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Delaunay Triangulations Efficiency for Conduction-Convection Problems

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***Abstract*—**This work is a comparative study on the effect of Delaunay triangulation algorithms on discretization error for conduction-convection conservation problems. A structured triangulation and many unstructured Delaunay triangulations using three popular algorithms for node placement strategies are used. The numerical method employed is the vertex-centered finite volume method. It is found that when the computational domain can be meshed using a structured triangulation, the discretization error is lower for structured triangulations compared to unstructured ones for only low Peclet number values, i.e. when conduction is dominant. However, as the Peclet number is increased and convection becomes more significant, the unstructured triangulations reduce the discretization error. Also, no statistical correlation between triangulation angle extremums and the discretization error is found using 200 samples of randomly generated Delaunay and non-Delaunay triangulations. Thus, the angle extremums cannot be an indicator of the discretization error on their own and need to be combined with other triangulation quality measures, which is the subject of further studies.

***Keywords*—**Conduction-convection problems, Delaunay triangulation, discretization error, finite volume method.

I. Introduction

THIS work, investigates the effect of triangulation methodology on the finite volume discretization error for conduction-convection problems. We also examine how different triangulation scheme impact on error changes for conduction dominated cases and convection dominated ones by studying the effect of the Péclet number on the discretization error.

Here, the triangulation approach, when vertices for triangulation are predetermined, is the Delaunay triangulation [1]; however, our focus is on algorithms that place the interior vertices, or Steiner points, with a given triangulation domain boundary. The following three node insertion algorithms are used in this study: The triangle splitting method of Weatherill and Hassan [2], the segment and triangle splitting approach of Shewchuk [3], commonly known as TRIANGLE, and advancing front technique by Lo [4], [5]. For brevity, hereafter the designations WH, TR and AF refer to node insertion methods of, respectively, Weatherill and Hassan, TRIANGLE and Advancing front.

Note that there are many other competitive algorithms in the literature that are not considered here. For example, Bern et al. [6], employing quadtree point insertion scheme, offer several algorithms for triangulation that guarantee satisfaction of user provided bounds on angles acuteness or obtuseness. Bern et al. [7] offered a circle packing point insertion algorithm that ensures all angles in the ensuing triangulation are less than $\pi/2$. Üngör [8] introduced the concept of "off-centers", and Dorado et al. [9] combined advancing front and circle packing algorithms. More details on point insertion and Delaunay triangulation can be found in [10], [11] and [5].

One can classify the mainstream Steiner point insertion algorithms based on the nature of Steiner points. For example, Weatherill and Hassan [2] approach introduces new points at the geometric centers of Delaunay triangles, while Shewchuk [3] scheme employs the midpoint of segments and circumcenters of skinny triangles for point insertion. The advancing front method uses the equilateral triangles that are built upon the advancing front segments. However, the earlier version of the advancing front method places new points on a pre-existing grid [4]. Üngör [8] has introduced the Steiner points that are 'off-centers' meaning that the location of new points is derived by moving the circumcenter in a fashion that the resulting triangle formed by using the vertices of the minimum-length edge of a skinny triangle and the 'off-center' would have the desired radius to edge ratio. Some triangulation algorithms rely on a background mesh like quadtrees [6], [12], [13]. Yet another certain algorithm for point insertion involves placing the desired number of nodes into random positions in the not-yet-meshed domain. For example, Centroidal Voronoi Tessellation (CVT) [14]-[16] places Steiner nodes at random positions, then draws the Voronoi diagram of the points and moves the points (sites of the Voronoi diagram) to the mass center of the Voronoi regions. This procedure is repeated until location of Voronoi sites and centroid of Voronoi regions converge. Another popular two-dimensional meshing approach is suggested by Persson and Strang [17] with the triangulation routine that is called DISTMESH. In this method, points are randomly inserted in the meshing domain. Then, in each iteration, first, a Delaunay triangulation performed on the point set and second assuming all triangle sides are behaving as mechanical springs with the same stiffness, the points are allowed to move so that points are in mechanical equilibrium. The iterations are repeated until no move is necessary for points to be in equilibrium. Both CVT and DISTMESH algorithms require determining the number of Steiner points before node insertion. The three algorithms studied here do not have such a constraint.

The finite volume method (FVM), which is also called the

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control-volume finite element method, is the preferred approach for solving partial differential equations describing fluid flow, heat transfer, and in general the conservation equations [18], [19]. Patankar [18] classifies the FVM as special case of the finite element method (FEM). It is true that FVM like its ancestor, FEM, divides the solution domain to subdomains and in certain versions it also uses shape functions to describe the dependent variables; however, the main difference between the methods lies in the way the governing equations are discretized: Unlike mainstream FEM approaches, FVM does not employ the variational principle or a weighted residual method, the most popular being the Galerkin method. FVM discretizes the conservation equations by writing the balance of the fluxes of the conserved quantity entering or exiting a cell, as well as source terms if present. Thereby, FVM is more loyal to the physical meaning of conservation equations that it tries to discretize. It is worthwhile to note that there have been attempts to modify FEM, particularly the Galerkin scheme, for computational fluid dynamics (CFD) problems and generally the conservation equations, most notably the Petrov-Galerkin method [20]. Mizukami and Hughes [21] developed a version of Petrov-Galerkin custom-made for triangulations. Martinez [22] compared different FVM and FEM schemes for convection-conduction equations and has reported no significant difference between the methods for steady-state linear conservation equations. However, since FVM is the basis for the contemporary mainstream approaches for solving CFD equations [19], namely the semi-implicit method pressure-linked equations (SIMPLE) [23] and its derivations [19]; it is also selected for this study.

FVM schemes dealing with conservation equation can be classified as vertex-centered or cell-centered methods [19], [24]. Vertex-centered methods, first suggested by Winslow [25] for Poisson equations, create the control volumes around the vertices of the mesh by connecting the centroid of adjacent faces or cells whereas in cell-centered methods the faces or cells, depending on the problem being two or three dimensional, play the role of control volumes. While the geometric concept embedded in the cell-centered idea is easier than its counterpart, the vertex-based scheme, the discretization methodology is more complicated to compensate for the fact that a line connecting neighboring cells is not necessarily perpendicular to their boundary. The vertex-based methods on the other hand, although require additional computational effort to obtain the cell geometry from the triangulation geometry, enjoy a straightforward discretization scheme. Further, the vertex-based methods can take advantage of the shape function concept borrowed from FEM: Much like Petrov-Galerkin schemes, vertex-based methods embed the upstream concept in the shape function. Due to aforementioned reasoning, the vertex-based method of Baliga and Patankar [26] is considered in this study. The work of Baliga and Patankar [26] is among the first extensions of grid-based FVM to triangular meshes. Soon afterwards, these authors have devised a version of the SIMPLE method suitable for triangulations [27], [28]. Note that Prakash [29] as well as Hookey et al. [30] improved the FVM for general conservation equations when a source term is present. The interested reader is referred to [22] and [19].

Babuška and Aziz [31] proved that the maximum angle of triangles matters the most for interpolation accuracy whereas the minimum angles do not have a direct effect. Obviously, algorithms applying a lower bound on minimum angle are also effective in reducing interpolation errors because bounding minimum angle results in an upper bound on maximum angle. This fact is responsible for considering Delaunay triangulations for two dimensions as Delaunay triangulations maximize the minimum angle [11] among all possible triangulations for a given set of vertices. Recently, Song et al. [32] borrowed the K-means clustering algorithm from machine learning to devise a routine that optimizes the minimum angle in a Delaunay triangulation used for CFD studies of oceans.

Fan and Ollivier-Gooch [33] compared FVM discretization and truncation errors for solving the Poisson equation when triangulations were produced by Ruppert's Delaunay triangulation [34], Marcum's advancing front local reconnection [35] and Engwirda's Frontal-Delaunay algorithms [36]. The triangulation algorithms in [35] and [36] produce a more regular mesh (a mesh with more equilateral triangles) compared to the one in [34]. Here, we extend the work of Fan and Ollivier-Gooch [33] by including the convection term to the governing conservation equation. Note that they [33] stated that using a more regular triangulation did not improve the discretization error of the solution; thus, our focus in this work remains on algorithms that create a Delaunay triangulation without any compromise to mesh regularity, even the Delaunay advancing front [5] that is employed here uses the advancing front technique for only node placement; however, the final triangulation is created by a full Delaunay triangulation algorithm as discussed in Section II *C*.

Gao et al. [37] used statistical correlation analysis to assess the correlation among different hexahedral mesh quality measures including the angle extremums as well as correlation between mesh quality measures and FEM discretization error for solving Poisson equation and partial differential equations of linear elasticity and Stokes problem. In this work we perform a similar correlation analysis of the relation between triangulation angle extremums and FVM discretization error for the conduction-convection problem.

The remainder of this paper is organized as follows. In Section II, the methodologies used to generate the triangulations are reviewed. In Section III, the test conduction-convection problem is described. Note that describing the FVM formulation and implementation will not be attempted here as the methodology in [26] is closely followed here. Section IV investigates effect of different triangulations on FVM discretization error. The conclusions are presented in Section V. Note that hereinafter by error we mean the discretization error which is the absolute difference between the numerical solution and the exact solution.

II. Triangulation Methodologies

As mentioned earlier, in this work the emphasis is on comparing the node insertion (Steiner nodes) algorithms. For all the algorithms considered here, lifting to the third-dimension algorithm [38] is used to construct a Delaunay triangulation of given planar points. For a comparative study of different algorithms for constructing a Delaunay triangulation of planar points, see [39]. Moreover, in assessing the efficiency of the generated meshes by the three above methods, our attention is on the error of the numerical results compared to the theoretical ones. Direct method (Gaussian elimination) is employed to solve the resulting algebraic equations from discretization of governing equations. See [40] for the effect of different triangulation algorithms on convergence of indirect (iterative) methods of solving the final algebraic equations for FEM and FVM.

A. Weatherill & Hassan Algorithm

The first method to consider is the method of splitting existing triangles to insert nodes and produce a finer triangulation. A version of this approach is prescribed by Weatherill and Hassan [2], designated by WH here, as follows: One starts with a Delaunay triangulation of only boundary points. For each boundary point, a length scale is defined equal to the average length of all adjacent edges to that point. Then a new point is inserted at the centroid of one of triangles. The length scale of the new point is calculated as interpolation of length scales of only boundary points. If the new point's distance to all existing points (boundary and inserted ones) is larger than the calculated length scale for that point multiplied by a factor, the new point is accepted, the Delaunay triangulation is updated to include the new point and the procedure continues until no new point can be accepted. Regarding the multiplication factor for length scales, Weatherill and Hassan suggest using one factor, α, for checking the distance of a new point to boundary points and a different one, β, for previously inserted points. The authors have used $(\alpha, \beta) = (1, 10)$ and $(\alpha, \beta) = (1, 0.1)$ in their illustrations. In this work the values $(\alpha, \beta) = (1, 0.1)$ are employed. Figs. 1 (a) and (d) show two triangulation samples using this approach.

B. Shewchuk's TRIANGLE Algorithm

Here, the TRIANGLE algorithm by Shewchuk [3] is used and designated by TR for convenience. This algorithm involves triangle splitting like WH as well as edge splitting. First, certain definitions for this approach are reviewed: A segment is an edge that is an input edge. Diametral circle is a circle centered at the mid point of a segment with diameter equal to segment length. A vertex is said to be visible to a segment if it is adjacent to an adjacent face of the segment. An encroached segment is a segment that a visible, but not adjacent, vertex lies on or inside its diametral circle. Segment splitting to subsegments is splitting a segment by inserting a new vertex at the segment center. The new subsegments are also considered a segment. A skinny triangle is a triangle that the ratio of its circumcenter to its minimum side length is larger than a predetermined constant denoted by B which is normally set to $\sqrt{2}$. Triangle Splitting is inserting a vertex at the circumcenter of a triangle.

The vertex insertion cycle consists of: 1) generating a Delaunay triangulation of all input points, 2) splitting all encroached segments and 3) finding a skinny triangle and insert a vertex at its circumcenter if the new vertex is not encroaching any existing segments. If the new vertex is rejected, the segments that it encroaches are split. Steps 2 and 3 are repeated until no skinny triangle or encroached segment remain.

An earlier and alternative version of triangle and edge splitting technique was presented by Ruppert [34] and Chew [41], [42]. These approaches are then combined and modified by Shewchuk [3]. For brevity, the TRIANGLE algorithm is labeled TR from this point on. Figs. 1 (b) and (e) show triangulations of a square and an annular circular region employing TR method.

C. Delaunay Advancing Front Algorithm

Another approach for inserting new points is the advancing front (AF) technique [4], [5]. In AF approach, a front is a polygon separating the meshed and not-yet-meshed regions. Initially, the boundary of the domain to be triangulated is the same as front. At each step of AF an edge of the front is selected for a new triangle insertion by adding a new point to non-meshed regions or using existing vertices of the front. The processed edge is removed from the front and two sides of the new triangle are added to the front. In this way the front is changed at each step and the iteration continues until the front is a triangle. There are several variations of this method [5]; however, in this work we only consider Delaunay-AF method (see section 3.7.3 of [5]) with a slight modification: For each point that is a candidate for node insertion, instead of creating a Delaunay triangulation of existing points, as prescribed in [5], the distance of the candidate point to all other existing points is checked, if the new point is closer than pre-selected value to any existing point, it is rejected and the closest node in the front to the rejected point location is chosen to build a new triangle on the front. After the front is reduced to a triangle, the triangulation is discarded, and a Delaunay triangulation is built on the original and inserted nodes. Figs. 1 (c) and (f) show two triangulations by AF method.

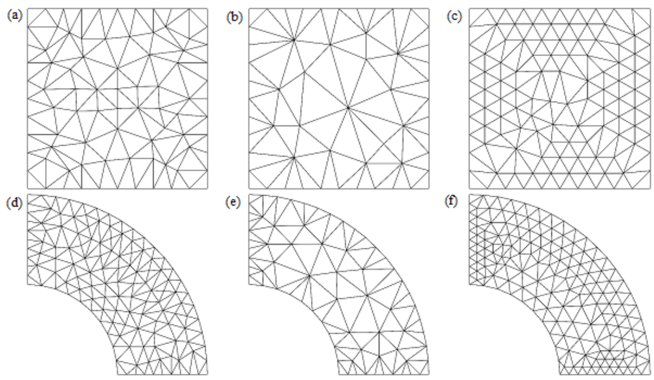

Fig. 1 Visual representation of three triangulation algorithms: (a) Triangulating a square using WH, (b) TR and (c) AF algorithms, triangulation of a quarter of the annular region between two concentric circles using (d) WH, (e) TR and (f) AF algorithms

Mavriplis [43] and Pirzdeh [44] devised an alternative AF scheme called advancing layer (AL) that is more suitable for boundary layer flows particularly when the flow is turbulent. In such flows, a thin layer of flow adjacent to the boundary, the viscous sublayer, experiences much sharper changes in velocity, compared to the rest flow farther from the boundary. The advancing layer algorithms create highly refined mesh regions possessing large aspect ratio cells near the boundary and then use the normal AF methodology for the rest of the flow domain. See [45] for the recent advances in AL method.

III. Problem Description

The equation that is considered in this work and solved by the FVM, is the steady-state conduction-convection conservation equation without a source term for a variable φ described as the following, in the non-dimensional form,

$$\boldsymbol{u} \cdot \nabla\varphi = \nabla \cdot (Pe^{-1}\nabla\varphi), \quad (1)$$

where $\boldsymbol{u}$ is the non-dimensional velocity of the fluid and is assumed to be known and ∇ is the non-dimensional gradient operator. Pe is the Péclet number, a non-dimensional group that describes the ratio of convection terms over the conduction terms and is normally expressed as:

$$Pe = LU/\kappa, \quad (2)$$

where L is a dimensional length scale, U is a dimensional velocity scale and κ is a kinematic diffusivity, with the dimension of length squared over time. It is assumed that the fluid is incompressible so that the velocity field obeys the continuity equation,

$$\nabla \cdot \boldsymbol{u} = 0. \quad (3)$$

Equation (1) is significant for all transport phenomena involving mass transfer, fluid flow (momentum transfer) and heat transfer [46]. The variable φ could represent a species concentration, for which the left-hand side of (1) describes the convection of the species by flow and the right side is the Fick's law with Pe^{-1} being the non-dimensional mass diffusivity. As yet another example, (1), by adding a pressure gradient and a transient velocity term, and exchanging the Pe number with the Reynolds number, becomes the Navier-Stokes equation that governs fluid flow of Newtonian liquids. Further, (1) also describes steady-state heat transfer when the viscous dissipation is ignored. In that case, the left-hand side of (1) is termed heat convection and the right-hand side, heat conduction or diffusion. In this text, the word convection or advection is used to describe the left-hand side of (1) in general and diffusion or conduction label the right-hand side of (1).

To assess the efficiency of different triangulation methods, the example 1, case 1 in [26] is used which is briefly described in the previous section. This problem was also used to do a comparative analysis among different finite difference methods by Runchal [47]. The configuration for this test problem is presented in Fig. 2. The geometry is of a solid hollow cylinder formed between two concentric cylinders of non-dimensional radiuses of $\sqrt{2}$ and $\sqrt{6+4\sqrt{2}}$. These values are selected in this way to make the lower left corner coordinates equal to (1,1) and the size of the square, that is the boundary of the computational domain, equal to $\sqrt{2}$.

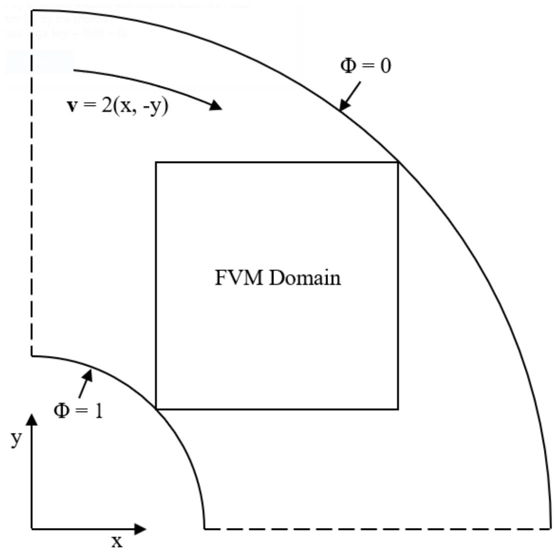

Fig. 2 The geometry of the problem considered in this work: The inner radius and outer radius are $\sqrt{2}$ and $\sqrt{6+4\sqrt{2}}$; the conduction-convection Equation (1) is solved in the square labelled "FVM Domain"

IV. Comparison oof FVM Accuracy Using Different Triangulations

As described above, to assess the efficiency of different triangulation methods, the example 1, case 1 in [26] is used. Figs. 3 (a) and (b) illustrate, respectively, the average of error percentage and maximum error percent of solving the aforementioned problem by using FVM.

In the averaging process, only internal nodes are considered as the error in boundary nodes vanishes since the problem is of Dirichlet type. As Fig. 3 shows, the error of all triangulation methods increases by increasing Pe, i.e., the error is higher as the convective term becomes more significant. At low Pe values where conduction is dominant, the FVM error using a structured triangulation error is less than those employing unstructured triangulations. However, as convection becomes more effective compared to conduction, the structured triangulation error grows more rapidly than unstructured ones. This trend can be explained by the fact that the resultant velocity distribution that derives the convection does not have a uniform direction in the domain and thus unstructured triangulations are more effective. Among the unstructured triangulations, it is observed that the advancing front method has the lowest average error. Looking at the maximum error percentage in Fig. 3, the AF and TR methods are indistinguishable at high Pe range.

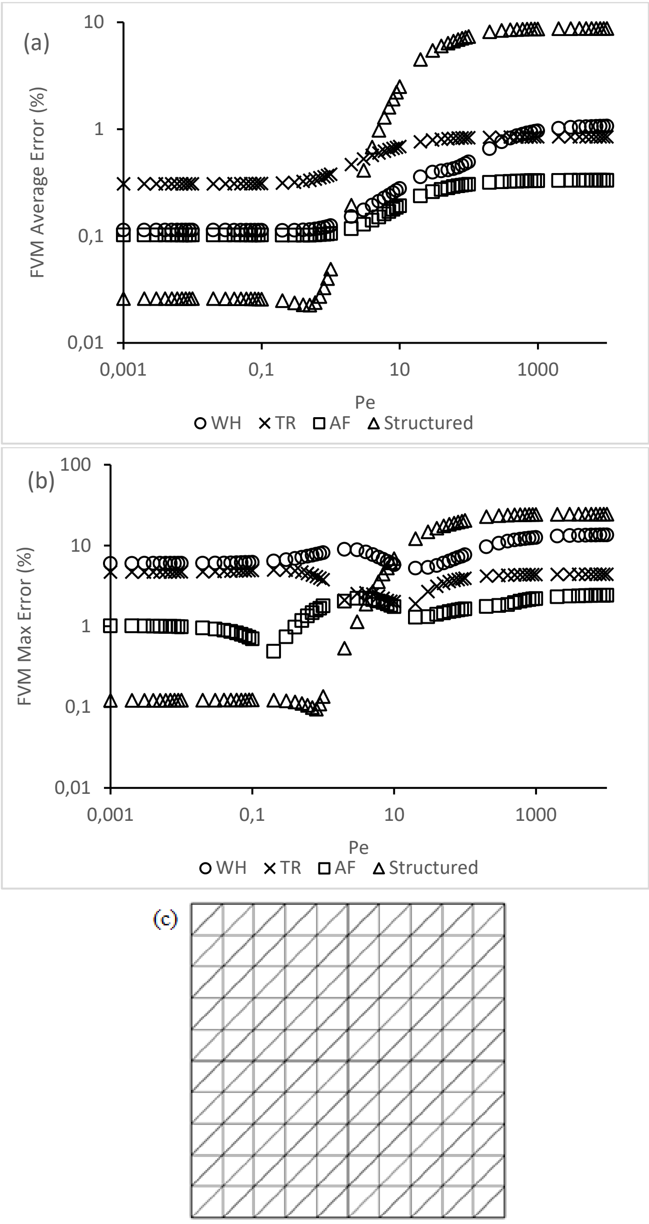

Fig. 3 (a) Average and (b) maximum error (%) for solving problem 1 case 1 (constant conductivity) in [26] for a square domain with 40 boundary points with FVM employing different triangulation approaches

One of the early triangulation measures proposed is the maximum triangle angle in a triangulation. Babuška and Aziz [31] proved that the maximum angle of triangles matters for interpolation accuracy whereas the minimum angles do not directly have any effect. Obviously, algorithms applying a lower bound on minimum angle are also effective in reducing interpolation errors because bounding minimum angle will result an upper bound on maximum angle. This fact is responsible for considering Delaunay triangulations for two dimensions as Delaunay triangulations maximize the minimum angle [11] among all possible triangulations for a given set of vertices. We wish to investigate how maximum or minimum angle of a triangulation is related to the numerical error for the problem considered here. To provide an answer, first, a set of triangulation samples with the same number of nodes are built and then the correlation between minimum and maximum angles with the FVM error is examined. Each triangulation sample is built by perturbing the position of internal points of the Steiner nodes obtained from the TR method on a square domain of size $\sqrt{2}\times\sqrt{2}$ with 40 boundary points, and then the new set of points are Delaunay triangulated. Here, a batch of 100 triangulations are built and the FVM problem is solved using these triangulations. For reader's visual inspection, six of the triangulation samples are depicted in Fig. 4. Next, the correlation of FVM error with maximum of minimum or minimum of maximum triangle angles will be examined using the correlation function. Recall that the correlation, or normalized covariance function is defined as [48]:

$$\sigma(e,q)=\frac{cov(e,q)}{\sigma(e)\sigma(q)}=\frac{\langle(e-\langle e\rangle)(q-\langle q\rangle)\rangle}{\sqrt{\langle(e-\langle e\rangle)^2\rangle\langle(q-\langle q\rangle)^2\rangle}}, \quad (4)$$

where e represents the error percentage (average or maximum) of the FVM compared to the exact solution, q is a triangulation measure, ⟨ ⟩ denotes average, $\sigma(e)$ and $\sigma(q)$ are the standard deviations for maximum error percentage and a quality measure in the sample of triangulations and $cov(e,q)$ is the covariance between e and q in that sample. σ(e,q) is in the range [-1, 1]. Values of $\sigma(e,q)$ equal or close to +1 (or -1) demonstrate favorable (adverse) correlation between e and q. On the other hand, values close to zero show absence of any correlation between e and q [48]. Fig. 5 presents the correlation between FVM error percentage with minimum or maximum in the triangulations. The problem that is solved by FVM is the example 1, case 1 (constant diffusivity) in [26]. As Fig. 5 shows, the correlations between FVM error and maximum or minimum angles are weak.

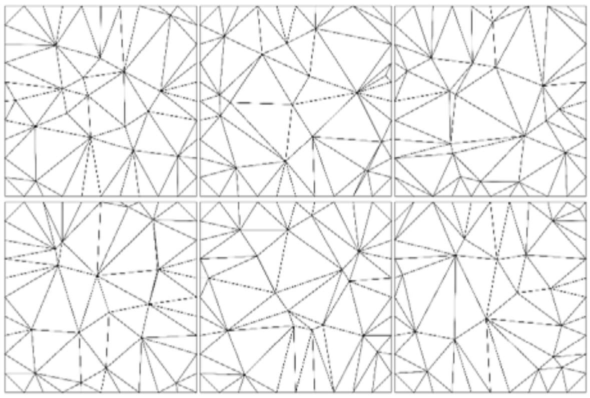

Fig. 4 Six samples of 100 triangulations that are generated by perturbing non-boundary points of a basic triangulation and then reapplying Delaunay triangulation on moved points

Further, to examine the effect of minimum or maximum angles on non-Delaunay triangulation, the same FVM problem is also solved using 100 non-Delaunay triangulations. The non-Delaunay triangulations are created by flipping ten randomly selected edges in the triangulation of a square domain of size

$\sqrt{2} \times \sqrt{2}$ with 40 boundary points distributed and triangulated using TR [3] approach which is depicted in Fig. 1 (b). Six samples of this set are shown in Fig. 6, and the correlation between FVM errors using these triangulations and minimum and maximum angles are presented in Fig. 7. This figure, similar to Fig. 5, shows low values of absolute correlation between FVM error and maximum or minimum angle.

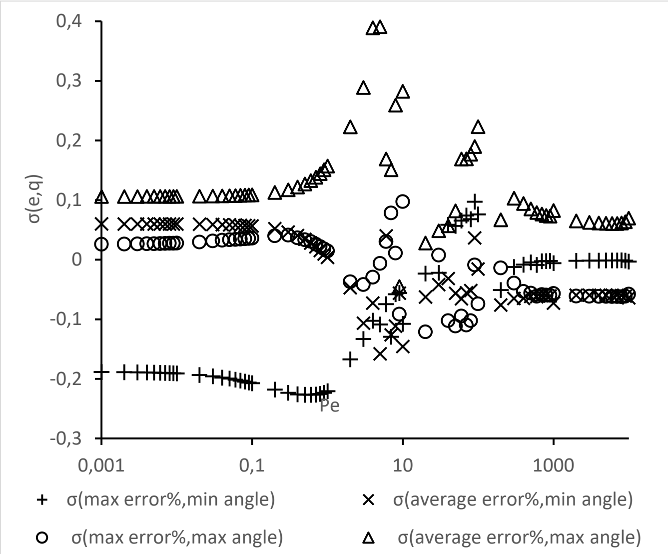

Fig. 5 The correlation between FVM error with minimum or maximum triangle angles, using a batch of 100 Delaunay triangulations, 6 samples of which are shown in Fig. 4

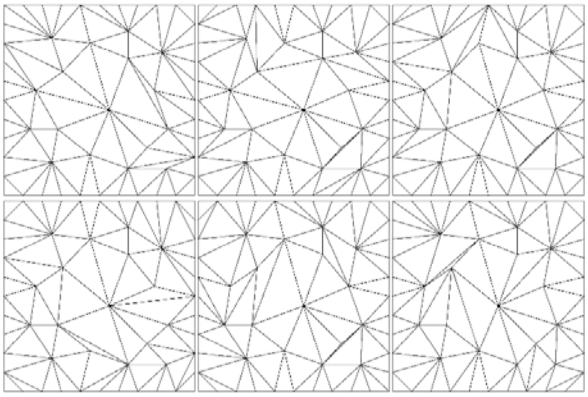

Fig. 6 Six samples of 100 *non-Delaunay* triangulations that are generated by flipping 10 randomly selected edges of a basic triangulation

To interpret trends in Figs. 5 and 7, we recall that the square of correlation between two variables is the same as R2 statistics for a least square linear regression of those variables [49]. Hence, low values of absolute correlation in Figs. 5 and 7 suggest absence of an exclusive linear relation between FVM error and minimum or maximum angles. However, one cannot rule out a nonlinear relation among these variables or a linear relation that also involves more variables.

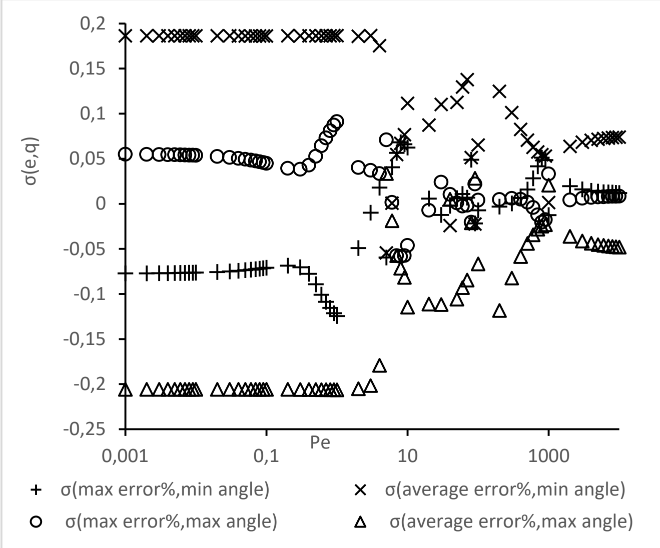

Fig. 7 The correlation between FVM error with minimum or maximum triangle angles, using a batch of 100 non-Delaunay triangulations, 6 samples of which are shown in Fig. 6

V. Conclusion

Triangulations resulting from three different Delaunay refinement algorithms and one structured triangulation were used to solve a conduction-convection problem by the vertex-based FVM scheme of Baliga and Patankar [26]. It is observed that for the regular domain boundary, a square, that is the case here, the discretization error of structured triangulation is lower for low Pe values, where conduction is dominant. However, for high Pe numbers, when convection is dominant, the unstructured triangulations compiled by triangle and edge splitting algorithm of Shewchuk [3] and advancing front of Lo [4], [5] correspond to lower discretization errors compared to the structured triangulation. Moreover, no direct statistical correlation is found between discretization error and maximum or the minimum angle in the triangulation for both Delaunay and non-Delaunay triangulations.

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