An Ecological Model for Three Species with Crowley–Martin Functional Response

Randhir Singh Baghel, Govind Shay Sharma

***Abstract*—**In this paper, we explore an ecosystem that contains a three-species food chain. The first and second species are in competition with one another for resources. However, the third species plays an important role in providing non-linear Crowley-Martin functional support for the first species. Additionally, the third species consumes the second species in a linear fashion, taking advantage of the available resources. This intricate balance ensures the survival of all three species in the ecosystem. A set of non-linear isolated first-order differential equations establish this model. We examine the system's stability at all potential equilibrium locations using the perturbed technique. Furthermore, by spending a lot of time observing the species in their natural habitat, the numerical illustrations at suitable parameter values for the model are shown.

***Keywords*—**Competition, predator, response function, local stability, numerical simulations.

I. Introduction

THE predator-prey model system is indeed one of the most important and widely studied topics in the field of ecology and biological systems. This model system explores the dynamics between predator and prey populations in an ecosystem, and it has been instrumental in understanding the complex interactions that occur in natural environments [34], [37]. Numerous ecological models can be used to explain a variety of global issues [1]. Many applications of real-world problems are expressed in ecological models [2], [38]. For instance, deer compete with one another for food in the same location, and when a deer is injured by a predator like a lion, it stays there for a while [4], [24], [25]. The study of ecological models has taken on a major position in mathematics and sparked a great deal of interest among writers. Numerous academics and experts have discussed the traditional food chain models that only include two tropic levels. The mathematical conversion of species interactions, such as prey-predator relationships, competition, mutualism, commensalism, commensalism, and so on, is given a lot of attention in order to understand species behavior both analytically and quantitatively [3], [39], [40]. Mathematically and biologically challenging and captivating, the discussion on the local and global stability of ecological models in various species interactions is particularly interesting [26], [27], [35], [36].

The authors are interested in the function that mathematics plays in understanding this kind of ecological model [5], [6], [28], [29]. It is both enjoyable and demanding from a mathematical and biological standpoint to discuss the global and local sustainability of ecological models in various sorts of interaction. Each animal-animal contact consists of a variety of exchanges. additional research on their interactions with its stable [7], [8], [10].

The quality of three food types that grow well and allow for interaction between a third type of competing species and their hosts is the main subject of this study [9], [11], [14], [30], [31]. Many scientists and academics dispute the traditional food model with just two tropical levels, including [34] and prey-predator relationships, rivalry, teamwork, etc. among animals [32], [33]. You must be very careful to alter their numbers when evaluating their interactions to properly analyze and mathematically analyze species behavior [15]-[18].

Debating the regional and global viability of ecological models in the interaction of different species, both mathematical and biological, is fascinating [19]-[21], [23]. A combination of the aforementioned interactions makes up an animal interaction [13], [12], [22].

II. Model Formulation

In the ecological setting for the proposed model's three species food chain, there are several interactions between them. The predator of Y is also present in this ecosystem. It is important to consider all of these factors when analyzing the overall health and sustainability of this environment. By taking a comprehensive approach, we can better understand the complex relationships between these species and work towards creating a more harmonious balance within the ecosystem. Type z uses y linearly and helps the first type to match x in the CMER type response.

The model equations for the ecosystem represented by the three species interacting in various ways consist of the following non-linear first-ordered differential equations:

$$\frac{dx}{dt} = x(1 - a_1 + \frac{a_2 z}{1+b_1x+cz+b_2xz})$$

$$\frac{dy}{dt} = ry\left(1 - \frac{d_1}{1+b_1x+cy+b_2xy}\right) - b_2yz$$

$$\frac{dz}{dt} = (-w + d_3y)$$

with nonnegative initial conditions x(0) ≥ 0, y(0) ≥ (0), and z(0) ≥ 0. The growth of the average of the first type of competition

Dr. Randhir Singh Baghel is with the Department of Mathematics, Poornima University, Jaipur-303905, Rajasthan, India. (e-mail: randhir.baghel@poornima.edu.in).

Dr. Govind Shay Sharma is with the Department of Mathematics, Vivekananda Global University, Jaipur, Rajasthan, India (e-mail: govind.sharma@vgu.ac.in).

has been steadily increasing over the past few years. Many factors have contributed to this trend, including advancements in technology, increased accessibility to information, and a growing interest in competitive activities. As more and more people become involved in these types of competitions, it is likely that we will continue to see growth in the average performance levels across various fields. This is a positive trend that reflects the dedication and hard work of individuals who are committed to achieving success in their chosen pursuits.

A. Analysis of the Model

Here, we analyze the stability of the proposed model by finding the equilibrium equation for the dynamic system. Equilibrium points of the system are necessary for examining the local stability of the ecological structure. The studied system has an equilibrium point fully washed out state or extinct state $E_1 = (0,0,0)$. It appears that only the initial species competing has become extinct $E_2 = \left(0, \frac{w}{d_2}, \frac{r}{d_3}\right)$. The third species has unfortunately lost its vibrancy solely due to the impact of the equilibrium state $E_3 = (\frac{1}{b_1}, \frac{1}{b_2}, 0)$. Interior or coexistence state: $E_4 = (x^*, y^*, z^*)$ where

$$x^* = (\frac{1}{1-a_1 y} - cz)(\frac{1}{b_1+b_2 z})$$

$$y^* = \frac{w}{d_3 z}$$

$$z^* = 1 - b_1 + b_2 xy + \frac{b_1 x}{d_2(y-c+x)}$$

In the following sections, we will discuss the local stability of differential equations using perturbation techniques.

B. Stability Analysis of Equilibrium Points

The Jacobian matrix for the equation at an equilibrium point $E = (\overline{x}, \overline{y}, \overline{z})$ is given by

$$JE = \begin{pmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ 0 & J_{32} & J_{33} \end{pmatrix}$$

where,

$$J_{11} = 1 - a_1 y + \frac{a_2 z}{1+b_1 x+cz+b_2 xz} - \frac{a_2 z x b_1 b_2}{1+b1x+cz+b2xz^2}$$

$$J_{12} = a_1 y;\ J_{13} = \frac{a_2 x}{1+b_1 x+cz+b_2 xz};$$

$$J_{21} = r(1 - \frac{d_1}{1+b_1 x+cy+b_2 xy})$$

$$J_{22} = \quad J_{23} = -d_2 y \quad J_{32} = d_3 \quad J_{33} = -w+d_3 y$$

Theorem1. When $b_2 > 1$ and $b_1 > 1$, there is an intermediate equation

$$E_4 = (x^*, y^*, z^*) > 1.$$

Proof. Let x*, y*, z* be the positive solutions of the equations

$$x^*(1 - a_1 y* + \frac{a_2 z*}{1+b_1 x*+cz*+b_2 x*z*})$$

$$ry^*(1-\frac{d_1}{1+b_1 x*+cy*+b_2 x*y*}) - d_2 y * z *$$

$$z^*(-w+d_3 y^*)$$

by solving these equations for x*, y* and z* we get

$$x^* = (\frac{1}{1-a_1 y} - cz)(\frac{1}{b_1+b_2 z})$$

$$y^* = \frac{w}{d_3 z}$$

$$z^* = 1 - b_1 + b_2 xy + \frac{b_1 x}{d_2(y-c+x)}$$

These will be positive when $\frac{1}{1-a_1 y} > cz$, $\frac{d_1}{1+b_1 x+cy+b_2 xy}1$. So, the interior equilibrium point $E_4 = (x^*, y^*, z^*)$ for the given model system, exist if $cz^*>1$ and $\frac{1}{1-a_1 y} cz *$.

Theorem2. The dynamical system (1) is always unstable at equilibrium points E_1, E_3, and E_4.

Proof. The eigenvalues of the dynamic system at the damping equilibrium point $E_1 = (0,0,0)$ are 1, r and -w. Therefore, the balance of the E_1 is the point of the saddle, so the engine is often unstable.

The corresponding Jacobian matrix at equilibrium point $E_3 = (\frac{1}{a_1}, \frac{1}{a_2}, 0)$ is

$$JE_3 = \begin{pmatrix} 0 & \frac{a_1}{a_2} & \frac{a_1}{1+a_2} \\ r\frac{a_1}{a_2} & 0 & -\frac{a_2}{a_1} \\ 0 & 0 & \frac{-wa+d_3}{a_1} \end{pmatrix}$$

The characteristics equation of JE_3 is

$$\left(\frac{-w+d_3}{a_1} - \lambda\right)(\lambda^2 - r) = 0$$

The eigenvalues of JE_3 are $\lambda_1 = \sqrt{r}, \lambda_2 = -\sqrt{r}, and\ \lambda_3 = \frac{d_3 - wa_1}{a_1}$. Therefore, point E_3 is a saddle point and hence the dynamical system is not stable or unstable. The Jacobian matrix at the coexistence state E4 = (x*, y*, z*) is

$$JE_4 = \begin{bmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & 0 & H_{23} \\ 0 & H_{32} & 0 \end{bmatrix}$$

where $H_{11} = -\frac{x*z*}{(1+x*)^2}$, H_{12}= -a1x*, $H_{13} = \frac{x*}{1+x*}$, H_{21}=-ra2y*, H_{23}= $-d_3 y^*$and $H_{32} = d_3 z^*$

The characteristic equation of JE_4 is

$$\lambda^3 - H_{11}\lambda^2 - (H_{23}H_{32} + H_{12}H_{21})\lambda + (H_{11}H_{32}H_{23} - H_{13}H_{21}H_{32}) = 0$$

The characteristic equation is rewritten in the following manner $\lambda^3 + \alpha_0\lambda^2 + \alpha_1\lambda + \alpha_2 = 0$, where

$$\alpha_0 = -H_{11}$$

$$\alpha_1 = (-H_{23}H_{32} + H_{12}H_{21})$$

$$\alpha_2 = H_{11}H_{23}H_{32} - H_{13}H_{21}H_{32}$$

According to Routh-Hurwitz criteria, the equilibrium point E_4 will be locally asymptotically stable if $\alpha_0 > 0, \alpha_2 > 0$ *and* $\alpha_0\alpha_1 - \alpha_2 > 0$ but we have

$$\alpha_0 = -H_{11}$$

$$\alpha_1 = (-H_{23}H_{32} + H_{12}H_{21})$$

$$\alpha_2 = H_{11}H_{23}H_{32} - H_{13}H_{21}H_{32} = \frac{d_3x^*y^*z^*}{(1+x^*)^2}\left(\frac{d_3z^*}{1+x^*} + ra_2\right) >$$

and

$$\alpha_0\alpha_1 - \alpha_2 = H_{11}H_{23}H_{32} - H_{13}H_{21}H_{32}$$

$$\frac{ra_2}{1+x^*}x^*y^*z^*\left(\frac{a_1x^*}{1+x^*} + d_3\right) < 0 \text{ always}$$

Hence the given dynamical system is also unstable at E_4 since it fails to satisfy the Routh-Hurwitz criteria.

Theorem3. The dynamical system at the boundary steady state at E_2 is Unstable trajectories are outward spirals, if $d_3d_2 + wd_2r > wd_2a_1$ closed orbits, if $d_3d_2 + wd_2r = wd_2a_1$. The trajectories are inward spiral, if $d_3d_2 + wd_2r < wd_2a_1$.

Proof. The Jacobian matrix for the system at $E_2 = (0, \frac{w}{d_3}, \frac{r}{d_2})$ is

$$JE_2 = \begin{pmatrix} 1 - a_1\frac{w}{d_3} + \frac{r}{d_2} & 0 & 0 \\ -ra_1\frac{w}{d_3} & 0 & -d_2\frac{w}{d_3} \\ 0 & d_3\frac{r}{d_2} & 0 \end{pmatrix}$$

The characteristic equation of which is

$$\left(\frac{d_2d_3 - wd_2a_1 + d_3r}{d_2d_3} - \lambda\right)(\lambda^2 + wr) = 0$$

Then the corresponding eigenvalues are

$$\lambda = \frac{d_2d_3 - wd_2a_1 + d_3r}{d_2d_3}, 0 \pm i\sqrt{wr}$$

In particular, we denote

$$\lambda_1 = \frac{d_2d_3 - wd_2a_1 + d_3r}{d_2d_3}, \lambda_2 = i\sqrt{wr} \text{ and } \lambda_3 = -i\sqrt{wr}$$

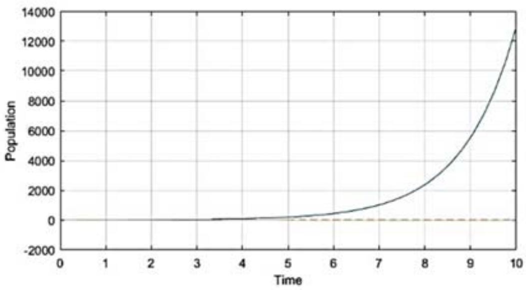
We can have the following observations of the dynamical system based on the eigenvalues λ_1.

- If $d_3d_2 + wd_2r > wd_2a_1$ then $\lambda_1 > 0$. Therefore, the system is unstable in the x direction and the orbit is circular in the y direction.
- If $d_3d_2 + wd_2r = wd_2a_1$ then $\lambda_1 = 0$. Hence, the dynamical system admits neutrally in the yz direction.
- $d_3d_2 + wd_2r < wd_2a_1$ then $\lambda_1 < 0$. So, the system is stable in the x direction and trajectories are circular orbits in the yz direction.

C. Numerical Solutions

Based on the selection of a specific design that meets the requirements in Theorem 3, the system of equations is numerically solved using the Rungue-Kutta fourth technique to assess animal behavior in the environment. The solution type is discovered using MATLAB program. The following inferences can be made from the current circumstances.

Case1. The requirement of Theorem 3 is satisfied by choosing the parameter values in the dynamical system provided as: $d_1 = 0.8$; $d_2 = 0.45$; $d_3 = 0.5$; $r = 0.04$; $a_1 = 0.5$; $a_2 = 0.23$; $b_2 = 0.4$; $b_2 = 2$; $c = 0.0001$; with basic circumstances $x(0) = y(0) = z(0) = 2$. Fig. 1, the representation of the growth rates of different species, indicates that the system is naturally unstable. The remaining two species y, z gradually go extinct over time, but only the first competitive species (x) persists in nature for a long period.



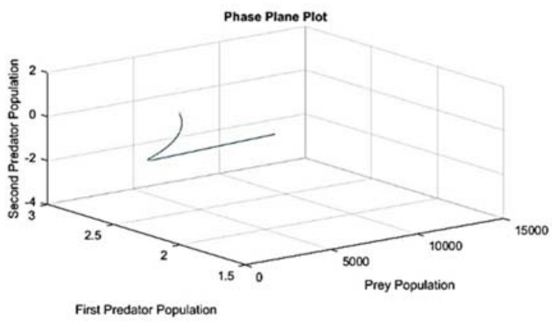


Fig. 1 Time series depicting the behavior for Case1

Case2. Now, at the parameter value $a_1 = 2.5$; $a_2 = 1.5$; $b_1 = 0.05$; $b_2 = 1.5$; $c = 2.9$; $d_1 = 2.08$; $d_2 = 0.02$; $d_3 = 0.5$; $r = 7.5$; the condition (iii) of Theorem 3 satisfied that the dynamical system is always unstable as can be evidenced in Fig. 2.

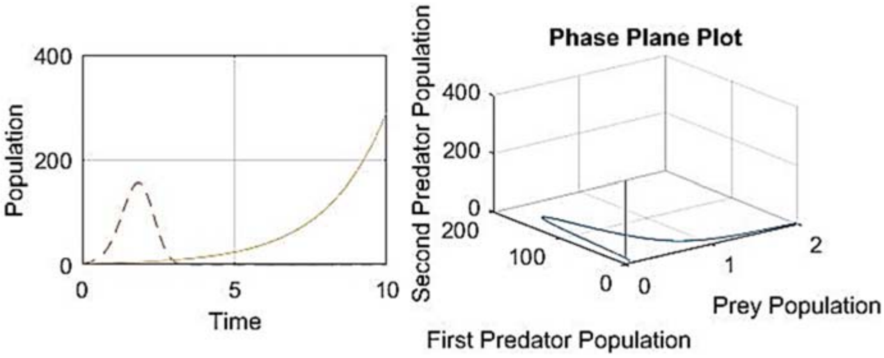


Fig. 2 Time series depicting the behavior of case 2

III. CONCLUSION

In this article, we look at the ecological model of a three-type food chain containing two species that compete with one another, including hosts and predators. A third species often helps the first species in these interactions by eating its rival in the CMR reaction to work in a competition for the availability of natural resources. At the conclusion, mathematical justifications are offered to justify the analytical findings made with sound knowledge. Finally, these data suggest that while such situations do arise in nature, their severity and diversity will prevent them from persisting over the long term. The two competing species x, y will survive as a result of their competition if the host species z survives.

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