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Pure Scalar Equilibria for Normal-Form Games

H. W. Corley

Abstract**—**A scalar equilibrium (SE) is an alternative type of equilibrium in pure strategies for an *n*-person normal-form game G. It is defined using optimization techniques to obtain a pure strategy for each player of G by maximizing an appropriate utility function over the acceptable joint actions. The players' actions are determined by the choice of the utility function. Such a utility function could be agreed upon by the players or chosen by an arbitrator. An SE is an equilibrium since no players of G can increase the value of this utility function by changing their strategies. SEs are formally defined, and examples are given. In a greedy SE, the goal is to assign actions to the players giving them the largest individual payoffs jointly possible. In a weighted SE, each player is assigned weights modeling the degree to which he helps every player, including himself, achieve as large a payoff as jointly possible. In a compromise SE, each player wants a fair payoff for a reasonable interpretation of fairness. In a parity SE, the players want their payoffs to be as nearly equal as jointly possible. Finally, a satisficing SE achieves a personal target payoff value for each player. The vector payoffs associated with each of these SEs are shown to be Pareto optimal among all such acceptable vectors, as well as computationally tractable.

Keywords**—**Compromise equilibrium, greedy equilibrium, normal-form game, parity equilibrium, pure strategies, satisficing equilibrium, scalar equilibria, utility function, weighted equilibrium.

I. Introduction

GAME theory is the study of strategic interactions among rational decision makers called players, whose decisions affect each other. Its systematic development began with [1], which described both noncooperative and cooperative games. Modern game theory is predominantly noncooperative [2], [3]. The players are assumed selfish, and the fundamental solution concept is the Nash equilibrium [4]-[6]. Its principal use is normative [7]-[9], i.e., to recommend decisions that the players should make.

In this paper, we consider *n*-person, one-shot, normal-form games. All payoffs are measured in the same units of some transferable utility allowing side payments among the players. The players are assumed rational in the sense that each player makes decisions consistent with a stated objective. Moreover, for any game, all players are assumed to have the same objective. For example, they may wish to maximize their individual payoffs.

The games studied here have both cooperative and noncooperative aspects. Such games are sometimes called semi-cooperative [10]-[18]. The noncooperative aspect is that each player tries to achieve his personal objective. The cooperative one is that the players either (i) agree on an appropriate utility function $\boldsymbol{T}$ for evaluating each possible joint pure strategy of a game or else (ii) let an arbitrator choose it.

Various utility functions $\boldsymbol{T}$ are presented for modeling player objectives. A joint pure strategy that maximizes $\boldsymbol{T}$ over the set of acceptable joint actions is deemed optimal for the players, and an SE is defined as such an action profile. An SE is called an equilibrium since no players can increase the value of $\boldsymbol{T}$ by changing their strategies.

The purpose of this paper is to reduce games to the selection of $\boldsymbol{T}$ that would determine the players' actions. Ties could be broken with secondary criteria. The SE approach addresses five problematic areas of noncooperative game theory.

(1) An SE always exists in pure strategies, which is not the case for both the Nash Equilibrium (NE) [2]-[6] and some other equilibria [19], [20].
(2) Mixed strategies are known to be difficult to calculate, interpret, and implement as noted in [1], [4], [18], [21]-[24], except possibly in repeated games or resource allocation games [25].
(3) SEs are not restricted to the strictly selfish motivation of the individual players as assumed for an NE.
(4) The SEs presented here are Pareto maxima among the acceptable set of vector payoffs, while NEs, for example, are frequently not. Pareto optimality is a desirable property for a solution concept [26], [27].
(5) For games with multiple pure equilibria, the value of an appropriate T could be calculated for each equilibrium and used as a secondary selection criterion or refinement mechanism. An equilibrium with the maximum value of T could then be selected. It would be an SE for the set of original multiple pure equilibria.

In Section II of this paper, some preliminary background is summarized. In Section III we define a greedy SE to model the situation where each player is motivated by selfishness as in an NE. In Section IV, we develop a general weighted SE that models the degree to which each player supports every player, including himself, in achieving as large a payoff as jointly possible. In Section V, we present a compromise SE to model fairness. In Section VI, we consider a parity SE giving the players approximately equal payoffs. In Section VII, we define a satisficing SE in which each player achieves a personal target payoff value. Each of these SEs is shown to be Pareto optimal and computationally tractable, and examples are presented. In Section VIII, some game theoretic axioms are stated, and the ones satisfied by the SEs of this paper are noted. Conclusions are offered in Section IX.

II. Preliminaries

We consider a standard n-person, noncooperative game in normal form. Let $G = \langle I, (S_i)_{i\in I}, (u_i)_{i\in I} \rangle$ denote such a game,

H.W. Corley is a co-founder and current member of the Center On Stochastic Modeling, Optimization, & Statistics (COSMOS), The University of Texas at Arlington, Arlington, TX 76019 USA (e-mail: corley@uta.edu).

where $I = \{1, \ldots, n\}$ is the set of players and S_i is the finite set of m_i pure strategies, or actions, for player i. For an action profile $s = (s_1, \ldots, s_n) \in S = \prod_{i\in I} S_i$, $u_i(s)$ is utility of player i; and the payoff matrix consists of the n-tuples $u(s) = (u_1(s), \ldots, u_n(s))$ ordered in the usual way. Assume that G has transferable utilities [28] so that players obtain equal utilities from equal payoffs. We refer to this property as assumption TU for G.

For subsequent reference, the Nash equilibrium (NE) and the more recent Berge equilibrium (BE) [19], [29], [30] are next defined for pure strategies using the standard notation s_{-i} for an incomplete strategy profile. In an NE of Definition 1, every player has a pure strategy that maximizes his own payoff for the other n-1 players' strategies. The opposite situation occurs in a BE of Definition 2, where every n-1 players have pure strategies that maximize the remaining player's payoff for his strategy. An NE models player selfishness, while a BE models mutual support or altruism.

Definition 1 (NE). The action profile s^* is an NE for G if and only if $u_i(s^*) = \max_{s_i \in S_i} u_i(s_i, s^*_{-i}),\ \forall i \in I$.

Definition 2 (BE). The action profile s^* is a BE for G if and only if $u_i(s^*) = \max_{s_{-i} \in S_{-i}} u_i(s_i^*, s_{-i}),\ \forall i \in I$.

We also define the security level of a player since this concept may have implications in the cooperative aspect of the games considered here.

Definition 3. An action profile $\hat{s}$ is a security profile for G if and only if $\hat{s}_i \in \arg\max_{s_i \in S_i} \min_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i}),\ \forall i \in I$. For each player $i \in I$, $L_i = \max_{s_i \in S_i} \min_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i})$ is the associated security level and $\hat{s}_i$ is an associated security action.

The value L_i is the least payoff that player i can be guaranteed to receive from his action in the game, regardless of what the other players' actions are. It is possible that $u_i(\hat{s}) > L_i$ since $\hat{s}_{-i}$ is not necessarily a worst response to $\hat{s}_i$ in a security profile. A player could justifiably not agree to an action profile in a negotiation for which he receives less than L_i. We denote the set of security profiles for G by $\Lambda = \{s \in S: u_i(s) \geq L_i\}$, which is nonempty by definition. Reference [27], for example, shows that any pure NE s^* for G is a member of Λ.

We now call $\Gamma = \langle G, \Omega, T\rangle$ the n-person, pure-strategy, normal-form, semi-cooperative game associated with the game G. In particular, Ω is the nonempty and finite set of feasible action profiles $s = (s_1, \ldots, s_n)$ acceptable to the n players or arbitrator. These parties may require that $\Omega \subseteq \Lambda$. In addition, $T: u(\Omega) \to R^1$ is a utility function agreed upon by the players or stipulated by an arbitrator according to the situation modeled by Γ.

An SE for the game Γ is an action profile $s^* \in \Omega$ determined by a utility function $T: u(\Omega) \to R^1$ on the n-tuples $u(s) = (u_1(s), \ldots, u_n(s))$. T induces a complete and transitive preference relation $\leq_T$ on $u(\Omega)$ as described in [28]. In other words, for all $s', s'' \in \Omega$,

(i) $u(s') <_T u(s'')$ if $T[u(s')] < T[u(s'')]$,

(ii) $u(s') =_T u(s'')$ if $T[u(s')] = T[u(s'')]$ and the players are indifferent between s' and s'',

(iii) $u(s') \leq_T u(s'')$ if either $u(s') <_T u(s'')$ or $u(s') =_T u(s'')$.

In particular, an SE s^* is an action profile that maximizes $T[u(s)]$ among the finite number of joint strategies in Ω.. If Γ has multiple SEs resulting from ties in the maximization, it is assumed that a negotiation among the players, similar to the one stipulating T and Ω, will choose a single s^*. If Γ is arbitrated, the arbitrator will select T, Ω, and a single SE. The following definition summarizes the previous discussion.

Definition 4 (SE). Let $T: u(\Omega) \to R^1$ be the utility function for a semi-cooperative game $\Gamma = \langle G, \Omega, T\rangle$. The joint action profile $s^* \in \Omega$ is an SE for Γ if and only if $T[u(s)] \leq T[u(s^*)]$ for all $s \in \Omega$. Equivalently, s^* is an SE if and only if s^* maximizes the composition $T \circ u$ over Ω.

Note again that the SE approach could be used to choose a single equilibrium for G from multiple ones. For example, the players or arbitrator could let Ω be the set of NEs for G and select an appropriate T. An NE could then be selected that is an SE for this Ω.

We next establish that an SE s^* for Γ will be Pareto maximal over Ω for a significant class of T. The following definitions are needed.

Definition 5. For the game Γ, the action profile $s'' \in \Omega$ dominates $s' \in \Omega$ if and only if $u_i(s') \leq u_i(s''),\ \forall i \in I$, and $u_j(s') < u_j(s'')$ for some $j \in I$. An action profile $s^* \in \Omega$ is a Pareto maximum for Γ if s^* is not dominated by any $s \in \Omega$. A Pareto maximum s^* is said to be Pareto maximal.

Definition 6. For the game Γ, the utility function $T: u(\Omega) \to R^1$ is said to be strictly increasing on $u(\Omega)$ if and only if $T[u(s')] < T[u(s'')]$ for any $s', s'' \in \Omega$ for which s'' dominates s'.

An immediate consequence of Definitions 5 and 6 is the next result.

Lemma 1. If s^* is an SE for Γ and if T is strictly increasing on $u(\Omega)$, then s^* is Pareto maximal for Γ.

Proof. Let $s^* \in \Omega$ be an SE for Γ. We prove the contrapositive. If s^* is not a Pareto maximum, there exists $s' \in \Omega$ that dominates s^*. But since T is strictly increasing, it follows from Definition 5 that $T[u(s^*)] < T[u(s')]$. Thus s^* is not an SE for Γ to establish the result. ■

III. Greedy Scalar Equilibrium

In this section it is assumed that each player of a game G is greedy and wants a payoff as high as jointly possible. A greedy scalar equilibrium (GSE) attempts to achieve this goal as a global maximum over all action profiles in Ω, as opposed to an NE, which does so only locally for each player, as described in Definition 1. For the n-person, normal-form game G with set of feasible action profiles Ω, let $M_i = \max_{s\in\Omega} u_i(s)$ and consider the utility function $T_g: u(\Omega) \to R^1$ for which

$$T_g[u(s)] = \prod_{i\in I} \frac{1}{M_i - u_i(s) + 1},\ s \in \Omega \qquad (1)$$

Because of assumption TU on G, the denominators $M_i - u_i(s) + 1$ in (1) are comparable. The number 1 in the denominators prevents a division by 0 if any $u_i(s) = M_i$, so $T_g[u(s)]$ is nonnegative and finite on Ω.

Definition 7 (GSE). The pure strategy profile s^* is a GSE for Γ if and only if s^* maximizes the utility function $T_g[u(s)]$ over Ω.

From Definition 7, a GSE always exists since Ω is a nonempty finite set. However, a pure NE modeling player selfishness may not. Moreover, from (1), it follows that $0 < T_g[u(s)] \leq 1$ for all $s \in \Omega$. Maximizing $T_g[u(s)]$ over Ω requires that each $u_i(s^*)$ be as close to M_i as jointly possible using (1). This maximization represents a discrete version of the maximization of $f(x_1, \ldots, x_n) = \prod_{i \in I} \frac{1}{x_i+1}$ over the region $x_i \geq 0, \forall i \in I$. In the continuous version, $\frac{\partial f}{\partial x_i} < 0, \forall i \in I$, over the feasible region, so the maximum is the point $x_i = 0, \forall i \in I$. We now establish that a GSE is a Pareto maximum.

Result 1. If s^* is a GSE for Γ, then s^* is Pareto maximal for Γ.

Proof. By Lemma 1, it suffices to show that $T_g: u(\Omega) \to R^1$ is strictly increasing on $u(\Omega)$. Let $s', s'' \in S$ be pure strategies such that s'' dominates s'. Thus

$$0 < \frac{1}{M_i - u_i(s') + 1} \leq \frac{1}{M_i - u_i(s'') + 1}, \forall i \in I,$$

and

$$0 < \frac{1}{M_j - u_j(s') + 1} < \frac{1}{M_j - u_j(s'') + 1}$$

for some $j \in I$. Since these fractions are all positive,

$$T_g[u(s')] = \prod_{i \in I} \frac{1}{M_i - u_i(s') + 1} < \prod_{i \in I} \frac{1}{M_i - u_i(s'') + 1} = T_g[u(s'')].$$

Thus T_g is strictly increasing on $u(\Omega)$. ■

A special case of Result 1 is the following corollary.

Corollary 1. If s^* is a GSE for Γ, then s^* is not dominated by any pure NE for Γ.

On the other hand, the next example shows that a GSE for Γ can dominate a pure NE for Γ. Moreover, Example 1 and Corollary 1 suggest that a GSE may sometimes model player selfishness better than a pure NE.

Example 1. Consider the two-person game G with the 3×3 payoff matrix of Table I, where $S = \{(a_i, b_j) : i, j = 1, 2, 3\}$. The security levels $L_1 = 2$ and $L_2 = 1$, so $\Omega = S$.

Immediately from Table I, $M_1 = 7$ and $M_2 = 6$. We calculate the greedy scalar matrix for Table I to give the GSE corresponding to the bold underlined number in Table II.

From Tables I and II, the unique GSE for Γ is (a_3, b_3) with payoff vector $(6,6)$. The two pure NEs are (a_1, b_1) and (a_2, b_3) with payoff vectors $(3,4)$ and $(7,4)$, respectively, while the two BEs are (a_1, b_1) and (a_3, b_3) with payoff vectors $(3,4)$ and $(6,6)$, respectively. This example illustrates that a GSE for Γ is not necessarily an NE and vice versa. Moreover, the GSE (a_3, b_3) dominates the NE (a_1, b_1). There is also a BE that is the unique GSE. Finally, suppose that the payoff for (a_3, b_3) was changed to (6,5) and Table II was recalculated. Then $T_g[(6,5)] = 0.2500$, (a_3, b_3) would not be a GSE, and no BE would be a GSE and vice versa.

TABLE I
PAYOFF MATRIX FOR EXAMPLE 1

		Player 2		
		b_1	b_2	b_3
Player 1	a_1	(3,4)	(2,2)	(2,1)
	a_2	(2,3)	(7,1)	(7,4)
	a_3	(2,1)	(5,6)	(6,6)

TABLE II
SCALAR MATRIX FOR EXAMPLE 1

		Player 2		
		b_1	b_2	b_3
Player 1	a_1	0.0667	0.0333	0.0278
	a_2	0.0417	0.1667	0.3333
	a_3	0.0278	0.3333	**0.5000**

TABLE III
PAYOFF MATRIX FOR EXAMPLE 2

		Player 2	
		b_1 (Defect)	b_2 (Cooperate)
Player 1	a_1 (Defect)	(1,1)	(5,0)
	a_2 (Cooperate)	(0,5)	(3,3)

TABLE IV
SCALAR MATRIX FOR EXAMPLE 2

		Player 2	
		b_1 (Defect)	b_2 (Cooperate)
Player 1	a_1 (Defect)	0.1111	×
	a_2 (Cooperate)	×	**1**

Example 2. Consider now the classic two-person Prisoner's Dilemma (PD) [31] game G with the payoff matrix of Table III. The associated greedy scalar matrix for Γ is shown in Table IV.

In this case, $L_1 = 1$ and $L_2 = 1$, so $\Omega = \{(a_1, b_1), (a_2, b_2)\}$. The symbol × denotes that the associated strategy profile is not in Ω. Then (a_2, b_2) is the only GSE. It is not an NE but is a BE. Thus, in this semi-cooperative PD game, the greedy decision is also mutually supportive.

Example 3. Let Table V be the payoff matrix of a Chicken game G in which two countries are involved in nuclear brinkmanship. The Straight pure strategy involves launching nuclear missiles, while the Swerve pure strategy avoids doing so. In this example, $L_1 = L_2 = -2$, but we choose to ignore security level considerations here. Thus $\Omega = S$. Table VI gives the greedy scalar matrix of Γ. The pure NEs are (a_1, b_2) and (a_2, b_1), which are also the only GSEs. The strategy pair (a_1, b_1) is a BE. The greedy decision is to have one player launch a missile and the other not. There is a tie between (a_1, b_2) and (a_2, b_1), but it is unlikely that either of the tied GSEs would be acceptable to both countries or to an arbiter.

TABLE V
PAYOFF MATRIX FOR EXAMPLE 3

		Player 2	
		b_1 (Swerve)	b_2 (Straight)
Player 1	a_1 (Swerve)	(0,0)	(-2,2)
	a_2 (Straight)	(2,-2)	(-3,-3)

TABLE VI
SCALAR MATRIX FOR EXAMPLE 3

		Player 2	
		b_1 (Swerve)	b_2 (Straight)
Player 1	a_1 (Swerve)	0.1111	**0.2000**
	a_2 (Straight)	**0.2000**	0.0278

Computational Procedure 1. An algorithm is now presented for finding the nonempty set of GSEs for Γ by maximizing $T_g[u(s)]$ over $\Omega = \Lambda$.

Step1. Enumerate the $s \in S$ (i.e., the cells of the payoff matrix) as $1, \ldots, \prod_{j \in I} m_j$, and let Q be a positive number much larger than any $|u_i(s)|$ for the matrix. Read a single player's payoff at a time from each cell in the order $1, \ldots, n$. The length of the input is $N = n \prod_{j \in I} m_j$ numbers. As these numbers are read, if any $u_i(s) < L_i$, set $u_i(s) = -Q$ so that $s \notin \Omega$ and s cannot be a GSE. After all cells have been read and all replacements have been made, every action profile $s \notin \Omega$ will have at least one $u_i(s)$ with value $-Q$. Moreover, every n number from the beginning of the input list represents an n-tuple $(u_1(s), \ldots, u_n(s))$ for some $s \in S$.

Step2. For $\forall i \in I$, compute $M_i = \max_{s \in \Omega} u_i(s)$, which is the maximum of the individual input numbers $i, i+n, i+2n, \ldots, i+n(\prod_{j \in I} m_j - 1)$, one or more of which have value at least L_i.

Step3. For each of the possible $\prod_{j \in I} m_j$ cells, i.e., joint actions $s \in S$, compute $T_g[v(s)] = \prod_{i \in I} \frac{1}{M_i - u_i(s) + 1}$.

Step4. Find the action profiles $s^* \in S$ that maximize $T_g[u(s)]$ in Step 3.

We now show that the worst-case time complexity of Computational Procedure 1 is linear in the input data. We use the fact that in the Random Access Machine model of [32], each elementary operation such as an addition, multiplication, replacement, and *if* statement is considered to take a single time step.

Result 2. The worst-case time complexity for obtaining all GSEs for Γ is $O(N)$ for $N = n \prod_{j \in I} m_j$, which is the number of individual player payoffs in the payoff matrix. i.e., the input data for the game.

Proof. It suffices to consider the case where $m_i = M, \forall i \in I$, so that the payoff matrix of G has M^n cells and nM^n player payoffs. The maximum possible number of replacements in the Step 1 is $nM^n - 1$ with time complexity $O(nM^n)$. Finding all the n maxima M_i in Step 2 has complexity $O(nM)$ as established by Blum et al. [33]. With all M_i computed, finding each term of the product in Step 3 for a given $s \in S$ takes $3n$ time steps. But there are M^n joint actions $s \in S$ and thus $M^n - 1$ multiplications. Hence, determining the product in Step 3 for each $s \in S$ has complexity $O(nM^n)$. Next finding the GSEs by taking the maximum in Step 4 over all $s \in S$ has complexity $O(M^n)$. It follows that all GSEs can be obtained in $O(nM^n) + O(nM) + O(nM^n) + O(M^n)$, which is $O(nM^n)$, to establish the result. ■

Comparing Result 2 for GSEs to the computation of other pure equilibria, we note that the problem of determining if pure NEs exist for a normal-form game and, if so, finding them is also $O(N)$ as shown by [34]. A similar result holds for BEs [35]. However, finding a mixed NE is a PPAD-complete problem [36], a different type of complexity than NP-completeness but still a compelling argument for computational intractability using current approaches or computers.

It should be noted that a mixed GSE (or any SE) can be defined. However, this approach will not be considered here. A mixed GSE would assign a probability $\sigma(s)$ to each possible action profile $s \in S$ instead of a probability σ_j to the event that a particular player chooses his jth pure strategy, as in the case of a standard mixed strategy. For example, an optimal mixed GSE σ^* would maximize the resulting expected $T_g[u(s)]$ and solve the following linear program in $\prod_{j \in I} m_j$ variables:

$$\underset{\sigma}{\text{maximize}}\ T_g[\sigma] = \sum_{s \in S} \sigma(s) \left[\prod_{i \in I} \frac{1}{M_i - u_i(s) + 1} \right],$$

where $\sigma(s) \geq 0, \forall s \in S$, and $\sum_{s \in S} \sigma(s) = 1$.

IV. Weighted Scalar Equilibrium

We now generalize the GSE, which models the selfishness of the pure NE. The weighted scalar equilibrium (WSE) of this section can also model the mutual support or altruism of the pure BE, as well as the degree to which any player contributes part of his own payoff to another player for some reason. The idea for the WSE comes from the fact that a BE for a two-player game G_1 is simply an NE for the two-player game G_2 with a payoff matrix obtained by interchanging the payoffs of the two players of G_1. This payoff matrix for G_2 is called the swap matrix [37] of the original payoff matrix for G_1. In other words, for the game G_2 with the swap payoff matrix, player 1 plays selfishly for player 2 and vice versa. For $s \in S$, any payoff vector $(u_1(s), u_2(s))$ of the original payoff matrix for G_1 could be written in the swap payoff matrix for G_2 as the vector $(\alpha_{11}u_1(s) + \alpha_{12}u_2(s), \alpha_{21}u_1(s) + \alpha_{22}u_2(s))$, where the scalar coefficients $\alpha_{ij}, i, j = 1, 2,$ have values $\alpha_{11} = \alpha_{22} = 0$ and $\alpha_{12} = \alpha_{21} = 1$.

More generally, for an arbitrary G and $s \in S$, the payoff vector $(u_1(s), \ldots, u_n(s))$ of G could be replaced by

$$\left(\sum_{i=1}^{n} \alpha_{i1} u_i(s), \ldots, \sum_{i=1}^{n} \alpha_{in} u_i(s) \right), \qquad (2)$$

where α_{ij} is the fraction of player i's payoff contributed exactly one time to player j in the original payoff matrix and $\sum_{i=1}^{n} \alpha_{ij} = 1, j = 1, \ldots, n.$ The resulting payoff matrix is called the alpha-transformed payoff matrix.

For $i, j = 1, \ldots, n,$ when $\alpha_{ii} = 1$ and $\alpha_{ij} = 0,\ i \neq j,$ a pure NE of the alpha-transformed payoff matrix would be a pure NE of the original game and hence model selfishness. When $\alpha_{ii} = 0$ and $\alpha_{ij} = 1/(n-1),\ i \neq j,$ a GSE of the alpha-transformed payoff matrix would model a pure BE since any player would be trying to make the payoffs of the other players as large as jointly possible. However, a GSE of this alpha-transformed would not necessarily be a BE of the original payoff matrix as seen in Example 4.

In the general case represented by the payoffs of (2), an alpha-transformed payoff matrix allows a player to distribute his payoff to the n players, including himself, in any way he desires. For this new payoff matrix, a GSE is then determined to obtain an SE that gives each player the largest individual alpha-transformed payoff jointly possible.

Definition 8 (WSE). For $M_k = \max_{s \in \Omega} \sum_{i=1}^{n} \alpha_{ik} u_i(s)$, the pure strategy profile s^* is a WSE for $\Gamma = \langle G, \Omega, T_w \rangle$ if and only if s^* maximizes over Ω the utility function

$$T_w[u(s)] = \prod_{k \in I} \frac{1}{M_k + 1 - \sum_{i=1}^{n} \alpha_{ik} u_i(s)}, \ s \in \Omega. \qquad (3)$$

Equation (3) is obtained by inserting (2) into (1) with a necessary change of subscripts. The following result follows as in the proof of Result 1 since $\sum_{i=1}^{n} \alpha_{ik} u_i(s') < \sum_{i=1}^{n} \alpha_{ik} u_i(s'')$ for any $s', s'' \in \Omega$ for which s'' dominates s'.

Result 3. If s^* is a WSE for the game associated with the alpha-transformed payoff matrix of $\Gamma = \langle G, \Omega, T_w \rangle$, then s^* is Pareto maximal for the alpha-transformed payoff matrix of Γ.

Example 4. Consider the game the three-player game G with the payoff matrix of Table VII.

TABLE VII
Payoff Matrix for Example 4

	s_3		t_3	
	s_2	t_2	s_2	t_2
s_1	(3,1,2)	(3,4,0)	(6,3,0)	(3,5,1)
t_1	(1,4,5	(2,2,3	(2,4,4	(-1,2,3)

TABLE VIII
Alpha-Transformed Payoffs for Example 4

	s_3		t_3	
	s_2	t_2	s_2	t_2
s_1	(1.5,2.5,2.0)	(2.0,1.5,3.5)	(1.5,3.0,4.5)	(3.0,2.0,4.0)
t_1	(4.5,3.0,2.5)	(2.5,2.5,2.0)	(4.0 3.0,3.0)	(2.5,1.0,0.5)

In Table VII, for players $i = 1, 2, 3$, the first strategy for player i is labeled s_i and the second as t_i. Let $\alpha_{ii} = 0,\ i = 1, 2, 3,$ and $\alpha_{ij} = 1/2,\ i \neq j.$ The associated alpha-transformed payoff matrix is shown in Table VIII.

The GSE for Table VIII is (t_1, s_2, s_3), which is thus the WSE for the original game G. However, (t_1, s_2, s_3) is not a pure BE of Table VII, though (t_1, s_2, t_3) is. On the other hand, the GSE (t_1, s_2, s_3) is also a BE of Table VII. The pure strategy profiles (t_1, s_2, s_3) and (t_1, s_2, t_3) both appear to model mutual support well for Table VII and to maximize the sum of the players' payoffs. The GSE (t_1, s_2, s_3) is Pareto maximal for Table VII by Result 3.

By including the degree to which players support each other, the WSE seems a superior model for mutual support or cooperation than either the BE or the general equilibrium of [20]. For given α_{ij}, the worst-case computational complexity for finding all WSEs is again $O(N)$ for $N = n\prod_{j\in I} m_j$.

V. Compromise Scalar Equilibrium

We next define a compromise scalar equilibrium (CSE) so that each player gets a fair payoff relative to the other players. For $\Omega = \Psi$, $m_i = \min_{s\in\Omega} u_i(s)$, and $M_i = \max_{s\in\Omega} u_i(s)$, we consider the utility function $T_c : u(\Omega) \to R^1$ for which

$$T_c[u(s)] = \prod_{i=1}^{n} \frac{u_i(s) - m_i + 1}{M_i - m_i + 1}, s \in \Omega. \tag{4}$$

From the definition of m_i and M_i, it follows that $0 < T_c[u(s)] \le 1$ for all $s \in \Omega$. The number 1 in the numerators of (4) prevents $T_c[u(s)]$ from being 0 if $u_i(s) = m_i$ for some i, while the number 1 in the denominators prevent a division by 0 if $m_i = M_i$. Maximizing $T_c[u(s)]$ over Ω requires that each $u_i(s^*)$ be as close to M_i as jointly possible. Thus, the notion of fairness for a CSE is that all players with $M_i > m_i$ will receive payoffs in approximately the same percentile of their payoff ranges over the feasible action profiles. Players with $M_i = m_i$ will receive M_i. The following definition extends that of [41].

Definition 9 (CSE). The pure strategy profile s^* is a CSE for $\Gamma = \langle G, \Omega, T_c \rangle$ if and only if s^* maximizes $T_c[u(s)]$ over Ω.

A CSE can be construed as a compromise between the players' selfishness and unselfishness. It differs from the fairness equilibrium of Rabin [38] for two players and from such notions of fairness as in Korth [39]. $T_c[u(s)]$ is a discrete analog of the Nash product for the two-person bargaining problem [40]. Moreover, maximizing $T_c[u(s)]$ over Ω is a discrete version of maximizing $f(x_1,...,x_n) = \prod_{i\in I} x_i$ over $0 < x_i \le 1, \forall i \in I$, where the maximum is at $x_i = 1, \forall i \in I$.

An algorithm to obtain all CSEs would be a simple modification of Computational Procedure 1 and have worst-case time complexity $O(N)$ for $N = n\prod_{j\in I} m_j$. Since T_c is easily shown to be strictly increasing on $u(\Omega)$, the next result follows directly from Lemma 1.

Result 4. If s^* is a CSE for $\Gamma = \langle G, \Omega, T_c \rangle$, then s^* is Pareto maximal for Γ.

Example 5. Consider the two-person PD game G of Table III with $\Omega = \{(a_1, b_1), (a_2, b_2)\}$ as in the associated Example 2. The compromise scalar matrix for Γ is shown in Table IX. The unique CSE is (a_2, b_2), which is not an NE but is a BE.

TABLE IX
Scalar Matrix for Example 5

		Player 2	
		b_1 (Defect)	b_2 (Cooperate)
Player 1	a_1 (Defect)	0.1111	×
	a_2 (Cooperate)	×	1

VI. Parity Scalar Equilibrium

In the parity scalar equilibrium (PSE) for Γ, the objective is to determine a pure strategy profile s for which the players' payoffs are as nearly equal as jointly possible. For $\Omega = S$, let $M = \max_{i\in I} \max_{s\in\Omega} u_i(s)$ and define $T_p : u(\Omega) \to R^1$ as

$$T_p[u(s)] = \prod_{i\in I} \frac{1}{M - u_i(s) + 1}, s \in \Omega. \tag{5}$$

Much as for (1), a pure strategy profile $s^* \in \Omega$ maximizing $T_p[u(s)]$ over Ω requires that the payoffs $u_i(s^*), i = 1,\dots,n,$ be as nearly equal as jointly possible using (5).

Definition 10 (PSE). The pure strategy profile s^* is a PSE for $\Gamma = \langle G, \Omega, T_p \rangle$ if and only if s^* maximizes $T_P[u(s)]$ over Ω.

Much as for Result 1, we have the following result.

Result 5. If s^* is a PSE for Γ, s^* is Pareto maximal for Γ.

Example 6. Performing the calculations of (5) for the game G of Table I yields a maximum value for the unique PSE (t_1, s_2, t_3) with associated payoffs (2, 4, 4), which is Pareto maximal. It should be noted that if the nonpositive utility function

$$\hat{T}_P[u(s)] = -\sum_{i=1}^{n} \sum_{j=i+1}^{n} [u_i(s) - u_j(s)]^2, s \in \Omega, \tag{6}$$

were used instead of (5), then maximizing (6) to get a PSE would have given (t_1, t_2, s_3) with payoffs (2,2,3), which is not Pareto maximal. The reason is that (5) and (6) approximate continuous problems slightly differently. In addition, $\hat{T}_P[u(s)]$ is not strictly increasing on $u(\Omega)$.

VII. Satisficing Scalar Equilibrium

Aspiration levels are widely used in decision theory [42] and will be used here in a satisficing scalar equilibrium (SSE) unrelated to the satisficing games of [43]. It will achieve the following three objectives.

(i) An SSE s^* gives each player $i \in I$ at least the targeted payoff level p_i required for the player to agree to the action profile s^*. It is assumed that $p_i \geq L_i, \forall i \in I$, with some $p_k > L_k$ to distinguish these aspiration levels from the security levels, which are always obtainable.

(ii) The SSE model focuses the players or arbitrator on the SSE scalar parameters $p_i, d_i, \forall i \in I$. For example, if all p_i cannot be simultaneously satisfied, they can be modified by an agreement of the players. An SSE can also give certain players higher relative payoffs than other players. Each player $i \in I$ will be assigned a weighting factory $d_i > 0$. If $d_j > d_k$, then player j will receive a higher payoff than player k by the factor $\frac{d_j}{d_k}$, if possible.

(iii) For $d_i > 0, \forall i \in I$, an SSE is Pareto maximal over the joint actions achieving the players' aspiration levels.

Definition 11 (SSE). For the fixed scalars $p_i, \forall i \in I$, and $d_i > 0, \forall i \in I$, let $\Omega = \{ s \in S : u_i(s) \geq p_i, i \in I\}$. Then the action profile s^* is an SSE for the game Γ if and only if s^* maximizes the utility function $T_S[u(s)] = \sum_{i\in I} d_i u_i(s)$ over Ω.

Result 6. Let the parameters $p_1,..., p_n, d_1,..., d_n$ be scalars with $d_i > 0, \forall i \in I$. Then any $s^* \in S$ solving

$$\underset{s\in S}{\text{maximize}} \sum_{i\in I} d_i u_i(s) \quad \textit{subject to} \quad u_i(s) \geq p_i, \forall i \in I, \tag{7}$$

is both an SSE and a Pareto maximum for the game Γ.

Proof. The action profile $s^* \in S$ is an SSE by Definition 11. Let $d_i > 0, \forall i \in I$, and $s', s'' \in \Omega$ be such that s' dominates s'', so $0 < \sum_{i\in I} d_i[u_i(s') - u_i(s'')] = \sum_{i\in I} d_i u_i(s') - \sum_{i\in I} d_i[u_i(s'')]$. Thus T_S is strictly increasing on $u(\Omega)$, and Lemma 1 gives the result. ■

One approach for determining feasible aspiration levels p_i satisfied by at least one $s \in S$ in Definition 11 is to select weights $d_i > 0, \forall i \in I$, and then $\underset{s\in S}{\text{maximize}} \sum_{i\in I} d_i u_i(s)$. The aspiration levels $p_i = u_i(s^*), \forall i \in I$, are then feasible. For $d_i = 1, \forall i \in I$, these aspiration levels might be construed as fair. This approach suggests the following counterpart to Result 6.

Result 7. If s^* is Pareto maximal for Γ, then for any $d_i > 0, \forall i \in I$, the action profile s^* is an SSE for the aspiration levels $p_i = u_i(s^*), \forall i \in I$.

Proof. Let s^* be Pareto maximal for Γ_n. Then s^* is obviously feasible for (3) with $p_i = u_i(s^*), \forall i \in I$. Suppose there exist $d_i' > 0, \forall i \in I$, for which s^* does not solve (1). Then there exists some $s' \in S$ for which $u_i(s') \geq p_i = u_i(s^*), \forall i \in I$. It follows that $\sum_{i\in I} d_i' u_i(s') > \sum_{i\in I} d_i' u_i(s^*)$ and hence $\sum_{i\in I} d_i'[u_i(s') - u_i(s^*)] > 0$. But since $d' > 0, \forall i \in I$, then $u_i(s') \geq u_i(s^*), \forall i \in I$, and $u_k(s') > u_k(s^*)$ for some k. Hence s^* is not a Pareto maximum for Γ in contradiction to the assumption. It follows that for any $d_i > 0, \forall i \in I$, s^* solves (3) for aspiration levels $p_i = u_i(s^*), \forall i \in I$, and is an SSE.■

Example 7. Consider now the payoff matrix of Example 1 in Table I. Note that $(p_1, p_2) = (7,5)$ yields no feasible action profiles. When $(p_1, p_2) = (5,4)$, the optimization problem (7) has a solution, and the feasible $s \in \Omega$ are (a_2, b_3) with payoff vector $(7,4)$, (a_3, b_2) with payoff vector $(5,6)$, and (a_3, b_3) with payoff vector $(6,6)$. Setting $d_1 = d_2 = 1$ in (7) gives the unique SSE $s^* = (a_3, b_3)$ with payoff vector $(6,6)$. For $(p_1, p_2) = (5,4)$ and $(d_1, d_2) = (0.7, 0.3)$, the satisficing scalar matrix of values $T_S[u(s)]$ for the feasible payoffs is shown in Table X, where cells with $\times$ have infeasible payoffs. The unique SSE for Table X is (a_2, b_3) with payoff profile $(7,4)$. Its scalar value is bolded and underlined.

TABLE X
Scalar Matrix for Example 7 with $(p_1, p_2) = (5,4)$, $(d_1, d_2) = (0.7,0.3)$

		Player 2		
		b_1	b_2	b_3
Player 1	a_1	×	×	×
	a_2	×	×	**<u>6.1</u>**
	a_3	×	5.3	6.0

For appropriate aspiration levels, an SSE always exists for Γ. Computational Procedure 1 can again be modified to determine if an SSE exists and to obtain them all if so. To do so, L_i is replaced by p_i and T_g by T_S for a given set of $d_i > 0$. Whenever $u_i(s) < p_i$, the replacement $u_i(s) = -Q$ in the maximization of (3) assures that s will not be an SSE. This modified procedure also has worst-case time complexity $O(N)$ for $N = n\prod_{j\in I} m_j$.

VIII. Axiomatic Considerations

A set of five axioms is now proposed for the SEs of $\Gamma = \langle G, \Omega, T\rangle$. Nash [40] gave versions of Axioms 1-4 for his bargaining problem. Axiom 5 was formulated by Kalai [45].

1. *Nondominance*: No player's payoff for an SE s^* of Γ is better for any other $s \in \Omega$ unless some different player's payoff is worse.
2. *Symmetry*: If the players of Γ cannot be distinguished before an SE is obtained, then an SE cannot distinguish between them.
3. *Invariance to Linear Transformations:* Let s^* be an SE of Γ for which all utilities $u_i(s), \forall i \in I, s \in \Omega$, and any aspiration levels $p_i, \forall i \in I$, are nonnegative. For every $\beta > 0$, then s^* will also be an SE for the game with all utilities and any aspiration levels linearly transformed to $\beta u_i(s)$ and $\beta p_i, \forall i \in I$.
4. *Independence of Irrelevant Alternative:* If $s^* \in \Omega_1$ is an SE for Γ with respect to a feasible set Ω_1 and if $s^* \in \Omega_2 \subset \Omega_1$, then s^* is an SE with respect to the feasible set Ω_2.
5. *Monotonicity.* If $s^* \in S$ is an SE for the feasible set Ω_1 and $s^{**} \in S$ is an SE for the feasible set $\Omega_2 \supset \Omega_1$, then $u_i(s^*) \le u_i(s^{**}), \forall i \in I$.

The Nash two-person bargaining solution [40] satisfies the above five axioms except for Axiom 5. The two-person bargaining solution of Kalai [44] satisfies all the axioms except Axiom 4, while the solution of Kalai and Smorodinsky [45] satisfies Axioms 1-3 but not Axioms 4-5. The SEs of this paper clearly satisfy Axioms 1-4. However, Axiom 5 is not satisfied by the SEs here as illustrated by the game of Example 2 when $\Omega_1 = \{(a_1,b_1),(a_2,b_2)\}$ and $\Omega_2 = S$. In that case, the GSE associated with Ω_1 is $s^* = (a_2, b_2)$ with $T_g[u(s^*)] = 1$. while one GSE associated with Ω_2 is $s^{**} = (a_1, b_2)$ with $T_g[u(s^*)] = 0.1667$. Note that $u_2(s^{**}) = 0 < u_2(s^*) = 3$.

IX. Conclusion

Mixed strategy solution concepts for normal-form games are problematic in their computation, interpretation, and application. Indeed, a mixed Nash equilibria for three or more players is usually only an approximation due to the PPAD computational complexity of the problem. In addition, according to [1], most games involve implicit negotiations in some fashion. The approach proposed here would require an agreement on a utility function T yielding a particular type of SE. The associated negotiation would represent a cooperative aspect for the game. There would also be a competitive aspect since each player would want any SE to be as good as possible for him according to his stated objective. In summary, this paper attempts to reduce the selection of pure strategies for the games considered here to the choice of an appropriate utility function T, of which five were offered.

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