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# Optimal Relaxation Parameters for Obtaining Efficient Iterative Methods for the Solution of Electromagnetic Scattering Problems

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*Abstract*—The approximate solution of a time-harmonic electromagnetic scattering problem for inhomogeneous media is required in several application contexts and its two-dimensional formulation is a Fredholm integral equation of second kind. This integral equation provides a formulation for the direct scattering problem but has to be solved several times in the numerical solution of the corresponding inverse scattering problem. The discretization of this Fredholm equation produces large and dense linear systems that are usually solved by iterative methods. To improve the efficiency of these iterative methods, we use the Symmetric SOR preconditioning and propose an algorithm to evaluate the associated relaxation parameter. We show the efficiency of the proposed algorithm by several numerical experiments, where we use two Krylov subspace methods, i.e. Bi-CGSTAB and GMRES.

*Keywords*—Fredholm integral equation, iterative method, preconditioning, scattering problem.

## I. Introduction

We consider the numerical solution of the Lippmann-Schwinger equation, see [10] for details. This is a Fredholm integral equation of the second kind resulting from electromagnetic scattering problems, whose approximate solution is required in several application contexts, see [16] for some examples. Hence, different strategies for solving the Lippmann-Schwinger equation have been studied [2]–[5], [7], [12].

Even if the considered integral equation provides a formulation for the direct scattering problem, it has to be solved several times also in the numerical solution of the corresponding inverse scattering problem. So, effective methods for solving this equation are also relevant to the corresponding inverse problem.

Discretization schemes, for this integral equation, give dense linear systems that are usually solved by using iterative methods as a consequence of their size [4], [17]. In order to decrease the computational cost of these methods and/or reduce the number of iterations and so improve the efficiency of the method, preconditioning techniques are usually used in these discretization schemes or directly in the integral formulation, see for example [5], [7], [8], [12]. Preconditioning techniques are key instruments to improve the efficiency and robustness of iterative methods, see [15] for a complete discussion. These techniques for linear systems are based on a quite simple idea: The original linear system is transformed into an equivalent linear system which is easier to solve than the original one. This transformation is given by a suitable matrix, which is usually called preconditioner, and the transformed linear system is called preconditioned system.

The computation of a good preconditioner is not an easy task. A quite general idea, for this computation, is to obtain the preconditioner from an approximation of the coefficient matrix under consideration, in such a way that the preconditioned system has coefficient matrix near the identity matrix, which is the most desirable of all the possible coefficient matrices. Moreover, this approximation has to be computed quickly and it has to be quite accurate in order to obtain an effective preconditioner. Thus, in general, this approximation is not easy to construct, theoretical results are rare and the performance of the preconditioners is usually unexpected. For example, the Jacobi preconditioner [15], that is the diagonal matrix arising from the diagonal entries of the coefficient matrix, is usually an effective preconditioner. On the contrary, the preconditioner arising from a low-rank approximation of the integral equation under study is not an effective preconditioner, even if this is an accurate approximation of the original integral equation, see [11] for details. As a consequence of this behavior, the preconditioner is sometimes derived from the properties of the original physical problem from which the linear system is obtained. In fact, some specialized methods allow to construct a preconditioner starting from the original coefficient matrix, see [15] for a detailed discussion.

In a previous paper [11], we studied some preconditioning techniques for the numerical solution of the integral equation under consideration. From this study, we concluded that Symmetric SOR preconditioning can be profitably used in these scattering problems when the relaxation parameter in the preconditioner is properly chosen. Here we present a simple interpolation algorithm for choosing the optimal relaxation parameter for the Symmetric SOR preconditioning. We note that other different preconditioning techniques have been proposed for analogous scattering problems, see [6], [9], [14] for some examples.

We first introduce some notations, $\mathbb{R}$ and $\mathbb{C}$ denote the sets of real numbers and complex numbers, respectively. Let $\nu$ be a positive integer, $\mathbb{R}^\nu$ and $\mathbb{C}^\nu$ denote the $\nu$-dimensional real Euclidean space and the $\nu$-dimensional complex Euclidean space, respectively. Let $\underline{x}$, $\underline{y} \in \mathbb{R}^\nu$, we denote with $\underline{x}^t\underline{y}$ the Euclidean scalar product of $\underline{x}$ and $\underline{y}$, the superscript $t$ means transposed, $\|\underline{x}\|$ denotes the Euclidean norm of $\underline{x}$ and $\|\underline{x}\|_\infty$ denotes the usual infinity norm of $\underline{x}$. Let $\mu$ be a positive

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integer, we denote with $\mathcal{M}_{\mathbb{C}}(\mu,\nu)$ the space of complex matrices having $\mu$ rows and $\nu$ columns. Let $A \in \mathcal{M}_{\mathbb{C}}(\nu,\nu)$ we denote with $\det(A)$ the determinant of $A$. Let $\det(A) \neq 0$, we denote with $A^{-1} \in \mathcal{M}_{\mathbb{C}}(\nu,\nu)$ the inverse of $A$. Let $\mathbb{S}^1 = \{\underline{x} \in \mathbb{R}^2 : \|\underline{x}\| = 1\}$. Let $z \in \mathbb{C}$, denote with $Re(z)$ and $Im(z)$ the real part and the imaginary part of $z$, respectively. Finally, we denote with $\imath$ the imaginary unit.

In Section II, we describe the integral formulation of the electromagnetic scattering problem and the corresponding discretization scheme. In Section III, we describe the preconditioning technique. In Section IV, we propose a simple strategy to numerically compute the optimal relaxation parameters depending on the wave number and the Krylov subspace method used and we numerically test such algorithm. In Section V, we give some conclusions.

## II. The Integral Equation

Let $(x_1, x_2, x_3)^t \in \mathbb{R}^3$, be a generic vector. We consider the propagation of a time-harmonic electromagnetic plane wave in an inhomogeneous medium having a refractive index $n$. We suppose that the refractive index is independent of the third Cartesian component $x_3$ and that the inhomogeneity is contained in $D \times \mathbb{R} \subset \mathbb{R}^3$ where $D \subset \mathbb{R}^2$ is a given compact set, so that $n = n(\underline{x})$, $\underline{x} = (x_1, x_2)^t \in \mathbb{R}^2$ and $n(\underline{x}) = 1$ for $\underline{x} \notin D$.

Moreover, supposing that the polarization vector of the incident plane wave is parallel to the third Cartesian axis, we have that the incident electric field has only one non-vanishing component, i.e. the third one, and this component depends only on $\underline{x} \in \mathbb{R}^2$. Under these symmetry assumptions, the interaction of the incident plane wave and the inhomogeneity generates a scattered electromagnetic wave, that can be supposed harmonic in time, and the corresponding electric field can be supposed to have only one non-vanishing Cartesian component, i.e. the third one, and this component depends only on $\underline{x} \in \mathbb{R}^2$. We denote with $u^i(\underline{x}) = e^{\imath k \underline{\alpha}^t \underline{x}}$, $\underline{x} \in \mathbb{R}^2$ the third Cartesian component of the incident electric field, where $k$ is the wave number, $\imath$ is the imaginary unit and $\underline{\alpha} \in \mathbb{S}^1$ is the propagation direction. We denote with $u^s(\underline{x}) \in \mathbb{C}$, $\underline{x} \in \mathbb{R}^2$, the third Cartesian component of the scattered electric field.

The problem of computing the scattered wave reduces to a two-dimensional boundary value problem, for the scalar unknown function $u(\underline{x}) = u^i(\underline{x}) + u^s(\underline{x})$, $\underline{x} \in \mathbb{R}^2$, that represents the total electric field, see [13] for a complete description of this reduction. So, from the integral formulation of such a boundary value problem, we have that, for $\underline{x} \in \mathbb{R}^2$, $u$ is the solution of the following Fredholm integral equation of the second kind:

$$u(\underline{x}) + \frac{\imath k^2}{4} \int_D H_0^{(1)}\left(k\left\|\underline{x} - \underline{y}\right\|\right) u(\underline{y}) m(\underline{y})\, d\underline{y} = u^i(\underline{x}), \quad (1)$$

where $m(\underline{x}) = 1 - n(\underline{x})$, $\underline{x} \in \mathbb{R}^2$, is called the contrast index of the medium and $H_0^{(1)}$ is the Hankel function of the first kind and order 0, see [1] for details.

Equation (1) is solved in two steps: 1) compute the solution $u(\underline{x})$, $\underline{x} \in D$ of (1) restricted to $\underline{x} \in D$, 2) from the knowledge of $u(\underline{x})$, $\underline{x} \in D$, compute $u(\underline{x})$, $\underline{x} \in \mathbb{R}^2$ by using (1). Note that step 2) is a trivial computation, so we describe only step 1), where we have to solve a Fredholm integral equation of the second kind. We discretize (1), restricted to $\underline{x} \in D$, in the following way. Let $\nu_1, \nu_2$ be two positive integers and let $J = \{(i_1, i_2), i_1 = 1, 2, \ldots, \nu_1, i_2 = 1, 2, \ldots, \nu_2\}$ be a corresponding set of indices, we denote with $\{Q_{i_1,i_2} \subset \mathbb{R}^2, (i_1, i_2) \in J\}$ a rectangular partition of a rectangle $R = [a_1, b_1) \times [a_2, b_2) \subset \mathbb{R}^2$ containing $D$. For $(i_1, i_2) \in J$ we have

$$u_{i_1,i_2} + \frac{\imath k^2}{4} \sum_{(j_1,j_2)\in J} u_{j_1,j_2} m_{j_1,j_2} \cdot \cdot \int_{Q_{j_1,j_2}} H_0^{(1)}(k\|\underline{x}_{i_1,i_2} - \underline{y}\|) d\underline{y} = u^i_{i_1,i_2}, \quad (2)$$

where, for $(i_1, i_2) \in J$, $\underline{x}_{i_1,i_2}$ is the center of the rectangle $Q_{i_1,i_2}$, $u^i_{i_1,i_2} = u^i(\underline{x}_{i_1,i_2})$, $u_{i_1,i_2} = u(\underline{x}_{i_1,i_2})$, $m_{i_1,i_2} = m(\underline{x}_{i_1,i_2})$.

Linear system (2) is rewritten as follows

$$A\underline{u} = \underline{b}, \quad (3)$$

where $\underline{b} \in \mathbb{C}^{\nu_1\nu_2}$ has components $u^i_{i_1,i_2}$, $(i_1, i_2) \in J$, $\underline{u} \in \mathbb{C}^{\nu_1\nu_2}$ has components $u_{j_1,j_2}$, $(j_1, j_2) \in J$, moreover, the coefficient matrix $A \in \mathcal{M}_{\mathbb{C}}(\nu_1\nu_2, \nu_1\nu_2)$ has the following form: $A = I + HM$, where $I \in \mathcal{M}_{\mathbb{C}}(\nu_1\nu_2, \nu_1\nu_2)$ is the identity matrix, $H \in \mathcal{M}_{\mathbb{C}}(\nu_1\nu_2, \nu_1\nu_2)$ has entry

$$\frac{\imath k^2}{4} \int_{Q_{j_1,j_2}} H_0^{(1)}(k\|\underline{x}_{i_1,i_2} - \underline{y}\|) d\underline{y}$$

at row $(i_1, i_2) \in J$ and at column $(j_1, j_2) \in J$ and $M \in \mathcal{M}_{\mathbb{C}}(\nu_1\nu_2, \nu_1\nu_2)$ is the diagonal matrix with entries $m_{j_1,j_2}$, $(j_1, j_2) \in J$.

Linear system (3) is large even for moderate values of discretization parameters $\nu_1$, $\nu_2$, so direct methods cannot be practically employed for its solution. On the other hand, linear system (3) is also dense, so iterative methods may result inefficient. Thus to speed up these methods we use preconditioning techniques that are illustrated in the following section.

## III. The Preconditioning Technique

A preconditioning scheme is any explicit or implicit transformation of the original system in order to obtain an equivalent linear system which is easier than the original one. We represent this transformation as follows

$$P^{-1}A\underline{u} = P^{-1}\underline{b}, \quad (4)$$

where the original linear system is assumed to be (3) and matrix $P$ denotes the preconditioner. For example, the scaling of the equations of a linear system in order to obtain unitary entries on the diagonal of the corresponding coefficient matrix is a preconditioning scheme; in this case, $P$ is a suitable diagonal matrix. Another example of preconditioning is obtained by the incomplete factorization of the original coefficient matrix $A$. The simplest incomplete factorization is given as follows:

$$A = LU - R, \quad (5)$$

where $L, U$ have the same nonzero structure as the lower and upper parts of $A$ and $R$ is the residual of the factorization. In this case $P^{-1}$ can be easily computed from matrices $L$ and $U$, discarding residual $R$. Unfortunately, the preconditioning arising from this incomplete factorization may be ineffective, so several alternative incomplete factorizations have been developed to overcome this problem, see [15], Chapter 10, for details. These are efficient methods to obtain preconditioners of sparse linear systems since they are specialized to avoid fill-in during the computation. But the preprocessing cost to compute the factors is usually high and this is more attractive when several systems with the same coefficient matrix must be solved, in fact in these cases the preprocessing cost can be amortized. For dense linear systems, such as the one under study, this preprocessing cost is higher than the one for sparse linear systems, so these factorization methods are usually ineffective.

For dense linear systems, a particularly attractive class of preconditioners is given by the standard iteration methods. Let $P$ and $N$ be matrices such that $A = P - N$, and $\det(P) \neq 0$. As in the usual relaxation methods, we have the following iteration

$$\underline{u}^{(l+1)} = P^{-1}N\underline{u}^{(l)} + P^{-1}\underline{b}, \tag{6}$$

where $\underline{u}^{(l)}$ is the generic $l$th approximation of the solution of (3). Formula (6) can be seen as the fixed-point iteration for the linear system

$$(I - P^{-1}N)\underline{u} = P^{-1}\underline{b}. \tag{7}$$

The coefficient matrix of this linear system can be rewritten as follows: $I - P^{-1}N = I - P^{-1}(P - A) = P^{-1}A$, so linear system (7) is equal to (4) and the relaxation scheme (6) is equivalent to the fixed-point iteration for the preconditioned linear system (4). Note that in place of the fixed-point iteration, we can use other iterative methods, such as for example the Krylov subspace methods.

Let us consider the coefficient matrix $A$ written as follows:

$$A = A_D - A_L - A_U \tag{8}$$

where $A_D$ is the diagonal of $A$, $-A_L$ is its strict lower part and $-A_U$ is its strict upper part. The Jacobi preconditioner is obtained from the following choice: $P = A_D$, $N = A_L + A_U$; so this preconditioner is given by the diagonal of the coefficient matrix. The Symmetric SOR preconditioner is obtained by the composition of two iterations, where we have: 1)$P_1 = \frac{1}{\omega}(A_D - \omega A_L)$, $N_1 = \frac{1}{\omega}(\omega A_U + (1-\omega)A_D)$; 2)$P_2 = \frac{1}{\omega}(A_D - \omega A_U)$, $N_2 = \frac{1}{\omega}(\omega A_L + (1-\omega)A_D)$. Note that $\omega \in (0, 2)$ is a relaxation parameter. Thus, we can easily see that the corresponding preconditioner is

$$P = P_\omega = \frac{1}{\omega(2-\omega)}(A_D - \omega A_L)A_D^{-1}(A_D - \omega A_U). \tag{9}$$

We note that the performance of the Symmetric SOR preconditioner is usually dependent on the choice of relaxation parameter $\omega$. In the next section, we present a simple interpolation argument to deal with such a problem.

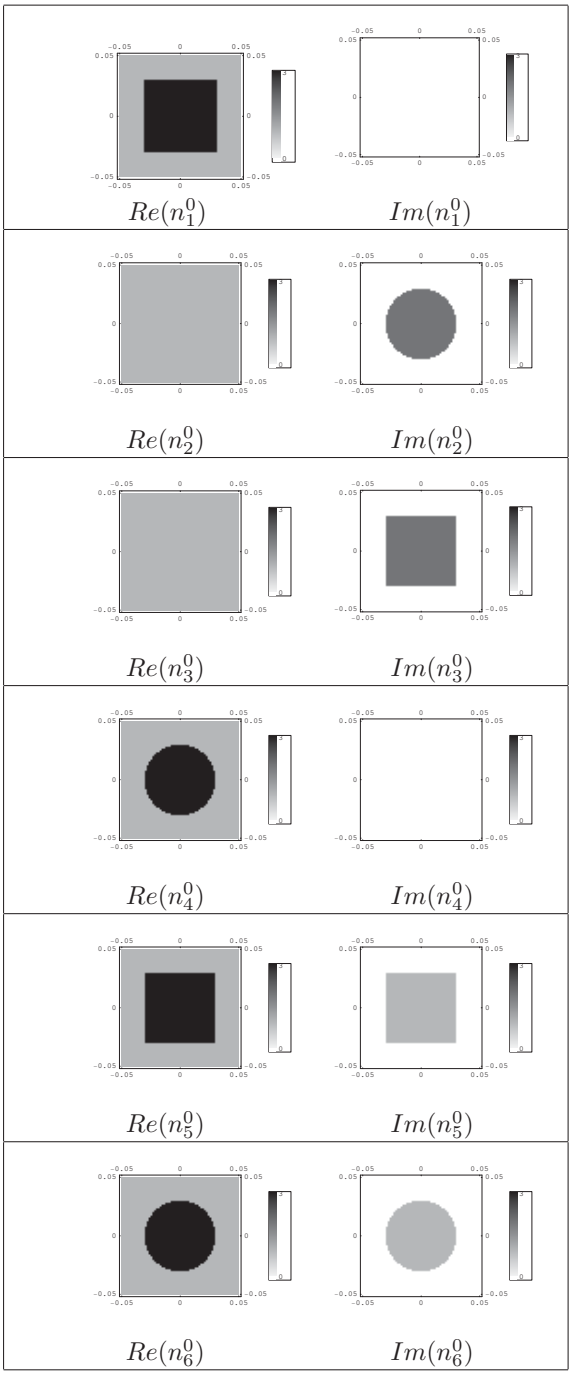


Fig. 1 The inhomogeneities considered in the numerical experiments for the computation of the optimal relaxation parameter

## IV. Relaxation Parameter Strategy and Numerical Experiments

We propose a simple method to evaluate the optimal relaxation parameter $\omega$ in the Symmetric SOR preconditioner. For this evaluation, we have taken into account six different inhomogeneities, $n_j^0$, $j = 1, 2, \ldots, 6$, and nine different electromagnetic plane waves having propagation direction $\underline{\alpha} = (\cos\frac{\pi}{3}, \sin\frac{\pi}{3})^t$ and wave number $k \in \Omega = \{1, 10, 20, 30, 40, 50, 100, 150, 200\}$. All the considered inhomogeneities are contained in a square centered at the

TABLE I
THE OPTIMAL PARAMETER $\omega^*$ OBTAINED BY BI-CGSTAB AND GMRES

| | Bi-CGSTAB | | | | | |
|---|---|---|---|---|---|---|
| $k$ | $n_1^0$ | $n_2^0$ | $n_3^0$ | $n_4^0$ | $n_5^0$ | $n_6^0$ |
| 1 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| 10 | 0.25 | 0.85 | 1.00 | 1.00 | 1.60 | 1.00 |
| 20 | 0.25 | 0.90 | 0.80 | 1.00 | 1.20 | 0.90 |
| 30 | 0.70 | 0.60 | 0.60 | 0.80 | 0.50 | 0.70 |
| 40 | 1.05 | 0.75 | 0.65 | 0.70 | 0.75 | 0.70 |
| 50 | 0.85 | 0.90 | 0.65 | 0.70 | 0.80 | 0.60 |
| 100 | 0.65 | 0.70 | 0.75 | 0.65 | 0.55 | 0.55 |
| 150 | 0.65 | 0.55 | 0.55 | 0.65 | 0.60 | 0.60 |
| 200 | 0.50 | 0.55 | 0.55 | 0.60 | 0.50 | 0.65 |

| | GMRES | | | | | |
|---|---|---|---|---|---|---|
| $k$ | $n_1^0$ | $n_2^0$ | $n_3^0$ | $n_4^0$ | $n_5^0$ | $n_6^0$ |
| 1 | 1.00 | 0.95 | 1.05 | 0.95 | 1.00 | 0.95 |
| 10 | 1.00 | 0.95 | 0.95 | 1.05 | 1.00 | 1.05 |
| 20 | 0.95 | 0.95 | 0.85 | 1.00 | 1.00 | 1.05 |
| 30 | 1.10 | 0.80 | 0.70 | 1.05 | 0.75 | 1.05 |
| 40 | 0.70 | 0.75 | 0.65 | 0.85 | 1.00 | 0.85 |
| 50 | 0.80 | 0.90 | 0.70 | 0.75 | 0.80 | 0.65 |
| 100 | 0.70 | 0.75 | 0.65 | 0.70 | 0.70 | 0.65 |
| 150 | 0.70 | 0.65 | 0.65 | 0.70 | 0.65 | 0.60 |
| 200 | 0.55 | 0.60 | 0.60 | 0.70 | 0.65 | 0.60 |

origin and having a side equal to 1, and their real and imaginary components are supposed in $[0, 3]$, see Fig. 1 for the graphical representation of the real and imaginary components of inhomogeneities $n_j^0$, $j = 1, 2, \ldots, 6$. Moreover, the discretization of (1) has been obtained with $\nu_1 = \nu_2 = 64$. The corresponding linear systems (3) were numerically solved by two different Krylov subspace methods, i.e. Bi-CGSTAB and GMRES, with Symmetric SOR preconditioning. For each linear system, we computed the optimal relaxation parameter $\omega = \omega^*$ by using the following simple algorithm. Let $L$ be a positive integer, for each $\omega_l = \frac{l}{L}$, $l = 1, 2, \ldots, 2L - 1$, we solved linear system (3) using preconditioner $P_l = P_{\omega_l}$; let $P_{\overline{\lambda}}$, with $\overline{\lambda} \in \{1, 2, \ldots, 2L - 1\}$, be the best preconditioner, i.e. the one attaining convergence with the smallest number $Itn$ of iterations, then $\omega^* = \omega_{\overline{\lambda}}$.

Table I reports for each one of the six inhomogeneities considered, $n_j^0$, $j = 1, 2, \ldots, 6$, these computed parameters $\omega^*$ obtained with $L = 10$ as a function of the wave number $k$, for the two different Krylov subspace methods employed: Bi-CGSTAB and GMRES.

Now, we use these numerically obtained optimal relaxation parameters, given in Table I, for evaluating the optimal relaxation parameter in different cases. For each $k \in \Omega$, we compute $\bar{\omega_1}(k)$ as the mean of the numerically computed optimal relaxation parameters in Table I associated with Bi-CGSTAB, and $\bar{\omega_2}(k)$ as the mean of the ones in Table I associated to GMRES. Moreover, for each $k \in [1, 200]$, and $h = 1, 2$, we define $\omega_h(k)$ as the linear spline that interpolates $\bar{\omega_h}(k)$, $k \in \Omega$. In this way, $\omega_h(k)$ gives an approximation of the relaxation parameter also for wave numbers $k \in [1, 200] \setminus \Omega$, for Bi-CGSTAB when $h = 1$ and for GMRES when $h = 2$.

In order to analyze the behavior of the relaxation parameters computed by $\omega_h(k)$, $k \in [1, 200]$, $h = 1, 2$, we consider ten inhomogeneities $n_j$, $j = 1, 2, \ldots, 10$, that are shown in Fig. 2, and wave numbers $k = 5, 85, 175$. We note that these inhomogeneities and these wave numbers are

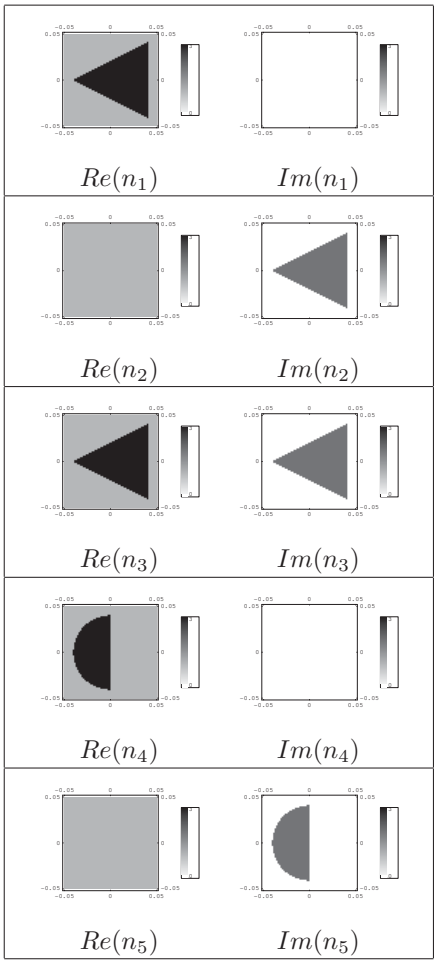


Fig. 2 The real and imaginary parts of the inhomogeneities $n_j$, $j = 1, 2, \ldots, 5$ considered in the numerical experiments

different from the ones considered in the construction of $\omega_h$, $h = 1, 2$. For each inhomogeneity and each wave number, we compute the solution of linear systems (3) by using Bi-CGSTAB and GMRES methods. We compare the results obtained with no preconditioner $P = I$, Jacobi preconditioner $P = A_D$, Symmetric SOR preconditioner $P = P_{\omega^*}$ with optimal relaxation parameter $\omega = \omega^*$, and Symmetric SOR preconditioner $P = P_{\omega_h}$ with relaxation parameter $\omega = \omega_h$, where $h = 1$ is used for Bi-CGSTAB method, and $h = 2$ is used for GMRES method. Note that the optimal relaxation parameter $\omega^*$ is numerically computed by the simple procedure described above with $L = 10$. These comparisons are given in terms of the following performance indices: $T$ the elapsed time (in seconds $[s]$) for the solution of the system; $Itn$ the number of iterations performed; $E$ the relative error in the computed solution. More precisely, given $\underline{\tilde{u}}$, the solution computed by one of the iterative methods mentioned above, and $\underline{u}_G$, the solution of the same linear system computed by the Gaussian elimination with partial pivoting, $E$ is defined as follows:

$$E = \frac{\|\underline{\tilde{u}} - \underline{u}_G\|_\infty}{\|\underline{u}_G\|_\infty}. \tag{10}$$

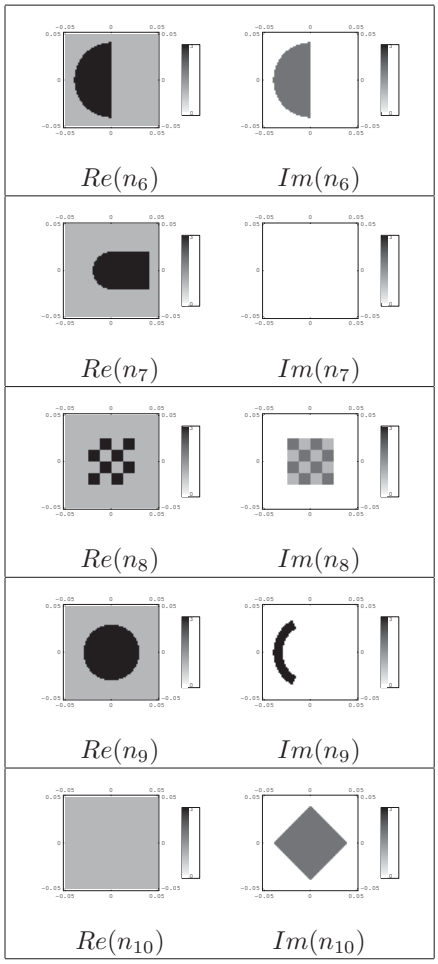


Fig. 3 The real and imaginary parts of the inhomogeneities $n_j$, $j = 6, \ldots, 10$ considered in the numerical experiments

The results are reported in Tables II-VI, where with the notation $x(y)$ we mean $x \cdot 10^y$. From these tables, we have that the Symmetric SOR preconditioning performs better than the other preconditioning techniques, that is no preconditioning and Jacobi preconditioning. Moreover, functions $\omega_h(k)$, $k \in [1, 200]$, $h = 1, 2$, provide satisfactory approximations of the optimal relaxation parameters, in fact, the computational time and the number of iterations, corresponding to $\omega_h$, $h = 1, 2$, are very close to the ones obtained for $\omega^*$.

## V. Conclusions

We considered the Symmetric SOR preconditioner in the numerical solution of an electromagnetic scattering problem. From the numerical results reported in this paper, we have that the Symmetric SOR gives an effective preconditioning technique for this scattering problem. In particular, the simple algorithm, proposed for the evaluation of the relaxation parameter, provides satisfactory results. Note that this is an important result since it gives an effective relation between the physical quantities in the scattering problem, i.e. the wave number and the relaxation parameter in the Symmetric SOR preconditioner. So, we are able to choose a quasi-optimal

TABLE II
The Results Obtained by Bi-CGSTAB

| | | $P = I$ | | | $P = A_D$ | | |
|---|---|---|---|---|---|---|---|
| $n$ | $k$ | $T$ | $Itn$ | $E$ | $T$ | $Itn$ | $E$ |
| $n_1$ | 5 | 3.6s | 3 | 2.7(-14) | 3.7s | 3 | 2.7(-14) |
| $n_1$ | 85 | 17.3s | 18 | 5.1(-10) | 17.4s | 18 | 5.0(-10) |
| $n_1$ | 175 | 44.9s | 48 | 2.8(-09) | 44.0s | 47 | 2.4(-09) |
| $n_2$ | 5 | 2.6s | 3 | 2.5(-14) | 2.7s | 3 | 2.4(-14) |
| $n_2$ | 85 | 7.7s | 11 | 1.7(-10) | 7.8s | 11 | 1.4(-10) |
| $n_2$ | 175 | 14.8s | 22 | 4.3(-10) | 14.8s | 22 | 4.7(-10) |
| $n_3$ | 5 | 2.7s | 3 | 2.1(-13) | 2.7s | 3 | 2.0(-13) |
| $n_3$ | 85 | 11.1s | 16 | 4.1(-10) | 11.1s | 16 | 3.0(-10) |
| $n_3$ | 175 | 20.8s | 31 | 1.1(-09) | 20.8s | 31 | 1.2(-09) |
| $n_4$ | 5 | 2.1s | 3 | 1.4(-14) | 2.1s | 3 | 1.4(-14) |
| $n_4$ | 85 | 9.0s | 17 | 5.3(-12) | 8.6s | 16 | 4.5(-10) |
| $n_4$ | 175 | 21.5s | 42 | 7.6(-10) | 21.0s | 41 | 1.0(-09) |
| $n_5$ | 5 | 2.1s | 3 | 1.3(-14) | 2.1s | 3 | 1.3(-14) |
| $n_5$ | 85 | 5.6s | 10 | 1.2(-10) | 5.6s | 10 | 1.1(-10) |
| $n_5$ | 175 | 10.6s | 20 | 1.7(-10) | 10.6s | 20 | 2.3(-10) |
| $n_6$ | 5 | 2.7s | 3 | 1.1(-13) | 2.8s | 3 | 1.1(-13) |
| $n_6$ | 85 | 11.1s | 15 | 6.6(-11) | 11.1s | 15 | 5.0(-11) |
| $n_6$ | 175 | 20.1s | 28 | 1.1(-09) | 20.9s | 29 | 8.3(-10) |
| $n_7$ | 5 | 1.9s | 3 | 2.7(-15) | 1.9s | 3 | 2.6(-15) |
| $n_7$ | 85 | 7.3s | 15 | 1.0(-10) | 7.3s | 15 | 9.5(-11) |
| $n_7$ | 175 | 20.6s | 44 | 6.1(-10) | 19.8s | 42 | 9.6(-10) |
| $n_8$ | 5 | 0.7s | 2 | 8.5(-12) | 0.7s | 2 | 7.9(-12) |
| $n_8$ | 85 | 2.1s | 8 | 2.8(-12) | 2.1s | 8 | 2.6(-12) |
| $n_8$ | 175 | 3.6s | 15 | 9.6(-11) | 3.6s | 15 | 6.7(-11) |
| $n_9$ | 5 | 3.0s | 3 | 2.2(-14) | 3.0s | 3 | 2.1(-14) |
| $n_9$ | 85 | 15.1s | 20 | 9.2(-10) | 15.1s | 20 | 5.3(-10) |
| $n_9$ | 175 | 45.0s | 62 | 3.5(-09) | 44.4s | 61 | 3.0(-09) |
| $n_{10}$ | 5 | 3.6s | 3 | 2.4(-14) | 3.6s | 3 | 2.3(-14) |
| $n_{10}$ | 85 | 11.8s | 12 | 1.0(-11) | 11.0s | 11 | 4.4(-10) |
| $n_{10}$ | 175 | 21.9s | 23 | 6.0(-11) | 21.0s | 22 | 41.7(-09) |

TABLE III
The Results Obtained by Bi-CGSTAB

| | | $P = P_{\omega^*}$ | | | $P = P_{\omega_1}$ | | |
|---|---|---|---|---|---|---|---|
| $n$ | $k$ | $T$ | $Itn$ | $E$ | $T$ | $Itn$ | $E$ |
| $n_1$ | 5 | 2.9s | 2 | 1.5(-10) | 3.0s | 2 | 2.76(-14) |
| $n_1$ | 85 | 11.2s | 10 | 5.8(-10) | 11.2s | 10 | 2.3(-11) |
| $n_1$ | 175 | 19.4s | 18 | 5.3(-10) | 20.5s | 19 | 2.3(-09) |
| $n_2$ | 5 | 2.3s | 2 | 1.2(-10) | 2.3s | 2 | 1.9(-14) |
| $n_2$ | 85 | 5.3s | 6 | 3.5(-10) | 5.3s | 6 | 6.0(-11) |
| $n_2$ | 175 | 7.5s | 9 | 1.8(-10) | 7.5s | 9 | 6.5(-10) |
| $n_3$ | 5 | 2.3s | 2 | 5.1(-11) | 2.3s | 2 | 2.0(-13) |
| $n_3$ | 85 | 6.8s | 8 | 2.2(-11) | 6.8s | 8 | 6.5(-11) |
| $n_3$ | 175 | 8.3s | 10 | 5.8(-10) | 8.4s | 10 | 4.5(-10) |
| $n_4$ | 5 | 1.8s | 2 | 8.5(-11) | 1.8s | 2 | 9.0(-15) |
| $n_4$ | 85 | 5.9s | 9 | 4.6(-10) | 5.9s | 9 | 1.0(-10) |
| $n_4$ | 175 | 8.8s | 14 | 1.6(-09) | 9.4s | 15 | 1.8(-09) |
| $n_5$ | 5 | 1.8s | 2 | 1.4(-15) | 1.8s | 2 | 2.5(-15) |
| $n_5$ | 85 | 4.1s | 6 | 4.5(-11) | 4.1s | 6 | 2.1(-12) |
| $n_5$ | 175 | 5.3s | 8 | 1.2(-10) | 5.3s | 8 | 1.4(-10) |
| $n_6$ | 5 | 2.3s | 2 | 1.1(-10) | 2.3s | 2 | 4.6(-14) |
| $n_6$ | 85 | 6.1s | 7 | 1.9(-10) | 6.2s | 7 | 1.2(-10) |
| $n_6$ | 175 | 7.7s | 9 | 2.8(-10) | 7.7s | 9 | 2.2(-10) |
| $n_7$ | 5 | 1.6s | 2 | 7.3(-11) | 1.6s | 2 | 3.4(-15) |
| $n_7$ | 85 | 5.3s | 9 | 4.0(-10) | 5.3s | 9 | 2.3(-11) |
| $n_7$ | 175 | 10.7s | 19 | 5.0(-11) | 10.7s | 19 | 5.1(-10) |
| $n_8$ | 5 | 0.7s | 2 | 6.5(-12) | 0.8s | 2 | 7.6(-16) |
| $n_8$ | 85 | 1.5s | 5 | 8.3(-11) | 1.8s | 6 | 1.2(-12) |
| $n_8$ | 175 | 2.0s | 7 | 5.3(-11) | 2.4s | 8 | 1.3(-11) |
| $n_9$ | 5 | 2.6s | 2 | 1.1(-12) | 2.6s | 2 | 5.9(-15) |
| $n_9$ | 85 | 9.2s | 10 | 2.9(-11) | 9.2s | 10 | 1.0(-10) |
| $n_9$ | 175 | 16.7s | 19 | 2.4(-09) | 16.7s | 19 | 2.3(-09) |
| $n_{10}$ | 5 | 2.9s | 2 | 9.6(-11) | 3.0s | 2 | 1.8(-14) |
| $n_{10}$ | 85 | 7.1s | 6 | 3.3(-10) | 7.1s | 6 | 8.1(-11) |
| $n_{10}$ | 175 | 10.1s | 9 | 1.5(-09) | 10.1s | 9 | 2.8(-10) |

TABLE IV
THE RELAXATION PARAMETERS $\omega^*$ AND $\omega_1$ USED TO OBTAIN TABLE III

| $n$ | $k$ | $\omega^*$ | $\omega_1$ | $n$ | $k$ | $\omega^*$ | $\omega_1$ |
|---|---|---|---|---|---|---|---|
| $n_1$ | 5 | 0.55 | 0.99 | $n_6$ | 5 | 0.65 | 0.99 |
| $n_1$ | 85 | 0.80 | 0.71 | $n_6$ | 85 | 0.60 | 0.71 |
| $n_1$ | 175 | 0.65 | 0.64 | $n_6$ | 175 | 0.60 | 0.64 |
| $n_2$ | 5 | 0.55 | 0.99 | $n_7$ | 5 | 0.40 | 0.99 |
| $n_2$ | 85 | 0.50 | 0.71 | $n_7$ | 85 | 0.80 | 0.71 |
| $n_2$ | 175 | 0.65 | 0.64 | $n_7$ | 175 | 0.65 | 0.64 |
| $n_3$ | 5 | 0.75 | 0.99 | $n_8$ | 5 | 0.05 | 0.99 |
| $n_3$ | 85 | 0.65 | 0.71 | $n_8$ | 85 | 0.75 | 0.71 |
| $n_3$ | 175 | 0.60 | 0.64 | $n_8$ | 175 | 0.65 | 0.64 |
| $n_4$ | 5 | 0.55 | 0.99 | $n_9$ | 5 | 1.10 | 0.99 |
| $n_4$ | 85 | 0.65 | 0.71 | $n_9$ | 85 | 0.70 | 0.71 |
| $n_4$ | 175 | 0.65 | 0.64 | $n_9$ | 175 | 0.60 | 0.64 |
| $n_5$ | 5 | 0.55 | 1.00 | $n_{10}$ | 5 | 0.55 | 0.99 |
| $n_5$ | 85 | 0.50 | 0.71 | $n_{10}$ | 85 | 0.55 | 0.71 |
| $n_5$ | 175 | 0.60 | 0.64 | $n_{10}$ | 175 | 0.45 | 0.64 |

TABLE V
THE RESULTS OBTAINED BY GMRES

| | | $P = I$ | | | $P = A_D$ | | |
|---|---|---|---|---|---|---|---|
| $n$ | $k$ | $T$ | $Itn$ | $E$ | $T$ | $Itn$ | $E$ |
| $n_1$ | 5 | 5.0s | 8 | 2.3(-14) | 5.0s | 8 | 2.0(-14) |
| $n_1$ | 85 | 15.1s | 30 | 5.4(-15) | 15.2s | 30 | 2.0(-14) |
| $n_1$ | 175 | 38.2s | 80 | 2.0(-13) | 38.3s | 80 | 2.5(-13) |
| $n_2$ | 5 | 3.6s | 8 | 2.5(-15) | 3.6s | 8 | 3.1(-14) |
| $n_2$ | 85 | 9.4s | 26 | 4.8(-15) | 9.1s | 25 | 3.5(-14) |
| $n_2$ | 175 | 13.6s | 40 | 46.0(-15) | 13.7s | 40 | 8.9(-15) |
| $n_3$ | 5 | 4.0s | 9 | 1.6(-15) | 3.7s | 8 | 2.0(-14) |
| $n_3$ | 85 | 10.8s | 30 | 5.6(-15) | 10.9s | 30 | 3.2(-14) |
| $n_3$ | 175 | 13.8s | 40 | 4.0(-11) | 13.8s | 40 | 4.7(-11) |
| $n_4$ | 5 | 2.8s | 8 | 1.5(-15) | 2.9s | 8 | 1.2(-14) |
| $n_4$ | 85 | 97.6s | 27 | 4.1(-15) | 7.7s | 27 | 1.5(-14) |
| $n_4$ | 175 | 10.7s | 40 | 7.1(-11) | 10.7s | 40 | 9.6(-11) |
| $n_5$ | 5 | 2.9s | 8 | 1.8(-15) | 2.9s | 8 | 2.0(-14) |
| $n_5$ | 85 | 6.9s | 24 | 4.3(-15) | 6.7s | 23 | 2.8(-14) |
| $n_5$ | 175 | 10.7s | 39 | 5.7(-15) | 10.7s | 39 | 1.4(-14) |
| $n_6$ | 5 | 3.8s | 8 | 1.4(-15) | 3.8s | 8 | 1.4(-14) |
| $n_6$ | 85 | 10.8s | 28 | 7.0(-15) | 11.0s | 27 | 2.5(-14) |
| $n_6$ | 175 | 14.7s | 40 | 5.0(-14) | 14.7s | 40 | 7.1(-14) |
| $n_7$ | 5 | 2.6s | 8 | 2.3(-15) | 2.3s | 7 | 1.5(-14) |
| $n_7$ | 85 | 6.8s | 26 | 3.6(-15) | 6.6s | 25 | 1.6(-14) |
| $n_7$ | 175 | 9.9s | 40 | 1.5(-12) | 9.9s | 40 | 1.9(-12) |
| $n_8$ | 5 | 1.2s | 7 | 1.4(-15) | 1.1s | 7 | 1.4(-15) |
| $n_8$ | 85 | 2.5s | 19 | 2.1(-15) | 2.4s | 18 | 3.6(-15) |
| $n_8$ | 175 | 3.6s | 29 | 2.5(-15) | 3.8s | 29 | 2.0(-15) |
| $n_9$ | 5 | 4.4s | 9 | 1.9(-15) | 4.1s | 8 | 2.5(-14) |
| $n_9$ | 85 | 13.1s | 33 | 7.5(-15) | 13.1s | 33 | 2.0(-14) |
| $n_9$ | 175 | 30.0s | 80 | 5.5(-10) | 30.0s | 80 | 3.3(-10) |
| $n_{10}$ | 5 | 5.0s | 8 | 1.8(-15) | 5.0s | 8 | 3.2(-14) |
| $n_{10}$ | 85 | 12.8s | 25 | 1.1(-14) | 12.9s | 25 | 4.0(-14) |
| $n_{10}$ | 175 | 19.3s | 40 | 1.1(-14) | 19.3s | 40 | 7.0(-15) |

TABLE VI
THE RESULTS OBTAINED BY GMRES

| | | $P = P_{\omega^*}$ | | | $P = P_{\omega_2}$ | | |
|---|---|---|---|---|---|---|---|
| $n$ | $k$ | $T$ | $Itn$ | $E$ | $T$ | $Itn$ | $E$ |
| $n_1$ | 5 | 4.6s | 6 | 1.8(-15) | 4.6s | 6 | 1.9(-15) |
| $n_1$ | 85 | 12.8s | 22 | 3.6(-15) | 12.8s | 22 | 5.4(-15) |
| $n_1$ | 175 | 20.1s | 36 | 8.1(-15) | 21.1s | 38 | 4.3(-15) |
| $n_2$ | 5 | 3.4s | 6 | 2.0(-15) | 3.4s | 6 | 1.8(-15) |
| $n_2$ | 85 | 8.0s | 18 | 3.4(-15) | 8.4s | 19 | 3.4(-15) |
| $n_2$ | 175 | 11.8s | 28 | 5.6(-15) | 11.8s | 28 | 4.1(-15) |
| $n_3$ | 5 | 3.5s | 6 | 1.5(-15) | 3.5s | 6 | 2.4(-15) |
| $n_3$ | 85 | 9.9s | 23 | 3.0(-15) | 9.9s | 23 | 3.9(-15) |
| $n_3$ | 175 | 14.1s | 34 | 3.7(-15) | 14.1s | 34 | 3.8(-15) |
| $n_4$ | 5 | 2.7s | 6 | 1.6(-15) | 2.7s | 6 | 1.6(-15) |
| $n_4$ | 85 | 6.8s | 20 | 3.8(-15) | 6.9s | 20 | 5.1(-15) |
| $n_4$ | 175 | 10.7s | 33 | 4.5(-15) | 11.0s | 34 | 3.0(-15) |
| $n_5$ | 5 | 2.7s | 6 | 2.7(-15) | 2.7s | 6 | 1.6(-15) |
| $n_5$ | 85 | 6.0s | 17 | 2.6(-15) | 6.0s | 17 | 2.3(-15) |
| $n_5$ | 175 | 8.3s | 25 | 2.8(-15) | 8.3s | 25 | 3.8(-15) |
| $n_6$ | 5 | 3.5s | 6 | 2.1(-15) | 3.5s | 6 | 2.6(-15) |
| $n_6$ | 85 | 9.0s | 20 | 2.8(-15) | 9.0s | 20 | 6.5(-15) |
| $n_6$ | 175 | 13.3s | 31 | 4.8(-15) | 12.9s | 30 | 2.6(-15) |
| $n_7$ | 5 | 2.2s | 5 | 2.0(-15) | 2.1s | 5 | 3.1(-15) |
| $n_7$ | 85 | 6.2s | 20 | 3.2(-15) | 6.2s | 20 | 5.1(-15) |
| $n_7$ | 175 | 9.7s | 33 | 5.7(-15) | 9.7s | 33 | 5.0(-15) |
| $n_8$ | 5 | 1.0s | 5 | 2.3(-15) | 1.0s | 5 | 3.2(-15) |
| $n_8$ | 85 | 2.2s | 14 | 1.5(-15) | 2.2s | 14 | 1.9(-15) |
| $n_8$ | 175 | 3.0s | 21 | 1.6(-15) | 3.1s | 21 | 4.2(-15) |
| $n_9$ | 5 | 3.9s | 6 | 3.6(-15) | 3.9s | 6 | 2.0(-15) |
| $n_9$ | 85 | 11.0s | 23 | 4.7(-15) | 10.6s | 22 | 5.7(-15) |
| $n_9$ | 175 | 17.3s | 38 | 6.2(-15) | 17.3s | 38 | 7.8(-15) |
| $n_{10}$ | 5 | 4.6s | 6 | 3.4(-15) | 4.6s | 6 | 2.4(-15) |
| $n_{10}$ | 85 | 11.3s | 19 | 3.3(-15) | 10.7s | 18 | 3.8(-15) |
| $n_{10}$ | 175 | 16.0s | 28 | 4.4(-15) | 16.5s | 29 | 3.3(-15) |

TABLE VII
THE RELAXATION PARAMETERS $\omega^*$ AND $\omega_2$ USED TO OBTAIN TABLE VI

| $n$ | $k$ | $\omega^*$ | $\omega_2$ | $n$ | $k$ | $\omega^*$ | $\omega_2$ |
|---|---|---|---|---|---|---|---|
| $n_1$ | 5 | 1.05 | 0.99 | $n_6$ | 5 | 0.95 | 0.99 |
| $n_1$ | 85 | 0.80 | 0.65 | $n_6$ | 85 | 0.60 | 0.71 |
| $n_1$ | 175 | 0.70 | 0.64 | $n_6$ | 175 | 0.55 | 0.64 |
| $n_2$ | 5 | 0.90 | 0.99 | $n_7$ | 5 | 1.00 | 0.99 |
| $n_2$ | 85 | 0.70 | 0.71 | $n_7$ | 85 | 0.70 | 0.71 |
| $n_2$ | 175 | 0.60 | 0.64 | $n_7$ | 175 | 0.70 | 0.64 |
| $n_3$ | 5 | 1.00 | 0.99 | $n_8$ | 5 | 1.05 | 0.99 |
| $n_3$ | 85 | 0.70 | 0.71 | $n_8$ | 85 | 1.00 | 0.71 |
| $n_3$ | 175 | 0.55 | 0.64 | $n_8$ | 175 | 0.80 | 0.64 |
| $n_4$ | 5 | 1.05 | 0.99 | $n_9$ | 5 | 1.05 | 0.99 |
| $n_4$ | 85 | 0.75 | 0.71 | $n_9$ | 85 | 0.65 | 0.71 |
| $n_4$ | 175 | 0.70 | 0.64 | $n_9$ | 175 | 0.65 | 0.64 |
| $n_5$ | 5 | 0.85 | 0.99 | $n_{10}$ | 5 | 0.85 | 0.99 |
| $n_5$ | 85 | 0.80 | 0.71 | $n_{10}$ | 85 | 0.65 | 0.71 |
| $n_5$ | 175 | 0.60 | 0.65 | $n_{10}$ | 175 | 0.45 | 0.64 |

relaxation parameter from the knowledge of the wave number used in the scattering problem.

The proposed method can be refined in different ways, taking into account the discretization parameters $\nu_1, \nu_2$ and other physical quantities; for example, for each iterative method, the evaluation of $\omega$ can be given by three different interpolating functions: one for insulating inhomogeneities ($Im(n) = 0$), one for perfectly conductive inhomogeneities ($Re(n) = 1$), one for the other inhomogeneities ($Im(n) \neq 0$, $Re(n) \neq 1$). So, depending on the physical nature of the inhomogeneities under consideration, we can select $\omega$ by the appropriate interpolation function. Finally, we note that the excellent results obtained for the Symmetric SOR preconditioning suggest further investigations of this preconditioning technique; in particular, an interesting development of the present paper is the theoretical study of the optimal relaxation parameter for the scattering problem under consideration, and the application of the Symmetric SOR preconditioning in the numerical solution of similar scattering problems.

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