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Explicit Solution of an Investment Plan for a DC Pension Scheme with Voluntary Contributions and Return Clause under Logarithm Utility

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***Abstract*—**The paper merged the return of premium clause and voluntary contributions to investigate retirees' investment plan in a defined contributory (DC) pension scheme with a portfolio comprising of a risk-free asset and a risky asset whose price process is described by geometric Brownian motion (GBM). The paper considers additional voluntary contributions paid by members, charge on balance by pension fund administrators and the mortality risk of members of the scheme during the accumulation period by introducing return of premium clause. To achieve this, the Weilbull mortality force function is used to establish the mortality rate of members during accumulation phase. Furthermore, an optimization problem from the Hamilton Jacobi Bellman (HJB) equation is obtained using dynamic programming approach. Also, the Legendre transformation method is used to transform the HJB equation which is a nonlinear partial differential equation to a linear partial differential equation and solves the resultant equation for the value function and the optimal distribution plan under logarithm utility function. Finally, numerical simulations of the impact of some important parameters on the optimal distribution plan were obtained and it was observed that the optimal distribution plan is inversely proportional to the initial fund size, predetermined interest rate, additional voluntary contributions, charge on balance and instantaneous volatility.

***Keywords*—**Legendre transform, logarithm utility, optimal distribution plan, return clause of premium, charge on balance, Weibull mortality function.

I. Introduction

THE DC pension scheme is designed to enable pension fund administrators manage and plan for its members' retirement needs. In this scheme, members (employees) and employers are mandated to contribute a certain percentage of employee's income into their retirement savings account (RSA). Following the Nigerian pension reform act 2004, the employee and the employer contribute 8% and 10% respectively of the employee's income into the employee's RSA [1]. Also, members are allowed to contribute voluntarily into their RSA account different from the compulsory contributions; this additional voluntary contribution may be constant or stochastic depending on the member's preference [2]. However, in the DC scheme, member's benefits depend solely on returns of investment during the accumulation phase. Hence, there is need to study and develop an investment plan capable of managing a portfolio in the financial market with risk free and risky assets that combine compulsory and voluntary contributions together. This has led to the study of optimal distribution plan which explains how investment portfolios are managed for maximum returns. Some of the researches which studied optimal distribution plan involving additional voluntary include [2], which studied the optimal distribution plan and the impact of additional voluntary contribution on the investment strategy; they observed that optimal distribution plan is a decreasing function of the additional voluntary contribution. In [3], the stochastic optimal investment plan under inflammatory market with minimum guarantee was studied. In their work, members contributed extra funds to amortize the pension fund. In [4], the effect of additional voluntary contribution on the optimal investment strategy in a DC pension with stochastic salary under affine interest model was studied; in their model, the extra contribution was constant. Also, the work of [4] was later extended by [5] from that of constant additional voluntary contribution to stochastic additional voluntary contribution to obtain the optimal distribution plan.

Apart from studying optimal distribution plan involving additional voluntary contributions, there are a lot of papers which investigated optimal distribution plan when the price process of the risky assets is modelled using GBM and constant elasticity of variance model. They include but not limited to [6]-[13] which studied the optimal distribution plan for portfolios whose risky assets were modelled using GBM model with constant volatility as in the Black-Scholes model. Following the extension of the GBM process, another volatility model known as the constant elasticity of variance (CEV) model has been used by some authors to model the stock market prices. They include but not limited to [14]-[16].

Considering the fact that not all members of the DC pension scheme stand a chance of seeing through the accumulation phase due to mortality risk, a good number of authors such as [17]-[22] have studied the optimal distribution plan with return of premium clause under different assumptions. They developed investment strategies that aimed at protecting the rights of members and their families in case of mortality during the accumulation period. They used the mortality force functions which include Abraham De Moivre's model [17]-[21] and Weilbull force function [22], [23] to form the pension wealth. In [17], the optimal distribution plan with return of contribution clause was studied where the risky asset was

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modelled using GBM under mean variance utility. Also, in [18], the optimal distribution plan with return of contribution clause was studied under Heston volatility model using mean-variance utility. In [19], the optimal distribution plan with return of premium under CEV model was studied using mean-variance utility function. Reference [20] used the Jump diffusion model to model the risky asset and determined the optimal distribution plan with return of premium for the pension plan member. Also, [24] and [25] studied the optimal distribution plan for a DC plan where the return of premium is with predetermined interest; they assumed the returned premiums are with predetermined interest from investment on risk free asset. In each of the above, they used the Abraham De Moivre model as their mortality force function. On the other hand, optimal distribution plan with return of premium clause was studied in [22] and [23] under Weibull force function as their mortality force function using different utility functions.

The main contribution of this paper is the combination of voluntary and compulsory contribution with return of premium clause using Weibull force function. Also, the logarithm utility is used instead of mean variance utility and Legendre transformation method is used instead of game theoretic approach. Finally, the charge on balance similar to [22] is introduce into this paper and solve for the optimal distribution plan.

II. Wealth Function with Voluntary Contributions, Charge on Balance and Return of Premium Clause

Let $\mathcal{B}_t(t)$ represents the price of the risk-free asset and its price process follows the following dynamics as in [17] and [25]

$$\begin{cases}\frac{d\mathcal{B}_t(t)}{\mathcal{B}_t(t)}=\mathfrak{K}dt\\ \mathcal{D}_0(0)=d_0>0\end{cases}, \tag{1}$$

where $\mathfrak{K}>0$ is the predetermined interest rate of the risk free asset.

Similarly, the pension fund administrator may also be willing to invest in a risky asset (stock) modelled by the GBM whose price process is given as follows as in [17] and [25]:

$$\begin{cases}\frac{d\mathcal{D}_t(t)}{\mathcal{D}_t(t)}=\mu dt+\vartheta dM_t\\ \mathcal{D}_t(0)=1\end{cases}, \tag{2}$$

where μ is the expected appreciation rate of $\mathcal{D}_t(t)$, ϑ is the volatility of the stock market price and M_t is the Brownian motion generating the available information in the market generated represented by $\mathcal{F}_t$ called the filtration in a complete probability space $(\Omega,\mathcal{F}_t,\mathcal{P})$, where Ω, is a real space and $\mathcal{P}$ a probability measure.

Next, we consider $\pi(t)$ to be the fraction of the accumulated wealth to be invested in risky asset and $1-\pi(t)$, the fraction to be invested in a risk-free asset.

Let a be the monthly contributions at a given time by the pension member, $\aleph_0$ the initial age during the accumulation phase, T the accumulation phase period, and $\aleph_0+T$ is the terminal age of the member. $\mathcal{V}_{\frac{1}{z},\aleph_0+t}$ is the mortality rate from time t to $t+\frac{1}{z}$, ta is the accumulated contributions at time t, $ta\mathcal{V}_{\frac{1}{z},\aleph_0+t}$ is the returned contributions to the death members' families within the accumulation period. Also, let ρ be the charge on balance which is determined based on the value by the pension fund administrators [26].

From the work of [17]-[21], if we consider the accumulation period $[t,t+\frac{1}{z}]$ and the investment strategy for the risky asset, the differential form of the fund size is given as:

$$\mathcal{X}\left(t+\frac{1}{z}\right)=\begin{bmatrix}\mathcal{X}(t)\begin{pmatrix}(1-\pi(t))\frac{\mathcal{B}_{t+\frac{1}{z}}}{\mathcal{B}_t}\\ +\pi(t)\frac{\mathcal{D}_{t+\frac{1}{z}}}{\mathcal{D}_t}\end{pmatrix}\\ -\rho\mathcal{X}(t)\frac{1}{z}+a\frac{1}{z}\\ +\varepsilon dM_t-ta\mathcal{V}_{\frac{1}{z},\aleph_0+t}\end{bmatrix}\left(\frac{1}{1-\mathcal{V}_{\frac{1}{z},\aleph_0+t}}\right) \tag{3}$$

$$\mathcal{X}\left(t+\frac{1}{z}\right)=\begin{bmatrix}\mathcal{X}(t)\begin{pmatrix}(1-\pi(t))\begin{pmatrix}\frac{\mathcal{B}_{t+\frac{1}{z}}}{\mathcal{B}_t}\\ -\frac{\mathcal{B}_t}{\mathcal{B}_t}\end{pmatrix}\\ +\pi(t)\begin{pmatrix}\frac{\mathcal{D}_{t+\frac{1}{z}}}{\mathcal{D}_t}\\ -\frac{\mathcal{D}_t}{\mathcal{D}_t}\end{pmatrix}\\ +1-\frac{\rho 1}{z}\end{pmatrix}\\ +\varepsilon dM_t+a\frac{1}{z}-ta\mathcal{V}_{\frac{1}{z},\aleph_0+t}\end{bmatrix}\begin{pmatrix}1+\\ \frac{\mathcal{V}_{\frac{1}{z},\aleph_0+t}}{1-\mathcal{V}_{\frac{1}{z},\aleph_0+t}}\end{pmatrix} \tag{4}$$

$$\mathcal{X}\left(t+\frac{1}{z}\right)-\mathcal{X}(t)=\begin{bmatrix}\mathcal{X}(t)\begin{pmatrix}(1-\pi(t))\left(\frac{\mathcal{B}_{t+\frac{1}{z}}}{\mathcal{B}_t}-\frac{\mathcal{B}_t}{\mathcal{B}_t}\right)\\ +\pi(t)\left(\frac{\mathcal{D}_{t+\frac{1}{z}}}{\mathcal{D}_t}-\frac{\mathcal{D}_t}{\mathcal{D}_t}\right)-\rho\frac{1}{z}\end{pmatrix}\\ +\varepsilon dM_t+a\frac{1}{z}-ta\mathcal{V}_{\frac{1}{z},\aleph_0+t}\end{bmatrix}\left(1+\frac{\mathcal{V}_{\frac{1}{z},\aleph_0+t}}{1-\mathcal{V}_{\frac{1}{z},\aleph_0+t}}\right) \tag{5}$$

$$\begin{cases}\mathcal{V}_{\frac{1}{z},\aleph_0+t}=1-\mathrm{Exp}\{-\int_0^{\frac{1}{z}}\mathcal{Q}(\aleph_0+t+e)de\}\\ \qquad=\mathcal{Q}(\aleph_0+t)i+O(\frac{1}{z}),\\ \frac{\mathcal{V}_{\frac{1}{z},\aleph_0+t}}{1-\mathcal{V}_{\frac{1}{z},\aleph_0+t}}=\mathcal{Q}(\aleph_0+t)\frac{1}{z}+O(\frac{1}{z})\\ \frac{1}{z}\to 0,\mathcal{V}_{\frac{1}{z},\aleph_0+t}=\mathcal{Q}(\aleph_0+t)dt,\\ \frac{\mathcal{V}_{\frac{1}{z},\aleph_0+t}}{1-\mathcal{V}_{\frac{1}{z},\aleph_0+t}}=A(\aleph_0+t)dt,\\ a\frac{1}{z}\to adt,\rho\frac{1}{z}\to\rho dt,\frac{\mathcal{B}_{t+\frac{1}{z}}}{\mathcal{B}_t}-\frac{\mathcal{B}_t}{\mathcal{B}_t}\to\frac{d\mathcal{B}_t(t)}{\mathcal{B}_t(t)},\\ \frac{\mathcal{D}_{t+\frac{1}{z}}}{\mathcal{D}_t}-\frac{\mathcal{D}_t}{\mathcal{D}_t}\to\frac{d\mathcal{D}_t(t)}{\mathcal{D}_t(t)}\end{cases} \tag{6}$$

Substituting (6) into (5) we have

$$d\mathcal{X}(t) = \begin{bmatrix} \mathcal{X}(t)\begin{pmatrix} \pi(t)\frac{d\mathcal{S}_t(t)}{\mathcal{S}_t(t)} \\ +\left(1-\pi(t)\right)\frac{dD_t(t)}{D_t(t)} \\ -\rho dt \end{pmatrix} \\ +\varepsilon dM_t + adt \\ -taQ(\aleph_0+t)dt \end{bmatrix}\begin{pmatrix} 1+ \\ \mathcal{Q}(\aleph_0+t)dt \end{pmatrix} \quad (7)$$

where $\mathcal{Q}(t)$ is the force function and ϑ is the maximal age of the life table. From [15], $\mathcal{Q}(t)$ is given as:

$$\mathcal{Q}(t) = kt^n \quad 0 \le t < T$$

This implies that

$$\mathcal{Q}(\aleph_0+t) = k(\aleph_0+t)^n \quad (8)$$

Substituting (1), (2) and (8) into (7), we have

$$d\mathcal{X}(t) = \begin{bmatrix} \begin{Bmatrix} \mathcal{X}(t)\begin{pmatrix} \pi(t)(\mu-\mathfrak{K})+(\mathfrak{K}-\rho) \\ +k(\aleph_0+t)^n \end{pmatrix} \\ +a(1-tk(\aleph_0+t)^n) \end{Bmatrix} dt \\ +(\mathcal{X}(t)\pi(t)\vartheta+\varepsilon)dM_t \\ \mathcal{X}(0)=x_0 \end{bmatrix} \quad (9)$$

$$d\mathcal{X}(t) = \begin{bmatrix} \begin{Bmatrix} \mathcal{X}(t)\begin{pmatrix} \pi(t)(\mu-\mathfrak{K})+(\mathfrak{K}-\rho) \\ +k(\aleph_0+t)^n \end{pmatrix} \\ +a(1-tk(\aleph_0+t)^n) \end{Bmatrix} dt \\ +(\mathcal{X}(t)\pi(t)\vartheta+\varepsilon)dM_t \\ \mathcal{X}(0)=x_0 \end{bmatrix} \quad (10)$$

III. Optimization Problem

A. Wealth Formulation and HJB Equation

Let $\pi(t)$ be the optimal portfolio strategy and we define the utility attained by the investor from a given state x at time t as

$$\mathcal{A}_{\pi(t)}(t,x) = E_{\pi(t)}\left[U(\mathcal{X}(T)) \mid \mathcal{X}(t)=x\right], \quad (11)$$

where t is the time and x is the wealth. The objective here is to determine the optimal portfolio strategy and the optimal value function of the investor given as

$$\pi(t)^* \text{and} \mathcal{A}(t,x) = \sup_{\pi(t)} \mathcal{A}_{\pi(t)}(t,x) \quad (12)$$

Respectively such that

$$\mathcal{A}_{\pi(t)^*}(t,x) = \mathcal{A}(t,x). \quad (13)$$

From [31], applying the Ito's lemma and maximum principle, the HJB equation which is a nonlinear PDE associated with (10) is obtained by maximizing $\mathcal{A}_{\varphi^*}(t,x)$ subject to the insurer's wealth as follows

$$\begin{pmatrix} \mathcal{A}_t + \begin{pmatrix} x(\mathfrak{K}-\rho+k(\aleph_0+t)^n) \\ +a(1-tk(\aleph_0+t)^n) \end{pmatrix}\mathcal{A}_x \\ +\sup_{P(t)}\begin{Bmatrix} \pi(t)(\mu-\mathfrak{K})\mathcal{L}_x \\ +\frac{1}{2}(x\pi(t)\vartheta+\varepsilon)^2\mathcal{A}_{xx} \end{Bmatrix} \end{pmatrix} = 0 \quad (14)$$

Differentiating (14) with respect to $\pi(t)$, we obtain the first order maximizing condition for (14) as

$$\pi(t)^* = -\frac{[(\mu-\mathfrak{K})\mathcal{A}_x + \vartheta\varepsilon\mathcal{A}_{xx}]}{x\vartheta^2\mathcal{A}_{xx}} \quad (15)$$

Substituting (15) into (14), we have

$$\begin{Bmatrix} \mathcal{A}_t + \begin{pmatrix} x(\mathfrak{K}-\rho+k(\aleph_0+t)^n) \\ -\left(\frac{\mu-\mathfrak{K}}{\vartheta}\right)\varepsilon + a(1-tk(\aleph_0+t)^n) \end{pmatrix}\mathcal{A}_x \\ -\frac{1}{2}\frac{(\mu-\mathfrak{K})^2}{\vartheta^2}\frac{\mathcal{A}_x^2}{\mathcal{A}_{xx}} \end{Bmatrix} = 0 \quad (16)$$

B. Legendre Transform and Dual Theory

The differential equation in (16) is a nonlinear partial differential equation and is difficult to solve with direct method hence, we introduce the Legendre transformation and dual theory to transform the nonlinear partial differential equation to a linear partial differential equation.

Theorem1. Let $\hbar: \mathcal{R}^n \to \mathcal{R}$ be a convex function for $\ell > 0$, define the Legendre transform

$$\mathcal{H}(\ell) = \max_x\{\hbar(x) - \ell x\}, \quad (17)$$

where $\mathcal{H}(\ell)$ is the Legendre dual of $\hbar(x)$.

Since $\hbar(x)$ is convex, from Theorem 1 and [27] and [28], the Legendre transform for the value function $\mathcal{A}(t,x)$ can be defined as follows

$$\hat{\mathcal{A}}(t,\ell) = \sup\left\{\begin{matrix}\mathcal{A}(t,x) \\ -\ell x\end{matrix} \,\middle|\, 0 < x < \infty\right\} \; 0 < t < T \quad (18)$$

where $\hat{\mathcal{A}}$ is the dual of $\mathcal{A}$ and $\ell > 0$ is the dual variable of x.

The value of x where this optimum is achieved is represented by $\hbar(t,\ell)$, such that

$$\hbar(t,\ell) = \inf\left\{x \mid \mathcal{A} \ge \ell x + \hat{\mathcal{A}}(t,\ell)\right\} \; 0 < t < T. \quad (19)$$

From (19), the function $\hbar$ and $\hat{\mathcal{A}}$ are very much related and can be referred to as the dual of $\mathcal{A}$ and are related thus

$$\hat{\mathcal{A}}(t,\ell) = \mathcal{A}(t,\hbar) - \ell\hbar. \quad (20)$$

where

$$\hbar(t,\ell) = x, \mathcal{A}_x = \ell, \; g = -\hat{\mathcal{A}}_\ell. \quad (21)$$

Differentiating (20) with respect to $t, and x$

$$\left\{\mathcal{A}_t = \hat{\mathcal{A}}_t, \; \mathcal{A}_x = \ell, \mathcal{A}_{xx} = \frac{-1}{\hat{\mathcal{A}}_{\ell\ell}}\right. \quad (22)$$

At terminal time T, we define the dual utility in terms of the original utility function $U(x)$ as

$$\hat{U}(\ell) = sup\{U(x) - \ell x \mid 0 < x < \infty\},$$

and

$$G(\ell) = sup\,\{x \mid U(x) \ge \ell x + \hat{U}(\ell).$$

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As a result, $\hat{\mathcal{A}}(t,\ell) = \mathcal{A}(t,\hbar) - \ell\hbar$.

$$G(\ell) = (U')^{-1}(\ell), \tag{23}$$

where G is the inverse of the marginal utility U and we note that $\mathcal{A}(T,x) = U(x)$. At terminal time T, we can define

$$\hbar(T,\ell) = \inf_{x>0}\{ x \mid U(x) \geq \ell x + \hat{\mathcal{A}}(t,\ell)\} \, and \, \hat{\mathcal{A}}(t,\ell) = \sup_{x>0}\{ U(x) - \ell x\}$$

so that

$$\hbar(T,\ell) = (U')^{-1}(\ell). \tag{24}$$

Substituting (22) into (15) and (16), we have

$$\begin{Bmatrix} \hat{\mathcal{A}}_t + \begin{bmatrix} \hbar(\mathfrak{K} - \rho + k(\aleph_0 + t)^n) \\ -\left(\frac{\mu-\mathfrak{K}}{\vartheta}\right)\varepsilon + a(1 - tk(\aleph_0 + t)^n) \end{bmatrix} \ell \\ -\frac{1}{2}\left(\frac{(\mu-\mathfrak{K})^2}{\vartheta^2}\right)\ell^2 \hat{\mathcal{A}}_{\ell\ell} \end{Bmatrix} = 0 \tag{25}$$

$$\pi(t)^* = -\left[\frac{(\mu-\mathfrak{K})\ell\hat{\mathcal{A}}_{\ell\ell}}{x\vartheta^2} + \frac{\varepsilon}{x\vartheta}\right] \tag{26}$$

From (21), differentiating (25) and (26) with respect to ℓ, we have

$$\begin{Bmatrix} \hbar_t - \begin{bmatrix} \hbar(\mathfrak{K} - \rho + k(\aleph_0 + t)^n) \\ -\left(\frac{\mu-\mathfrak{K}}{\vartheta}\right)\varepsilon + a(1 - tk(\aleph_0 + t)^n) \end{bmatrix} \\ -\ell\hbar_\ell(r - \rho + k(\aleph_0 + t)^n) \\ +\ell\hbar_\ell\frac{(\mu-\mathfrak{K})^2}{\vartheta^2} + \frac{1}{2}\left(\frac{(\mu-\mathfrak{K})^2}{\vartheta^2}\right)\ell^2\hbar_{\ell\ell} \end{Bmatrix} = 0, \tag{27}$$

$$\pi(t)^* = -\left[\frac{(\mu-\mathfrak{K})}{x\vartheta^2}\ell\hbar_\ell + \frac{\varepsilon}{x\vartheta}\right]. \tag{28}$$

where, $\mathcal{A}(T,\ell) = U(\ell)$ and $U(\ell)$ is the marginal utility of the investor. Next, we proceed to solve (27) for $\hbar$ considering an insurer with logarithm utility, then we substitute the solution into (28) for the optimal investment plan using change of variable approach.

IV. The Optimal Distribution Plan Under Logarithm Utility

Here, we consider an insurer with utility function exhibiting constant relative risk aversion (CRRA). Since our interest here is to determine the optimal distribution plan for pension fund member with return of premium, voluntary contribution, charge on balance and CRRA utility, we choose the logarithm utility function similar to the one in [29] and [30].

From [29] and [30], the logarithm utility function is given as

$$U(x) = lnx, x > 0$$

From (24),

$$\hbar(T,\ell) = (U')^{-1}(\ell) = \frac{1}{\ell} \tag{29}$$

Next, we conjecture a solution to (27) similar to the one in [21] with the form:

$$\begin{cases} \hbar(t,\ell) = \frac{1}{\ell}f(t) + g(t) \\ f(T) = 1, \ g(T) = 0, \end{cases} \tag{30}$$

$$\hbar_t = \frac{1}{\ell}f_t + g_t, \ \hbar_\ell = -\frac{1}{\ell^2}f, \quad \hbar_{\ell\ell} = \frac{2}{\ell^3}f\} \tag{31}$$

Substituting (31) into (27), we have

$$\begin{Bmatrix} \begin{bmatrix} g_t - (\mathfrak{K} - \rho + k(\aleph_0 + t)^n)g + \\ \left(\frac{\mu-\mathfrak{K}}{\vartheta}\right)\varepsilon - a(1 - tk(\aleph_0 + t)^n) \end{bmatrix} \\ +\frac{1}{\ell}\begin{bmatrix} f_t + (\mathfrak{K} - \rho + k(\varphi_0 + t)^n)f \\ -(\mathfrak{K} - \rho + k(\aleph_0 + t)^n)f \end{bmatrix} \end{Bmatrix} = 0 \tag{32}$$

Splitting (32) we have

$$\begin{cases} g_t - (\mathfrak{K} - \rho + k(\aleph_0 + t)^n)g \\ +\left(\frac{\mu-\mathfrak{K}}{\vartheta}\right)\varepsilon - a(1 - tk(\aleph_0 + t)^n) = 0 \\ g(T) = 0 \end{cases} \tag{33}$$

$$\begin{cases} f_t = 0 \\ f(T) = 1 \end{cases} \tag{34}$$

Solving (34) for f, we obtain

$$f(t) = 1 \tag{35}$$

also, solving (33) for g, we have

$$g(t) = \frac{a\begin{bmatrix} \frac{k}{n+2}\begin{bmatrix} (\aleph_0+T)^{n+2} \\ -(\aleph_0+t)^{n+2} \end{bmatrix} - \frac{k\aleph_0}{n+1}\begin{bmatrix} (\varphi_0+T)^{n+1} \\ -(\aleph_0+t)^{n+1} \end{bmatrix} \\ +\left(\left(\frac{\mu-\mathfrak{K}}{\vartheta}\right)\varepsilon+1\right)(T-t) \end{bmatrix}}{(\mathfrak{K}-\rho)(T-t) - \frac{k}{n+1}[(\aleph_0+T)^{n+1} - (\aleph_0+t)^{n+1}]} \tag{36}$$

Substituting (35) and (36) into (30) and (31), we obtain

$$\hbar(t,\ell) = \frac{1}{\ell} + \frac{a\begin{bmatrix} \frac{k}{n+2}\begin{bmatrix} (\aleph_0+T)^{n+2} \\ -(\aleph_0+t)^{n+2} \end{bmatrix} - \frac{k\aleph_0}{n+1}\begin{bmatrix} (\aleph_0+T)^{n+1} \\ -(\aleph_0+t)^{n+1} \end{bmatrix} \\ +\left(\left(\frac{\mu-\mathfrak{K}}{\vartheta}\right)\varepsilon+1\right)(T-t) \end{bmatrix}}{(\mathfrak{K}-\rho)(T-t) - \frac{k}{n+1}[(\aleph_0+T)^{n+1} - (\aleph_0+t)^{n+1}]} \tag{37}$$

$$\hbar_\ell = -\frac{1}{\ell^2}, x = \hbar \tag{38}$$

also, from (37), we have

$$\frac{1}{\ell} = \hbar - \frac{a\begin{bmatrix} \frac{k}{n+2}\begin{bmatrix} (\aleph_0+T)^{n+2} \\ -(\aleph_0+t)^{n+2} \end{bmatrix} - \frac{k\aleph_0}{n+1}\begin{bmatrix} (\aleph_0+T)^{n+1} \\ -(\aleph_0+t)^{n+1} \end{bmatrix} \\ +\left(\left(\frac{\mu-\mathfrak{K}}{\vartheta}\right)\varepsilon+1\right)(T-t) \end{bmatrix}}{(\mathfrak{K}-\rho)(T-t) - \frac{k}{n+1}[(\aleph_0+T)^{n+1} - (\aleph_0+t)^{n+1}]} \tag{39}$$

Substituting (38) and (39) into (28), we obtain the optimal distribution plan for a member with logarithm utility function.

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$$\pi(t)^* = \frac{1}{x\vartheta}\left[\frac{(\mu-\mathfrak{K})}{\vartheta}\left(x - \frac{a\left[\begin{array}{l}\frac{k}{n+2}\left[\begin{array}{l}(\aleph_0+T)^{n+2}\\-(\aleph_0+t)^{n+2}\end{array}\right]\\-\frac{k\aleph_0}{n+1}\left[\begin{array}{l}(\aleph_0+T)^{n+1}\\-(\aleph_0+t)^{n+1}\end{array}\right]\\+\left(\left(\frac{\mu-\mathfrak{K}}{\vartheta}\right)\varepsilon+1\right)(T-t)\end{array}\right]}{\left(\begin{array}{l}(\mathfrak{K}-\rho)(T-t)\\-\frac{k}{n+1}\left[\begin{array}{l}(\aleph_0+T)^{n+1}\\-(\aleph_0+t)^{n+1}\end{array}\right]\end{array}\right)}\right) - \varepsilon\right]. \qquad (40)$$

V. Numerical Simulations and Discussion

In this section, we present some numerical simulations to illustrate the impact of some sensitive parameters on the optimal distribution plan of a pension scheme with return clause, charge on balance and voluntary contributions. In achieving this, (40) was used as our focus equation with the following parameters: $\mu = 0.2$, $\mathfrak{K} = 0.1$, $x = 1$, $\vartheta = 0.6$, $\rho = 0.05$, $\varepsilon = 0.1$, $\aleph_0 = 10$, $T = 20$, $k = 0.01$, $n = 0.001$.

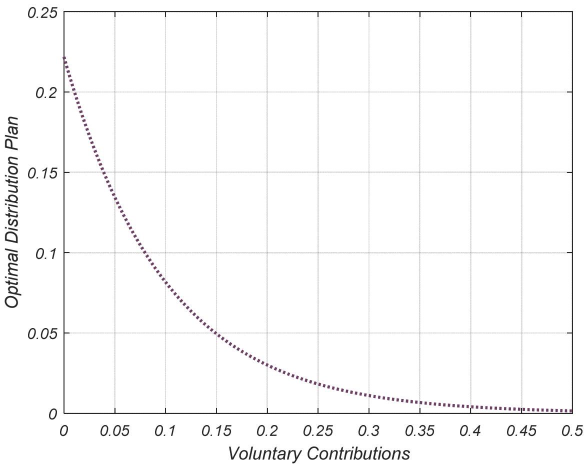

Fig. 1 Impact of voluntary on optimal distribution plan

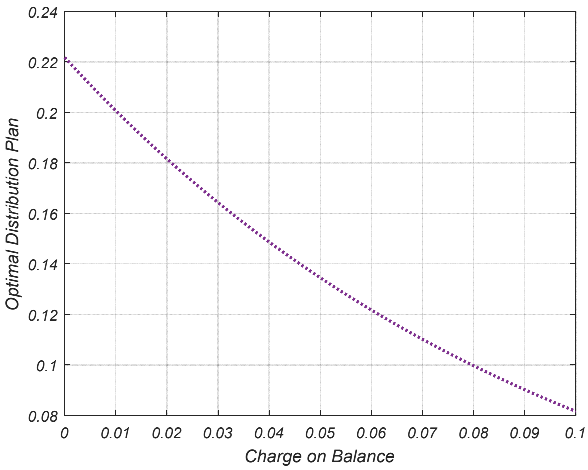

Fig. 2 Impact of charge on balance on optimal distribution plan

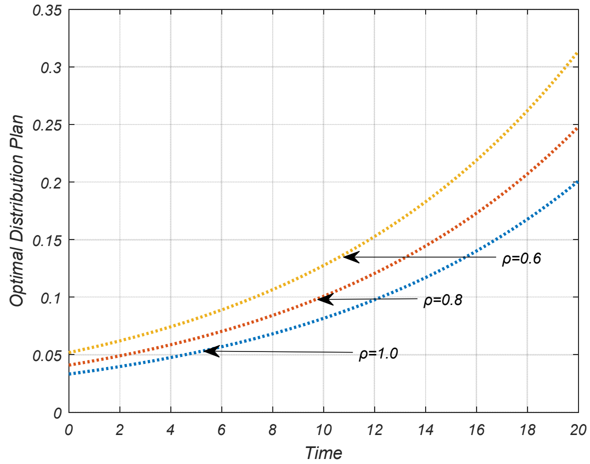

Fig. 3 Impact of instantaneous volatility on optimal distribution plan

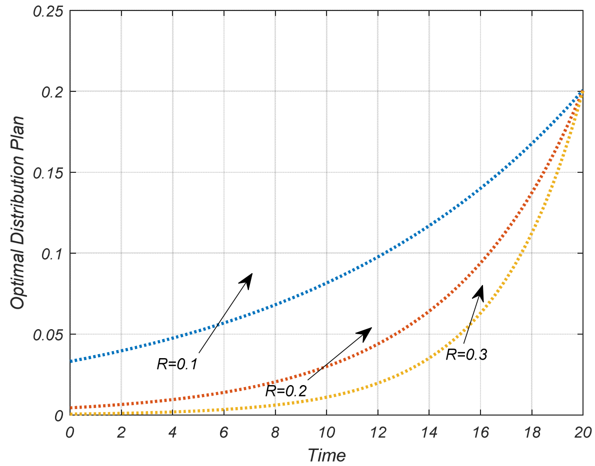

Fig. 4 Impact of risk free interest rate optimal distribution plan

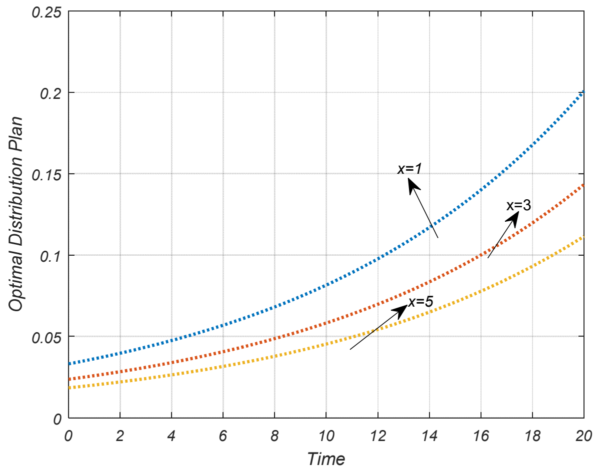

Fig. 5 Impact of Initial fund size on optimal distribution plan

Fig. 1 gives the graph of optimal distribution plan against additional voluntary contribution. It was observed that the optimal distribution plan for a pension plan member decreases as the additional voluntary contributions increase and vice versa. The implication of Fig. 1 shows that when there is more

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Open Science Index, Physical and Mathematical Sciences Vol:17, No:8, 2023 publications.waset.org/10013228.pdf

money in the pension plan member's RSA account, the pension plan member may be cautious on how to invest in risky asset to avoid losing what they have already; hence a reduction in the proportion of his/her wealth to be invested in the risky asset. On the other hand, if the money in the member's RSA account is not sufficient to take care of the member's need after retirement, the member may be willing to invest more in the risky asset with aim of expecting higher returns before retirement time.

Fig. 2 gives the graph of optimal distribution plan against charge on balance of the member's asset under investment. It was observed that the optimal distribution plan used by the pension plan member decreases as charge on balance increases. The implication of Fig. 2 is that, the higher the charge on balance on investment of the risky asset, the more members of the scheme will be discouraged from investing more in risky asset and vice versa.

Fig. 3 gives the graph of optimal distribution plan against instantaneous volatility and it was observed that the optimal distribution plan for the risky asset decreases as the instantaneous volatility increases. We deduced from the graph that more volatile asset may dread the member knowing full well that they stand a chance of losing what they have and vice versa.

Also, Fig. 4 gives the graph of optimal distribution plan against the risk-free interest rate. It is observed that the optimal distribution plan decreases as the risk-free interest rate increases; this shows that members will prefer to invest in risky asset when the risk-free interest rate is not attractive. However, if the risk-free interest rate is attractive enough, members may invest more in the risk-free asset, thereby reducing their investment in the risky asset.

Fig. 5 gives the graph of optimal distribution plan against initial fund size. It was observed that the optimal distribution plan for the risky asset decreases as the initial fund size parameter of the member increases. The implication of Fig. 5 is that if the initial fund size at the time of investment is reasonably large, members may want to be very careful and cautious and may be discouraged from taking more risks thereby reducing the proportion of his or her wealth to be invested in the risky asset and vice versa.

VI. Conclusion

In this work, the optimal distribution plan with return of premium clause was studied under mortality risk using Weibull force function. We considered a portfolio with voluntary contributions, mandatory contribution, one risk free and a risky asset modeled by GBM model. Furthermore, an optimization problem was obtained from the HJB equation by the method of dynamic programming and we used the Legendre transformation and dual theory to reduce the HJB equation to a linear partial differential equation (PDE). The resultant linear PDE was then solved by method of separation of variable using logarithm utility function. From Figs. 1 to 5, it was observed that as the initial fund size, predetermined interest rate, additional voluntary contributions, charge on balance and instantaneous volatility increase, the optimal distribution plan decreases.

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