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Fixed Point of Lipschitz Quasi Nonexpansive Mappings

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***Abstract*—**In this article, we study demiclosed and strongly quasi-nonexpansive of a sequence generated by the proximal point algorithm for a finite family of quasi-nonexpansive mappings in Hadamard spaces. $\triangle$-convergence of iterations for the sequence of strongly quasi-nonexpansive mappings as well as the strong convergence of the Halpern type regularization of them to a common fixed point of sequence are also established. Our results generalize and improve several previously known results of the existing literature.

Keywords—Fixed point, Hadamard space, proximal point algorithm, quasi-nonexpansive sequence of mappings, resolvent.

I. Introduction

A complete metric space is a Hadamard space if is geodesically connected, and if every geodesic triangle in it is at least as thin as its comparison triangle in the Euclidean plane. The precise definition of the Hadamard spaces is given in Section II. Some examples of the Hadamard spaces are pre-Hilbert spaces, Hadamard manifolds, $\mathbb{R}$-trees and many others. It is natural that a large number of notions known in these spaces can be extended to Hadamard spaces. Moreover, the extension of some theories, such as monotone operator theory, fixed point theory and many others, from such spaces to Hadamard spaces is useful and valuable.

The proximal point algorithm is one of the most effective iterative methods for approximating the solution of a variational inequality associated with a maximal monotone operator, or finding a zero of a maximal monotone operator. This algorithm was first introduced by Martinet [15] for minimizing a convex function on $\mathbb{R}^n$, and then systematically applied by Rockafellar [17] for maximal monotone operators. Rockafellar [17] proved the weak convergence of the generated sequence to a zero of the maximal monotone operator, with appropriate assumptions on the parameters and errors. As an advantage of this method, it is extendable to nonlinear spaces, like Riemannian manifolds and metric spaces of nonpositive curvature. Ferreira and Oliveira [11], Li et al. [14], and Ahmadi and Khatibzadeh [1] studied the proximal point algorithm in Hadamard manifolds. Bacak [3] studied the proximal point method in nonpositive curvature metric spaces. He proved $\triangle-$convergence of the generated sequence of the algorithm to a minimum point of the convex function. The proximal point algorithm also has been used for minimization of a quasi-convex function in Hadamard manifolds in [7], [16], [18], [19].

Some authors considered the asymptotic behavior of the product of resolvents of convex functions or more generally, maximal monotone operators, that the generated sequence converges weakly to an element of the intersection of zeros of a finite family of maximal monotone operators in Hilbert spaces. This iterative scheme is called the method of alternating (cyclic) resolvents, which was studied by Boikanyo and Morosanu [8] and Bauschke et al. [6] for two maximal monotone operators in Hilbert spaces. In the special case, when the monotone operators are the normal cone of convex and closed sets, this method is reduced to alternating projection method studied by von Numann in Hilbert spaces and extended by Bacak et al. [4] in Hadamard spaces.

In [13], the weak convergence of an iteration of a sequence of strongly quasi-nonexpansive mappings as well as the strong convergence of its Halpern regularization in $CAT(0)$ spaces were studied. The authors in [13] showed that some iterative methods like Mann and Ishikawa iterations for quasi nonexpansive mappings as well as proximal point algorithms for quasi-convex and pseudo-convex function can be extracted from the asymptotic behavior of iterations of strongly quasi-nonexpansive sequences. In this paper, we apply the tools of [13] for studying the asymptotic behavior of the product of resolvents of quasi-nonexpansive mappings which extend the results in the literature of alternating (cyclic) resolvent methods.

In the next section, we recall some preliminaries and tools which are needed in the paper.

II. Preliminaries of Hadamard Spaces

Let (X,d) be a metric space. For $x,y\in X$ a mapping $\theta\colon[0,l]\to X$, where $l\geq 0$, is called a geodesic with endpoints x,y, if $\theta(0)=x,\theta(l)=y$, and $d\big(\theta(t),\theta(t')\big)=|t-t'|$ for all $t,t'\in[0,l]$. If, for every $x,y\in X$, a geodesic with endpoints x,y exists, then we call (X,d) a geodesic metric space and it is said to be unique geodesic if between any two points there is exactly one geodesic.

A subset E of a uniquely geodesic space X is said to be convex when for any two points $x,y\in E$, the geodesic joining x and y is contained in E. The image of θ is called a geodesic segment which is denoted by $[x,y]$ for unique geodesic spaces. Let X be a unique geodesic metric space. For each $x,y\in X$ and for each $\lambda\in[0,1]$, the unique point $m_\lambda\in[x,y]$ such that:

$$d(x,m_\lambda)=\lambda d(x,y), \tag{1}$$

and,

$$d(y,m_\lambda)=(1-\lambda)d(x,y). \tag{2}$$

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We will use the notation $(1-\lambda)x\oplus\lambda y$ for the unique point m_λ satisfying in the above statement (see Lemma 2.3 in [2]).

A geodesic triangle $\triangle:=\triangle(x_1,x_2,x_3)$ in a geodesic metric space (X,d) consists of three points $x_1,x_2,x_3\in X$ as vertices and three geodesic segments joining each pair of vertices as edges. A comparison triangle for the geodesic triangle $\triangle$ is the triangle $\overline{\triangle}:=\overline{\triangle}(x_1,x_2,x_3):=\triangle(\overline{x_1},\overline{x_2},\overline{x_3})$ in the Euclidean space $\mathbb{R}^2$ such that $d(x_i,x_j)=d_{\mathbb{R}^2}(\bar{x}_i,\bar{x}_j)$ for all $i,j=1,2,3$. Let $\triangle$ be a geodesic triangle in X and let $\overline{\triangle}$ be a comparison triangle for $\triangle$. Then $\triangle$ is said to satisfy the $CAT(0)$ inequality if for all $x,y\in\triangle$ and all comparison points $x,y\in\overline{\triangle}$,

$$d(x,y)\le d_{\mathbb{R}^2}(\bar{x},\bar{y}). \quad (3)$$

Let x,y, and z be points in X and y_0 be the midpoint of the segment $[y,z]$; then, the $CAT(0)$ inequality implies:

$$d^2(x,y_0)\le\frac{1}{2}d^2(x,y)+\frac{1}{2}d^2(x.z)-\frac{1}{4}d^2(y,z). \quad (4)$$

Inequality (4) is known as the CN-inequality of Bruhat and Titis [9].

Definition1. A geodesic metric space is said to be a $CAT(0)$ space if all geodesic triangles satisfy the $CAT(0)$ inequality. Equivalently, X is called a $CAT(0)$ space if and only if it satisfies the CN-inequality.

$CAT(0)$ spaces are examples of uniquely geodesic spaces, and a complete $CAT(0)$ space is called a Hadamard space.

Now we define the notion of $\triangle-$convergence in $CAT(0)$ spaces as an alternative concept for weak convergence in these settings, which is weaker than the metric convergence (strong convergence). In a Hadamard space X, for a bounded sequence $\{x_n\}$ and $x\in X$, we set:

$$r(x,\{x_n\}):=\lim_{n\to\infty}\sup d(x,x_n). \quad (5)$$

The asymptotic radius of $\{x_n\}$ is defined as follows:

$$r(\{x_n\}):=\inf\{r(x,\{x_n\}):x\in X\}, \quad (6)$$

and the asymptotic center is the set:

$$A(\{x_n\}:=\{x\in X: r(x,\{x_n\}=r(\{x_n\})\}. \quad (7)$$

It is known that in a Hadamard space, $A(\{x_n\}$ is singleton [10].

Definition1. A sequence $\{x_n\}$ is said to be $\triangle-$convergence (or weak convergent) to x if x is the unique asymptotic center of every subsequence $\{x_{n_j}\}$ of $\{x_n\}$. The point x is called $\triangle-lim$ (or weak limit) of $\{x_n\}$. We denote $\triangle-$convergence or weak convergence of $\{x_n\}$ to x by $x_n\rightharpoonup x$.

Lemma1. [12, Proposition 3.6] Let X be a Hadamard space. Then, every bounded, closed and convex subset of X is $\triangle$-compact; that is, every bounded sequence in it, has a $\triangle$-convergent subsequence.

Let C be a nonempty subset of a Hadamard space (X,d). A mapping $T:C\to X$ is called nonexpansive if for each $x,y\in C$, $d(T\,x,T\,y)\le d(x,y)$, and if $X=C$, then T is nonexpansive self-mapping. A point $x\in C$ is called a fixed point of T if $T\,x=x$. We write:

$$F(T):=\{x\in C: T(x)=x\}. \quad (8)$$

The set of fixed points of a nonexpansive self-mapping on a closed convex subset of a $CAT(0)$ space is closed and convex. It is well known that a nonexpansive self-mapping on a nonempty, closed, convex and bounded subset of a $CAT(0)$ space has a fixed point.

Definition3. A mapping $T:C\to C$ is called quasi-nonexpansive iff $F(T)=\emptyset$ and $d(T\,x,p)\le d(x,p),\forall x\in C$ and $\forall p\in F(T)$.

We recall next the definition of firmly quasi-nonexpansive mappings.

Definition44. A mapping $T:C\to C$ is called firmly quasi-nonexpansive if $F(T)\neq\emptyset$ and,

$$d^2(Tx,p)\le d^2(x,p)-d^2(x,Tx),\forall(x,p)\in C\times F(T). \quad (9)$$

Obviously, firmly quasi-nonexpansive $\Rightarrow$ quasi-nonexpansive.

Definition5. A sequence $\{T_n\}$ of quasi-nonexpansive mappings is said to be strongly quasi-nonexpansive iff $\cap_n F(T_n)\neq\emptyset$ and $d(x_n,T_nx_n)\to 0$, whenever $\{x_n\}$ is a bounded sequence in C and $\left(d(x_n,q)-d(T_nx_n,q)\right)\to 0$, for some $q\in\cap_n F(T_n)$. When $T_n\equiv T$, T is called a strongly quasi-nonexpansive mapping.

Definition6. A sequence $T_n: C\to C$ of quasi-nonexpansive mappings with $\cap_n F(T_n)\neq\emptyset$ is called demiclosed iff for each subsequence x_{n_j} of the sequence $\{x_n\}\subset C$ such that $x_{n_j}\rightharpoonup p\in C$ and $d(x_{n_j},T_{n_j}x_{n_j})\longrightarrow 0$, we have $p\in\cap_n F(T_n)$.

It is easy to see that whenever $T_n\equiv T$ then definition 6 is exactly the definition of the demiclosedness of T.

In [13, Theorem 2.3 and Corollary 2.4] the authors proved weak convergence of the sequence $x_{n+1}=T_nx_n$, where $\{T_n\}$ is a demiclosed and strongly quasi-nonexpansive sequence with $\cap_n F(T_n)\neq\emptyset$. We recall the results here for convenience of the author.

Theorem1. [13, Theorem 2.3 and Corollary 2.4] Suppose that $T_n: X\to X$ is a sequence of strongly quasi-nonexpansive and demiclosed mappings and $x_1\in X$. We define $x_{n+1}=T_nx_n$. Then the sequence $\{x_n\}$ converges weakly to an element of $\cap_n F(T_n)$.

III. Resolvent of Nonexpansive Mappings

The resolvent operator J_λ^T for a nonexpansive mapping T has been defined in the literature for Hadamard spaces (see [5]). The definition for a Lipschitz mapping is similar, but it exists only for some parameters λ.

Let $C\subset X$ be nonempty, closed and convex. Suppose that $T: C\to C$ is L -Lipschitz with $L>1$ such that $d(Tx,Ty)\le\alpha d(x,y)$. For $\lambda>0$ and $x\in C$, we define $T_\lambda^x: C\to C$ as:

$$T_\lambda^x\, y = \frac{1}{1+\lambda} x \oplus \frac{\lambda}{1+\lambda} Ty. \quad (10)$$

In the sequel, let $y_1, y_2 \in C$, then we have:

$$d(T_\lambda^x y_1, T_\lambda^x y_2) = d\left(\frac{1}{1+\lambda} x \oplus \frac{\lambda}{1+\lambda} Ty_1, \frac{1}{1+\lambda} x \oplus \frac{\lambda}{1+\lambda} T_{y_2}\right) \quad (11)$$

$$\leq \frac{\lambda}{1+\lambda} d(Ty_1, Ty_2) \leq \frac{L\lambda}{1+\lambda} d(y_1, y_2). \quad (12)$$

Now if $\frac{L}{1+\lambda} < 1$, then T_λ^x is a contraction, i.e. if $\lambda < \frac{1}{L-1}$ then T_λ^x has a unique fixed point which we denote it by $J_\lambda^T x$ and and it is called the resolvent of T of order $\lambda > 0$ at x. In fact, $\{J_\lambda^T\} = F(T_\lambda^x)$. If $J_\lambda^T = x$, then we have:

$$x = \frac{1}{1+\lambda} x \oplus \frac{\lambda}{1+\lambda} Tx \implies Tx = x. \quad (13)$$

Now, let $Tx = x$, then we get $x = \frac{1}{1+\lambda} x \oplus \frac{\lambda}{1+\lambda} Tx$, i.e. J_λ^T which implies that $F(J_\lambda^T) = F(T)$.

Let $T : C \to C$ be a L-Lipschitz and quasi-nonexpansive mapping, where C is nonempty, closed and convex and $\lambda_n < \frac{1}{L-1}$, we define the sequence $\{x_n\}$ as:

$$x_{n+1} = J_{\lambda_n}^T x_n = \frac{1}{1+\lambda_n} x_n \oplus \frac{\lambda_n}{1+\lambda_n} T J_{\lambda_n}^T x_n. \quad (14)$$

IV. Main Results

In this section, we study $\triangle$-convergence of iterations for a sequence of strongly quasi-nonexpansive mappings as well as the strong convergence of the Halpern type regularization of them.

Lemma2. Suppose X be a Hadamard space and $T_i : X \to X, i = 1,2,\dots,N$ be L-Lipschitz with $L > 1$, and quasi-nonexpansive mappings with $\cap_{i=1}^N F(T_i) \neq \emptyset$. If $\lambda_k < \frac{1}{L-1}$ and $liminf\,\lambda_k > \lambda > 0$, then $J_{\lambda_k}^{T_N} \dots J_{\lambda_k}^{T_1}$ is strongly quasi-nonexpansive sequence.

Lemma3. Suppose X be a Hadamard space and $T_i : X \to X, i = 1,2,\dots,N$ be L-Lipschitz with $L > 1$, demiclosed and quasi-nonexpansive mappings. If $\cap_{i=1}^N F(T_i) \neq \emptyset$ and $\{\lambda_k\}$ is a positive sequence such that $liminf\,\lambda_k > \lambda > 0$ and $\lambda_k < \frac{1}{L-1}$ then $J_{\lambda_k}^{T_N} \dots J_{\lambda_k}^{T_1}$ is demiclosed.

Theorem2. Let X be a Hadamard space and $T_i : X \to X$, for $i = 1,2,\dots,N$ be L-Lipschitz and quasi-nonexpansive mappings with $\cap_{i=1}^N F(T_i) \neq \emptyset$. If $liminf\,\lambda_k > \lambda > 0$ and $\lambda_k < \frac{1}{L-1}$, then the sequence generated by:

$$x_{k+1} = J_{\lambda_k}^{T_N} \dots J_{\lambda_k}^{T_1} x_k, \quad (15)$$

converges weakly to an element of $\cap_{i=1}^N F(T_i)$.

Theorem3. Let X be a Hadamard space and $T_i : X \to X$, for $i = 1,2,\dots,N$ be L-Lipschitz and quasi-nonexpansive mappings with $\cap_{i=1}^N F(T_i) \neq \emptyset$. If $\lambda_k < \frac{1}{L-1}$ and $liminf\,\lambda_k > \lambda > 0$, α_k satisfies the conditions:

1. $0 \leq \alpha_k \leq 1$,
2. $\alpha_k \to 0 \; as \; k \to \infty$,
3. $\sum_{k=1}^\infty \alpha_k = +\infty$,

and u is an arbitrary element of X, then the sequence generated by:

$$x_{k+1} = \alpha_k u \oplus (1-\alpha_k) J_{\lambda_k}^{T_N} \dots J_{\lambda_k}^{T_1} x_k, \quad (16)$$

converges strongly to an element of $\cap_{i=1}^N F(T_i)$, which is the nearest point to u.

V. Conclusion

In this paper, we apply the same tools of [13] for studying the asymptotic behavior of the product of resolvents of quasi-nonexpansive mappings s in Hadamard spaces. The results are new even in Hilbert spaces and extend the results in the literature of alternating (cyclic) resolvent methods. Even for one quasi-nonexpansive mapping, the results extend the results of subsection 3.2 of [13].

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