, offering new perspectives and challenges in understanding these numbers.

In recent developments, the analysis of sequences and the distribution of specific number types have garnered attention, leading to significant findings about the structure and properties of mathematical constructs. The foundational work of Erdos et al. [1] has paved the way for deepening our understanding of these areas, particularly in the context of real numbers and their formal series representations.

In the real numbers case many authors studied the sequence $\{y_{k+1} - y_k\}_{k\geq 0}$ where $(y_k)_k$ is an increasing sequence $0 = y_0 < y_1 < \ldots, m \geq 1$ of numbers of the form $y = \varepsilon_n q^n + \cdots + \varepsilon_0$ with $n \geq 0$ et $\varepsilon_i \in \{0, \ldots, m\}$. Erdos et al. [1] were the first to analyse the properties of the sequence $\{y_{k+1} - y_k\}_{k\geq 0}$. They proved that for $q > A$ (where A is the golden number: the root > 1 of the equation $X^2 - X - 1$), there exists an infinity of k such that $y_{k+1} - y_k = 1$. Furthermore, for $q < A$ and q a Pisot number, they proved that $y_{k+1} - y_k$ does not converge to 0.

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Furthermore, in [2], Erdos and Joo proved that $y_{k+1} - y_k \leq 1$ for all k. And if q is a Pisot number satisfying the equation $X^r - X^{r-1} - \ldots - X - 1 = 0$ with $r > 1$, then for $m = 1$, $\inf_k (y_{k+1} - y_k) = \frac{1}{q}$. The analogue of our work in the real numbers case was treated in [3], [4], [1] and [5]. In [3], Bugeaud has proved for $1 < q < 2$ that q is a Pisot number if and only if $l^m(q) > 0$ for all m where

$$l^m(q) = \inf\{|y| : y \in \Lambda_m, y \neq 0\}, \quad m = 1, 2, \cdots$$

and

$$\Lambda_m = \{y = \varepsilon_n q^n + \varepsilon_{n-1} q^{n-1} + \cdots + \varepsilon_1 q + \varepsilon_0 : -m \leq \varepsilon_i \leq m\}.$$

In the same way, Erdos et al. [4] proved that if $q < A$ then $l^2(q) > 0$ if and only if q is a Pisot number where A is the golden number.

Erdos et al. in [1] and [8] another proof of the result of Bugeaud [3]. In [6], Komornik et al. had determined the exact values of $l^m(q)$ for some Pisot number q. They started by giving a simpler proof to the theorem of Erdos et al. [2]. Next, they proved that $l^1(q) = q^2 - 2$ if q is the Pisot number solution of the equation $X^3 - X^2 - 1$. They ended by showing that for $A^{k-2} < m \leq A^{k-1}$ and $k \geq 1$, $l^m(A) = \mid F_k A - F_{k+1} \mid$ where A is the golden number and $(F_k)_k$ is the Fibonacci sequence ($F_0 = 0$, $F_1 = 1$ and $F_k = F_{k-1} + F_{k-2}$ for $k = 2, 3, \cdots$).

In [7] and [9], Feng and Wen proved that the sequence $\{y_{k+1} - y_k\}_{k\geq 0}$ can take an infinity of different values for q a Pisot number and $m \geq q - 1$. They even gave an algorithm which determinate the value of $\inf_k (y_{k+1} - y_k)$.

A. Pisot and Salem Numbers in Formal Power Series

Let $\mathbb{F}$ be a finite field with characteristic $p > 0$. We denote by $\mathbb{F}[X]$ the ring of polynomials over $\mathbb{F}$ and $\mathbb{F}(X)$ the field of rational fractions with coefficients in $\mathbb{F}$. We equip $\mathbb{F}(X)$ with the metric absolute value $\mid \cdot \mid$ defined by $\mid \frac{A}{B} \mid = q^{\deg(A) - \deg(B)}$ for $A, B \in \mathbb{F}[X]$, $B \neq 0$ and $\mid 0 \mid = 0$ which confer to $\mathbb{F}(X)$ a structure of metric field where the complete is identified to the formal power series field X^{-1} denoted $\mathbb{F}((X^{-1}))$ An element $f \in \mathbb{F}((X^{-1}))$ takes the form:

$$f = \sum_{i=n_0}^{+\infty} a_i X^{-i} \quad with \quad a_i \in \mathbb{F}, \quad n_0 \in \mathbb{Z} \quad and \quad a_{n_0} \neq 0.$$

Hence we define $\deg(f) = -n_0$ and $\mid f \mid = q^{\deg(f)}$, we note $[f]$ the polynomial part of f and $\{f\} = f - [f]$ its fractional part.

A formal power series f is algebraic over $\mathbb{F}[X]$, if it is a root of a monic irreducible polynomial over $\mathbb{F}[X]$.

An algebraic formal power series f with absolute value > 1 is said to be Pisot (respectively, Salem) if all its conjuguates have an absolute value < 1 (resp. ≤ 1 and at least one conjuguate with absolute value $= 1$).

A characterization of Pisot and Salem series is given by the following theorem Batemann and Duquette:

Theorem 1. *Let $\beta \in \mathbb{F}((X^{-1}))$ be algebraic such that $|\beta| > 1$ and its minimal polynomial is $P(Y) = Y^d + A_{d-1}Y^{d-1} + \cdots + A_0$, where $A_i \in \mathbb{F}[X]$ for $i = 0, \cdots, d-1$. Then β is a Pisot* (resp. Salem) if and only if $|A_{d-1}| > \max\limits_{k\neq d-1} |A_k|$ (resp. $|A_{d-1}| = \max\limits_{k\neq d-1} |A_k|$).

Theorem 2. *Let $\beta \in \mathbb{F}((X^{-1}))$ satisfying $|\beta| > 1$. Then β is a Pisot if and only if there exists $\lambda \in \mathbb{F}((X^{-1})) \setminus \{0\}$ such that $\lim\limits_{n\to+\infty} \{\lambda\beta^n\} = 0$.*

B. β-Development of a Formal Series

The β-development is a generalization of the standard form of a formal power series in $\mathcal{B}_0$ in base X with another base in $\beta \in \mathbb{F}((X^{-1}))$ with $|\beta| > 1$.

Definition 1. *Given a formal power series β such that $|\beta| > 1$. We call β-representation of f ($f \in \mathcal{B}_0$), every sequence $(a_i(X))_{i\geq 1}$ of $\mathbb{F}[X]$ such that*

$$f = \sum_{i=1}^{\infty} \frac{a_i(X)}{\beta^i}.$$

The particular development that we obtain with the following algorithm:

$$\begin{cases} \gamma_1 = \beta f, & a_1(X) = [\gamma_1], \\ \gamma_n = \beta\{\gamma_{n-1}\} & \text{and} \quad a_n(X) = [\gamma_n], \quad \text{for } n \geq 2 \end{cases} \tag{1}$$

is called β-development of f which will be denoted by $d_\beta(f)$.

Remark 1. *It is obvious that $\deg(a_i) < \deg(\beta)$, for $i \geq 2$ and $\deg(a_1) \leq \deg(\beta)$.*

The object of this manuscript is the studying of $\Lambda_m = \Lambda_m(\beta)$ the set of formal power series ω with at least one representation of the form

$$\omega = \varepsilon_n\beta^n + \varepsilon_{n-1}\beta^{n-1} + ... + \varepsilon_1\beta + \varepsilon_0$$

for $n \in \mathbb{N}$ such that $\deg(\varepsilon_i) \leq m$, for all $0 \leq i \leq n$ and the value of:

$$l^m(\beta) = \inf\{|\omega| : \omega \in \Lambda_m, \omega \neq 0\}.$$

Remark 2. *We give some properties of the set Λ_m.*

i) *Let $t \in \mathbb{N}$, for all $s \in \mathbb{N}$ and $v \in \Lambda_t$ we have $\beta^s v \in \Lambda_t$.*
ii) *For all m, $t \in \mathbb{N}$, with $m \leq t$, we have*

$$\Lambda_m \subset \Lambda_t, \quad \Lambda_m \pm \Lambda_t = \Lambda_{\max(m,t)} \quad \text{and} \quad \Lambda_t\Lambda_m \subset \Lambda_{m+t}.$$

The main object of this paper is to prove the following two theorems which characterize Pisot and Salem formal power series, β, using $l^m(\beta)$ and give the values of $l^m(\beta)$ when β is a Pisot or a Salem formal power series.

Theorem 3. *Let β be a Pisot or a Salem formal power series and d be the degree of its minimal polynomial. Then, for $m \in \mathbb{N}$, we have*

i) $l^m(\beta) = 1$ *if* $m < \deg(\beta)$,
ii) $l^m(\beta) \geq |\beta|^{-(m-\deg(\beta)+1)(d-1)}$ *if* $m \geq \deg(\beta)$.

Theorem 4. *Let β be a Pisot formal power series such that $|\beta| > 1$ and satisfying the algebraic equation*

$$\beta^d + A_{d-1}\beta^{d-1} + \cdots + A_0 = 0,$$

with $\deg(A_{d-1}) = 1$ and $A_i \in \mathbb{F}$, $i \neq d-1$. Then β is a Pisot formal power series and d is the degree of its minimal polynomial. Moreover

$$l^m(\beta) = |\beta|^{-m(d-1)}.$$

It is noted that the theorems give an analogue result of Bugeaud [3] in the real numbers case.

II. Proof of Theorem 3

The proof of Theorem 3 is based on the following lemmas:

Lemma 1. *Let $P(Y) = A_dY^d + \cdots + A_0$, with $A_i \in \mathbb{F}[X]$, $A_d \neq 0$ and $|A_{d-1}| > |A_i|$, for all $i \neq d-1$. Then P has a unique root β over $\mathbb{F}((X^{-1}))$ with absolute value > 1. Moreover, $[\beta] = -[\frac{A_{d-1}}{A_d}]$.*

Proof: b Suppose that $P(Y) = A_dY^d + \cdots + A_0$, where for all $i = 0, \cdots, d$ we have $A_i \in \mathbb{F}[X]$. by substitution in P the variable $Y = \lambda Z$, where $\lambda = -\frac{A_{d-1}}{A_d}$, we obtain

$$\frac{-1}{A_{d-1}\lambda^{d-1}P(\lambda Z)} = Z^{d-1}(Z-1) + L(Z),$$

where $L(Z)$ is a polynomial where every coefficient is of absolute value < 1. It leds to Hensel's Lemma [7] for the polynomial $P(Y)$ has a unique root in $\mathbb{F}((X^{-1}))$ of the form $\beta = \lambda z$, with $|z-1| < 1$. And, for such z, we have $|z| = 1$ and since the coefficients of L are with absolute value $< \frac{1}{\lambda}$, we have $|z-1| = |L(z)| < \frac{1}{\lambda}$. Then P is a unique root $\beta = \lambda z$ such that $|\beta - \lambda| < 1$, and $[\beta] = [\lambda]$. ■

Lemma 2. *Let β be a Pisot formal power series and d be the degree of its minimal polynomial. Then if $m \geq \deg(\beta)$, we have*

$$\Lambda_m \subset \frac{1}{\beta^{d-1}}\Lambda_{m-1}.$$

Proof: Let β a Pisot formal power series, then β satisfies an algebraic equation of the form

$$\beta^d + A_{d-1}\beta^{d-1} + \ldots + A_0 = 0 \tag{2}$$

with

$$\deg(A_{d-1}) > \max_{j\neq d-1} \deg(A_j). \tag{3}$$

Let $v \in \Lambda_m$, then $v = \varepsilon_n\beta^n + \cdots + \varepsilon_0$, with $\deg(\varepsilon_i) \leq m$, for $0 \leq i \leq n$. The Euclidean divisions of ε_i by A_{d-1}, we obtain

$$\varepsilon_i = A_{d-1}\varepsilon_i' + \varepsilon_i''$$

with

$$\deg(\varepsilon_i') = \deg(\varepsilon_i) - \deg(A_{d-1}) \leq m - \deg(A_{d-1})$$

and
$$\deg(\varepsilon_i'') < \deg(A_{d-1}).$$

We get
$$v = \sum_{i=0}^{n}(A_{d-1}\varepsilon_i' + \varepsilon_i'')\beta^i = A_{d-1}v' + v'', \tag{4}$$
where
$$v' = \sum_{i=0}^{n}\varepsilon_i'\beta^i \in \Lambda_{m-\deg(A_{d-1})} \text{ and } v'' = \sum_{i=0}^{n}\varepsilon_i''\beta^i \in \Lambda_{\deg(A_{d-1})-1}. \tag{5}$$

Using (4) and (2), it comes
$$\begin{aligned} v &= \frac{\beta^{d-1}A_{d-1}v' + \beta^{d-1}v''}{\beta^{d-1}} \\ &= \frac{-(\beta^d + A_{d-2}\beta^{d-2} + \cdots + A_1\beta + A_0)v' + \beta^{d-1}v''}{\beta^{d-1}}. \end{aligned}$$

By (3), (5), Remark 2 and the Lemma 1 we obtain $\deg(A_{d-1}) = \deg(\beta)$, and
$$v \in \frac{1}{\beta^{d-1}}\Lambda_{m-1}.$$

■

Lemma 3. *Let β be a Salem formal power series and d be the degree of its minimal polynomial. Let $m \geq \deg(\beta)$, then*
$$\Lambda_m \subset \frac{\Lambda_{m-1}}{B},$$
where $B \in \Lambda_0$ et $\mid B \mid = \mid \beta \mid^{d-1}$.

Proof: Let β be a Salem formal power series, then β satisfy the equation
$$\beta^d + A_{d-1}\beta^{d-1} + \ldots + A_0 = 0, \tag{6}$$
such that
$$\deg(A_{d-1}) = \max_{i \neq d-1}\deg(A_i). \tag{7}$$

Let $A_{p_1}, \ldots, A_{p_k}$, be the coefficients of (6) satisfying
$$\deg(A_{p_j}) < \deg(A_{d-1}) \quad \text{with} \quad 0 \leq p_j \leq d \quad \text{and} \quad j \in \{1, \ldots, k\}$$
and we denote by $A_{d_1}, \ldots, A_{d_s}$ the other coefficients satisfying
$$\deg(A_{d_t}) = \deg(A_{d-1}) \quad \text{with} \quad 0 \leq d_t \leq d-1 \quad \text{and} \quad t \in \{1, \ldots, s\}.$$
Then for every $t \in \{1, \ldots, s\}$, there exists $\lambda_t \in \mathbb{F} \setminus \{0\}$ and $\mu_t \in \mathbb{F}[X]$ with $\deg(\mu_t) < \deg(A_{d-1})$, such that
$$A_{d_t} = \lambda_t A_{d-1} + \mu_t. \tag{8}$$
It is clear that (6) is equivalent to
$$A_{d_1}\beta^{d_1} + \cdots + A_{d_s}\beta^{d_s} = -(A_{p_1}\beta^{p_1} + \cdots + A_{p_k}\beta^{p_k}). \tag{9}$$
We replace in (9) the A_{d_t} by their expressions which appears in (8), we obtain
$$BA_{d-1} = C, \tag{10}$$
where
$$\begin{cases} B = & \lambda_1\beta^{d_1} + \cdots + \lambda_s\beta^{d_s}, \\ C = & -(A_{p_1}\beta^{p_1} + \cdots + A_{p_k}\beta^{p_k}) - (\mu_1\beta^{d_1} + \cdots + \mu_s\beta^{d_s}), \end{cases}$$
it is clear that
$$B \in \Lambda_0, \quad |B| = |\beta|^{d-1} \quad (d-1 \in \{d_1, \ldots, d_s\}) \quad \text{and}$$
$$C \in \Lambda_{\deg(A_{d-1})-1}. \tag{11}$$
Let now $v \in \Lambda_m$, then $v = \varepsilon_n\beta^n + \cdots + \varepsilon_0$, with $\deg(\varepsilon_i) \leq m$ for $0 \leq i \leq n$. From the first part of the proof of Lemma 2 represented by the relation (4), we can write v in the form $v = v'A_{d-1} + v''$ where v', v'' are the expressions giving by (5), what gives with (10),
$$v = \frac{v'C + v''B}{B}.$$
Using (5), (II) and Remark 2, we obtain
$$v \in \frac{\Lambda_{m-1}}{B}.$$

■

We are now ready to give the proof of Theorem 3:

i) Let β be a formal power series such that $|\beta| > 1$ and $\omega \in \Lambda_m$, then there exists $n \in \mathbb{N}$ such that $\omega = \varepsilon_n\beta^n + \cdots + \varepsilon_1\beta + \varepsilon_0$, with $\deg(\varepsilon_i) \leq m < \deg(\beta)$. If $n \geq 1$ and $\varepsilon_n \neq 0$, we have $|\varepsilon_i\beta^i| < |\varepsilon_n\beta^n|$ for all $i < n$. Then
$$|\omega| = |\varepsilon_n\beta^n| \geq |\beta| > 1.$$
By consequence, $l^m(\beta) = \inf\{|\varepsilon_0| : \varepsilon_0 \in \mathbb{F}[X], \deg(\varepsilon_0) \leq m\} = |1| = 1$.

ii) If $m \geq \deg(\beta)$, by Lemma 2 and Lemma 3, we obtain
$$l^m(\beta) \geq \frac{1}{|\beta|^{d-1}}l^{m-1}(\beta).$$
After $(m - \deg(\beta))$ iterations, it comes with i)
$$\begin{aligned} l^m(\beta) &\geq \frac{1}{|\beta|^{(m-\deg(\beta)+1)(d-1)}}l^{\deg(\beta)-1}(\beta) \\ &= \frac{1}{|\beta|^{(m-\deg(\beta)+1)(d-1)}}. \end{aligned}$$

III. Proof of Theorem 4

For the proof of Theorem 4, we need irreducibility criteria:

Lemma 4. *Let $P(Y) = Y^d + A_{d-1}Y^{d-1} + \cdots + A_0$, with $\deg(A_{d-1}) > \deg(A_i)$ for $i \neq d-1$ and $A_0 \neq 0$. Then P is irreducible over $\mathbb{F}[X]$ and it is the minimal polynomial of a Pisot formal power series.*

Proof: By Lemma 1, P has a unique root β such that $|\beta| > 1$ and $[\beta] = A_{d-1}$. Let $\beta_1 = \beta, \beta_2, \cdots, \beta_d$, the roots of P in the algebraic closure of $\mathbb{F}((X^{-1}))$.

Since P is a monic polynomial, then $\sum_{i=1}^{d}\beta_i^k \in \mathbb{F}[X]$, for all $k \in \mathbb{N}$, then
$$\lim_{n \to +\infty}\{\beta^n\} = 0,$$
with Theorem B, β is a Pisot. Let then $H(Y) = Y^n + B_{n-1}Y^{n-1} + \cdots + B_0$ be the minimal polynomial of β. It is clear by Lemma 1 that $[\beta] = B_{n-1}$ and $[B_{n-1}] = A_{d-1}$.

Let now $P(Y) = H(Y)Q(Y)$, with $Q(Y) = Y^m + C_{m-1}Y^{m-1} + \cdots + C_0$. We suppose that $m \geq 1$, then
$$C_{m-1} + B_{n-1} = A_{d-1} \quad \text{and} \quad C_0B_0 = A_0 \tag{12}$$

$$\sum_{\substack{i+j=s\\0\le i\le n\\0\le j\le m}} B_iC_j = A_s; \quad s\in\{1,2,\cdots,d-2\} \tag{13}$$

Since $B_{n-1} = A_{d-1}$,then by (12), $C_{m-1} = 0$. Let $i_0 \in \{0,1,\cdots,m\}$ such that $\deg(C_{i_0}) = \max_{0\le i\le m} \deg(C_i)$.

If $C_{i_0} \neq 0$, then $\deg(B_{n-1}C_{i_0}) > \deg(B_iC_j)$, $(i,j) \neq (n-1, i_0)$. Then,

$$\deg(\sum_{\substack{i+j=n+i_0-1\\0\le i\le n\\0\le j\le m}} B_iC_j) = \deg(B_{n-1}C_{i_0}) \geq \deg(A_{d-1}).$$

which is a contradiction (13).

Finally, we obtain $P(Y) = Y^m H(Y)$, $m \geq 1$, what contradict the fact that $A_0 \neq 0$ then, $P(Y) = H(Y)$. ■

Now we can prove Theorem 4. We show that $l^m(\beta) \leq |\beta|^{-m(d-1)}$ then it is Theorem 3 that gives the conclusion. We just prove that

$$\Lambda_m \supset \frac{1}{\beta^{d-1}}\Lambda_{m-1}.$$

It is clear that the last inclusion is totally assured if we prove that $\beta^{-(d-1)} \in \Lambda_1$ and as a result, it is the Remark 2.

Let β be a Pisot such that $\deg(\beta) = 1$ satisfying an algebraic equation of the form (6), with $\deg(A_{d-1}) = \deg(\beta) = 1$ (Lemma 1) and that $A_i \in \mathbb{F}$, for $i \neq d-1$ and $A_0 \neq 0$. By (6), we have

$$\beta^{-1} = -A_0^{-1}(\beta^{d-1} + A_{d-1}\beta^{d-2} + \cdots + A_1) \in \Lambda_1,$$

we can show that $\beta^{-i} \in \Lambda_1$ for $1 \leq i < d-1$ and

$$\beta^{-i} = -A_0^{-1}(\beta^{d-i} + A_{d-1}\beta^{d-i-1} + \cdots + A_{i+1}\beta + A_i) \tag{14}$$

$$-A_0^{-1}(\frac{A_1}{\beta^{i-1}} + \frac{A_2}{\beta^{i-2}} + \cdots + \frac{A_{i-1}}{\beta}).$$

It sufficient to use the recurrence on i dividing (14) each time by β, that gives with Remark 2

$$\beta^{-(d-1)} = -A_0^{-1}\beta - A_0^{-1}A_{d-1} - \frac{A_{d-2}A_0^{-1}}{\beta} - \ldots - \frac{A_2A_0^{-1}}{\beta^{d-3}} - \frac{A_1A_0^{-1}}{\beta^{d-2}} \in \Lambda_1.$$

Finally, we obtain

$$l^m(\beta) \leq \frac{1}{|\beta|^{d-1}} l^{m-1}(\beta).$$

By Theorem 3

$$l^m(\beta) = \frac{1}{|\beta|^{d-1}} l^{m-1}(\beta). \tag{15}$$

After m iterations of (15)and using the result i) in Theorem 3 we have

$$l^m(\beta) = \frac{1}{|\beta|^{m(d-1)}}.$$

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