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Improved Triple Integral Inequalities of Hermite-Hadamard Type

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Abstract—In this paper, we present the concept of preinvex functions on the co-ordinates on an invex set and establish some triple integral inequalities of Hermite-Hadamard type for functions whose third order partial derivatives in absolute value are preinvex on the co-ordinates. The results presented here generalize the obtained results in earlier works for functions whose triple order partial derivatives in absolute value are convex on the co-ordinates on a rectangular box in $\mathbb{R}^3$.

Keywords—Co-ordinated preinvex functions, Hermite-Hadamard type inequalities, partial derivatives, triple integral.

I. Introduction

LET J be an nonempty interval of real numbers. A function $f : J \to \mathbb{R}$ is said to be convex on the interval J, if the following inequality holds:

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)y \tag{1}$$

for every $x, y \in J$ and $\lambda \in [0,1]$. If the reversed inequality in (1) holds, then f is concave. One of the most famous inequalities for convex functions is the Hermite-Hadamard inequality. This double integral inequality states that if $f : J \to \mathbb{R}$ is a convex function, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}, \tag{2}$$

where $J \subseteq \mathbb{R}$ is an nonempty interval and a, b belong to J with $a < b$. Both inequalities in (2) hold in the reversed direction if f is a concave function. Over the last decade the double Hermite-Hadamard integral inequality (2) has been extended, refined and generalized using novel and innovative techniques (see for example [5], [8], [17], [22], [31], [32] and the references therein). A significant class of convex sets is that of invex sets introduced by Mohan et al. [16]. In [28], the authors introduced the concept of preinvex functions as a generalization of convex functions.

We recall the following definitions which are well known in literature: Let K be a nonempty and closed subset of $\mathbb{R}^n$ and let $f : K \to \mathbb{R}$ and $\eta : K \times K \to \mathbb{R}^n$ be continuous functions. In [16], the concept of invex sets was introduced as follows:

Definition 1. (invex set) The set K is said to be invex with respect to the mapping $\eta(.,.)$, if

$$x + t\eta(y,x) \in K,$$

for every $x, y \in K$ and $t \in [0,1]$.

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Notice that every convex set is invex with respect to the mapping $\eta(x,y) = x - y$, but there exist invex sets which are not convex (see for example [1] and [16]).

Definition 2. (preinvex function) The function $f : K \to \mathbb{R}$ is said to be preinvex on K with respect to the mapping $\eta(.,.)$, if

$$f(u + t\eta(v,u)) \leq (1-t)f(u) + tf(v),$$

for every $u, v \in K$ and $t \in [0,1]$.

It is trivial that every convex function is preinvex with respect to the mapping $\eta(.,.)$, but there exist preinvex functions which are not convex, (see for example [22]). Recently, several Hermite-Hadamard type inequalities have been obtained for preinvex functions (see [11], [18]).

Let $\Delta =: [a,b] \times [c,d] \subseteq \mathbb{R}^2$ be a bidimensional interval with $a < b$ and $c < d$. A mapping $f : \Delta \to \mathbb{R}$ is convex on the rectangle Δ from the plane $\mathbb{R}^2$, if

$$f(\lambda x + (1-\lambda)z, \lambda y + (1-\lambda)w) \leq \lambda f(x,y) + (1-\lambda)f(z,w),$$

holds for every $(x,y), (z,w) \in \Delta$ and $\lambda \in [0,1]$.

In [6], Dragomir introduced the concept of convex functions on the co-ordinates on the rectangle Δ as follows: A mapping $f : \Delta \to \mathbb{R}$ is said to be convex on the co-ordinates on the rectangle Δ if the partial mappings

$$f_y : [a,b] \to \mathbb{R}, f_y(u) = f(u,y)$$

and

$$f_x : [c,d] \to \mathbb{R}, f_x(v) = f(x,v)$$

are convex where defined for all $x \in [a,b], y \in [c,d]$.

In [12], the authors presented a formal definition for co-ordinated convex functions in following form: A mapping $f : \Delta \to \mathbb{R}$ is said to be convex on the co-ordinates on the rectangle Δ, if

$$\begin{aligned} & f(tx + (1-t)y, su + (1-s)w) \\ & \leq tsf(x,u) + t(1-s)f(x,w) \\ & + s(1-t)f(y,u) + (1-t)(1-s)f(y,w), \end{aligned}$$

holds for every $(x,u), (y,w) \in \Delta$ and $t, s \in [0,1]$. Clearly, every convex function on the rectangle Δ is convex on the co-ordinates on the rectangle Δ, but converse may not be true (see for example [6]). For several recent results concerning Hermite-Hadamard type inequalities for functions that satisfy different classes of convexity on the co-ordinates on the rectangle Δ from the plane $\mathbb{R}^2$, we refer the interested reader to [2], [6], [10], [12]–[14], [21], [23]–[26]. Let K_1 and K_2 be two nonempty subsets of $\mathbb{R}^n$ and let $\eta_1 : K_1 \times K_1 \to \mathbb{R}^n$ and $\eta_2 : K_2 \times K_2 \to \mathbb{R}^n$ be two continuous functions. The concept of preinvex functions on $K_1 \times K_2$ and co-ordinated preinvex functions on $K_1 \times K_2$ were introduced by [15] as follows:

Definition 3. Let $K_1 \times K_2$ be an invex set with respect to the mappings $\eta_1(.,.)$ and $\eta_2(.,.)$. We say that $f : K_1 \times K_2 \to \mathbb{R}$ is a preinvex function, if

$$f(u + t\eta_1(x, u), v + t\eta_2(y, v)) \leq (1 - t)f(x, y) + tf(u, v),$$

for all $(x, y), (u, v) \in K_1 \times K_2$ and $t \in [0, 1]$.

Definition 4. Let $K_1 \times K_2$ be an invex set with respect to the mappings $\eta_1(.,.)$ and $\eta_2(.,.)$. We say that $f : K_1 \times K_2 \to \mathbb{R}$ is a preinvex function on the co-ordinates, if the partial mappings

$$f_y : K_1 \to \mathbb{R}, f_y(u) = f(u, y)$$

and

$$f_x : K_2 \to \mathbb{R}, f_x(v) = f(x, v)$$

are preinvex with respect to the mappings η_1 and η_2 respectively for every $y \in K_2$ and $x \in K_1$.

Clearly, any convex function on the co-ordinates is preinvex on the co-ordinates. Furthermore, there exist preinvex functions on the co-ordinates which are not convex on the co-ordinates (see for example [15]). In the same article, the authors established several Hermite-Hadamard type inequalities for functions whose second order partial derivatives in absolute value are preinvex on the co-ordinates. In [27], the authors defined convex functions and co-ordinated convex functions on a rectangular box $\Omega := [a, b] \times [c, d] \times [e, f]$ in $\mathbb{R}^3$ as follows:

Definition 5. The mapping $f : \Omega \to \mathbb{R}$ is a convex function on the rectangular box Ω, if

$$\begin{aligned} &f(\lambda x + (1 - \lambda)z, \lambda y + (1 - \lambda)w, \lambda u + (1 - \lambda)v) \\ &\quad \leq \lambda f(x, y, u) + (1 - \lambda)f(z, w, v), \end{aligned}$$

for all $(x, y, u), (z, w, v) \in \Omega$ and $\lambda \in [0, 1]$.

Definition 6. We say that $f : \Omega \to \mathbb{R}$ is a convex function on the co-ordinates on Ω if for every $(x, y, z) \in \Omega$, the partial mappings,

$$f_x : [c, d] \times [e, f] \to \mathbb{R}, \;\; f_x(v, w) = f(x, v, w), \; x \in [a, b];$$

$$f_y : [a, b] \times [e, f] \to \mathbb{R}, \;\; f_y(u, w) = f(u, y, w), \; y \in [c, d];$$

and

$$f_z : [a, b] \times [c, d] \to \mathbb{R}, \;\; f_z(u, v) = f(u, v, z), \; z \in [e, f]$$

are convex.

In [27], the authors established the Hermite-Hadamard type inequality for co-ordinated convex functions on a rectangular box in $\mathbb{R}^3$.

The aim of this paper is to introduce the concept of co-ordinated preinvex functions defined on an open invex set and to establish some inequalities of Hermite-Hadamard type for functions whose third order partial derivatives in absolute value are preinvex on the co-ordinates. The presented results generalize the obtained results in earlier works for functions whose third order partial derivatives in absolute value are convex on the co-ordinates on a rectangular box in $\mathbb{R}^3$. Main aim of the present paper is to obtain several inequalities to functions that defined on an invex set of $\mathbb{R}^3$ and they generalize the obtained results to functions that defined on an invex set of $\mathbb{R}^2$.

II. Main Results

The goal of this paper is to introduce the notion co-ordinated preinvex functions on an open invex set which is a generalization of the notion co-ordinated convex functions on a rectangular box in $\mathbb{R}^3$ given in Definition6 and to establish some inequalities of Hermite-Hadamard type for these class functions.

Throughout this paper, let K_1, K_2 and K_3 be three nonempty subsets of $\mathbb{R}^n$, let $\eta_1 : K_1 \times K_1 \to \mathbb{R}^n$, $\eta_2 : K_2 \times K_2 \to \mathbb{R}^n$ and $\eta_3 : K_3 \times K_3 \to \mathbb{R}^n$ be three continuous functions and let $\Gamma = K_1 \times K_2 \times K_3$.

Definition 7. We say that Γ is an invex set with respect to the mappings $\eta_1(.,.)$, $\eta_2(.,.)$ and $\eta_3(.,.)$, if

$$(u + t\eta_1(x, u), v + t\eta_2(y, v), w + t\eta_2(z, w)) \in \Gamma$$

for all $(x, y, z), (u, v, w) \in \Gamma$ and $t \in [0, 1]$.

Definition 8. Let Γ is an invex set with respect to the mappings $\eta_1(.,.)$, $\eta_2(.,.)$ and $\eta_3(.,.)$. We say that $f : \Gamma \to \mathbb{R}$ is a preinvex function on Γ, if

$$\begin{aligned} &f(u + t\eta_1(x, u), v + t\eta_2(y, v), w + t\eta_2(z, w)) \\ &\leq (1 - t)f(x, y, z) + tf(u, v, w) \end{aligned}$$

for all $(x, y, z), (u, v, w) \in \Gamma$ and $t \in [0, 1]$.

Definition 9. Let Γ is an invex set with respect to the mappings $\eta_1(.,.)$, $\eta_2(.,.)$ and $\eta_3(.,.)$. We say that $f : \Gamma \to \mathbb{R}$ is a preinvex function on the co-ordinates, if the partial mappings

$$f_z : K_1 \times K_2 \to \mathbb{R}, f_z(u, v) = f(u, v, z),$$

$$f_y : K_1 \times K_3 \to \mathbb{R}, f_y(u, w) = f(u, y, w)$$

and

$$f_x : K_2 \times K_3 \to \mathbb{R}, f_x(v, w) = f(x, v, w)$$

are preinvex with respect to the mappings $\eta_1(.,.)$, $\eta_2(.,.)$ and $\eta_3(.,.)$, respectively, for every $z \in K_3$, $y \in K_2$ and $x \in K_1$.

Lemma 1. Every preinvex mapping $f : \Gamma \to \mathbb{R}$ is co-ordinated preinvex on Γ.

Proof: Let $f : \Gamma \to \mathbb{R}$ is preinvex on Γ. Defining the partial mappings as follows:

$$f_x : K_2 \times K_3 \to \mathbb{R}, f_x(y, z) = f(x, y, z), x \in K_1;$$

$$f_y : K_1 \times K_3 \to \mathbb{R}, f_y(x, z) = f(x, y, z), y \in K_2;$$

$$f_z : K_1 \times K_2 \to \mathbb{R}, f_z(x, y) = f(x, y, z), z \in K_3.$$

Let $t \in [0,1]$ and $(x_1,x_2),(y_1,y_2) \in K_2 \times K_3$. The preinvexity of f on Γ follows that

$$\begin{aligned} f_x & (y_1+t\eta_1(x_1,y_1), y_2+t\eta_2(x_2,y_2)) \\ = & \ f(x, y_1+t\eta_1(x_1,y_1), y_2+t\eta_2(x_2,y_2)) \\ = & \ f(x+t\eta_1(x,x), y_1+t\eta_1(x_1,y_1), y_2+t\eta_2(x_2,y_2)) \\ \leq & \ (1-t)f(x,x_1,x_2)+tf(x,y_1,y_2) \\ = & \ (1-t)f_x(x_1,x_2)+tf_x(y_1,y_2). \end{aligned}$$

Thus, the mapping f_x is a preinvex function. The preinvexity of functions f_y and f_z can be proved in a similar way. ■

Note that every co-ordinated convex function is co-ordinated preinvex; however, the converse is not generally true. See the following example:

Example 1. Consider the function $f : \Gamma \to \mathbb{R}$ defined by $f(u,v,w) = -|u||v||w|$. The function f is not co-ordinated convex, but it is clear that the function f is co-ordinated preinvex with respect to the mappings η_1, η_2 and η_3 defined as follows:

$$\eta_1(u,z) = \begin{cases} u-z, & u,z \geq 0 \text{ or } u,z \leq 0 \\ z-u, & \text{otherwise,} \end{cases}$$

$$\eta_2(v,y) = \begin{cases} v-y, & v,y \geq 0 \text{ or } v,y \leq 0 \\ y-v, & \text{otherwise} \end{cases}$$

and

$$\eta_3(w,x) = \begin{cases} w-x, & w,x \geq 0 \text{ or } w,x \leq 0 \\ x-w, & \text{otherwise.} \end{cases}.$$

To obtain us main results, we need to prove the following new lemma:

Lemma 2. Let $\Gamma \subseteq \mathbb{R}^3$ be an open invex set with respect to the mappings $\eta_1(.,.)$, $\eta_2(.,.)$ and $\eta_3(.,.)$. If $f : \Gamma \to \mathbb{R}$ be a mapping having third partial derivatives and

$$\frac{\partial^3 f}{\partial t \partial s \partial h} \in L\Big([a, a+t\eta_1(b,a)] \times [c, c+s\eta_2(d,c)] \times [e, e+h\eta_3(f,e)]\Big)$$

with $\eta_1(b,a) > 0$, $\eta_2(d,c) > 0$ and $\eta_3(f,e) > 0$, where $a,b \in K_1$, $c,d \in K_2$ and $e,f \in K_3$. Then one has the following equality:

$$\frac{\eta_1(b,a)\eta_2(d,c)\eta_3(f,e)}{8}\int_0^1\int_0^1\int_0^1(1-2t)(1-2s)(1-2h) \tag{3}$$
$$\frac{\partial^3 f(a+t\eta_1(b,a), c+s\eta_2(d,c), e+h\eta_3(f,e))}{\partial t \partial s \partial h} dt ds dh$$
$$= \frac{1}{\eta_1(b,a)\eta_2(d,c)\eta_3(f,e)}\int_a^{a+\eta_1(b,a)}\int_c^{c+\eta_2(d,c)}\int_e^{e+\eta_3(f,e)} f(x,y,z) dz dy dx - A + B - C,$$

where

$$\begin{aligned} A = \frac{1}{8}\Big[& f(a+\eta_1(b,a), c+\eta_2(d,c), e+\eta_3(f,e)) \\ & + f(a, c+\eta_2(d,c), e+\eta_3(f,e)) \\ & + f(a+\eta_1(b,a), c, e+\eta_3(f,e)) \\ & + f(a, c, e+\eta_3(f,e)) + f(a+\eta_1(b,a), c+\eta_2(d,c), e) \\ & + f(a, c+\eta_2(d,c), e) + f(a+\eta_1(b,a), c, e) + f(a,c,e)\Big], \end{aligned}$$

$$\begin{aligned} B = & \frac{1}{4\eta_3(f,e)}\int_e^{e+\eta_3(f,e)} f(a+\eta_1(b,a), c+\eta_2(d,c), z) \\ & + f(a, c+\eta_2(d,c), z) + f(a+\eta_1(b,a), c, z) + f(a,c,z) dz \\ & + \frac{1}{4\eta_2(d,c)}\int_c^{c+\eta_2(d,c)} f(a+\eta_1(b,a), y, e+\eta_3(f,e)) \\ & + f(a, y, e+\eta_3(f,e)) + f(a+\eta_1(b,a), y, e) + f(a,y,e) dy \\ & + \frac{1}{4\eta_1(b,a)}\int_a^{a+\eta_1(b,a)} f(x, c+\eta_2(d,c), e+\eta_3(f,e)) \\ & + f(x, c, e+\eta_3(f,e)) + f(x, c+\eta_2(d,c), e) + f(x,c,e) dx \end{aligned}$$

and

$$\begin{aligned} C = & \frac{1}{2\eta_2(d,c)\eta_3(f,e)} \\ & \int_e^{e+\eta_3(f,e)}\int_c^{c+\eta_2(d,c)} f(a+\eta_1(b,a), y, z) + f(a,y,z) dy dz \\ & + \frac{1}{2\eta_1(b,a)\eta_3(f,e)} \\ & \int_e^{e+\eta_3(f,e)}\int_a^{a+\eta_1(b,a)} f(x, c+\eta_2(d,c), z) + f(x,c,z) dx dz \\ & + \frac{1}{2\eta_1(b,a)\eta_2(d,c)} \\ & \int_c^{c+\eta_3(d,c)}\int_a^{a+\eta_1(b,a)} f(x, y, e+\eta_3(f,e)) + f(x,y,e) dx dy. \end{aligned}$$

Proof: In order to prove (3), we set

$$I_1 = \int_0^1 (1-2t)\frac{\partial^3 f(a+t\eta_1(b,a), c+s\eta_2(d,c), e+h\eta_3(f,e))}{\partial t \partial s \partial h} dt.$$

By integration by parts with respect to t over the interval $[0,1]$, one can obtain

$$\begin{aligned} I_1 = & (1-2t)\frac{\partial^2 f(a+t\eta_1(b,a), c+s\eta_2(d,c), e+h\eta_3(f,e))}{\eta_1(b,a)\partial s \partial h}\Big|_0^1 \\ & + \frac{2}{\eta_1(b,a)}\int_0^1 \frac{\partial^2 f(a+t\eta_1(b,a), c+s\eta_2(d,c), e+h\eta_3(f,e))}{\partial s \partial h} dt \\ = & \frac{-\partial^2 f(a+\eta_1(b,a), c+s\eta_2(d,c), e+h\eta_3(f,e))}{\eta_1(b,a)\partial s \partial h} \\ & \frac{-\partial^2 f(a, c+s\eta_2(d,c), e+h\eta_3(f,e))}{\eta_1(b,a)\partial s \partial h} \\ & + \frac{2}{\eta_1(b,a)}\int_0^1 \frac{\partial^2 f(a+t\eta_1(b,a), c+s\eta_2(d,c), e+h\eta_3(f,e))}{\partial s \partial h} dt. \end{aligned}$$

Putting $I_2 = \int_0^1 (1-2s) I_1 ds$. Therefore

$$I_2=\int_0^1(1-2s)\left[\frac{-\partial^2 f(a+\eta_1(b,a),c+s\eta_2(d,c),e+h\eta_3(f,e))}{\eta_1(b,a)\partial s\partial h}\right.$$
$$\frac{-\partial^2 f(a,c+s\eta_2(d,c),e+h\eta_3(f,e))}{\eta_1(b,a)\partial s\partial h}$$
$$\left.+\frac{2}{\eta_1(b,a)}\int_0^1\frac{\partial^2 f(a+t\eta_1(b,a),c+s\eta_2(d,c),e+h\eta_3(f,e))}{\partial s\partial h}dt\right]ds.$$

Similarly, integrating by parts with respect to s over the interval $[0,1]$, we have

$$\begin{aligned}
I_2 &= \frac{1}{\eta_1(b,a)\eta_2(d,c)} \\
&\Big[\frac{\partial f(a+\eta_1(b,a),c+\eta_2(d,c),e+h\eta_3(f,e))}{\partial h} \\
&+\frac{\partial f(a,c+\eta_2(d,c),e+h\eta_3(f,e))}{\partial h} \\
&+\frac{\partial f(a+\eta_1(b,a),c,e+h\eta_3(f,e))}{\partial h} \\
&+\frac{\partial f(a,c,e+h\eta_3(f,e))}{\partial h}\Big] \\
&-\frac{2}{\eta_1(b,a)\eta_2(d,c)} \\
&\Big[\int_0^1\frac{\partial f(a+\eta_1(b,a),c+s\eta_2(d,c),e+h\eta_3(f,e))}{\partial h} \\
&+\frac{\partial f(a,c+s\eta_2(d,c),e+h\eta_3(f,e))}{\partial h}ds \\
&+\int_0^1\frac{\partial f(a+t\eta_1(b,a),c+\eta_2(d,c),e+h\eta_3(f,e))}{\partial h} \\
&+\frac{\partial f(a+t\eta_1(b,a),c,e+h\eta_3(f,e))}{\partial h}dt\Big] \\
&+\frac{4}{\eta_1(b,a)\eta_2(d,c)} \\
&\int_0^1\int_0^1\frac{\partial f(a+t\eta_1(b,a),c+s\eta_2(d,c),e+h\eta_3(f,e))}{\partial h}dsdt.
\end{aligned}$$

Finally, taking $I_3=\int_0^1(1-2h)I_2dh$. So,

$$\begin{aligned}
I_3 &= \frac{1}{\eta_1(b,a)\eta_2(d,c)}\int_0^1(1-2h) \\
&\Big[\frac{\partial f(a+\eta_1(b,a),c+\eta_2(d,c),e+h\eta_3(f,e))}{\partial h} \\
&+\frac{\partial f(a,c+\eta_2(d,c),e+h\eta_3(f,e))}{\partial h} \\
&+\frac{\partial f(a+\eta_1(b,a),c,e+h\eta_3(f,e))+\partial f(a,c,e+h\eta_3(f,e))}{\partial h}\Big] \\
&dh-\frac{2}{\eta_1(b,a)\eta_2(d,c)}\int_0^1(1-2h) \\
&\Big[\int_0^1\frac{\partial f(a+\eta_1(b,a),c+s\eta_2(d,c),e+h\eta_3(f,e))}{\partial h} \\
&+\frac{\partial f(a,c+s\eta_2(d,c),e+h\eta_3(f,e))}{\partial h}ds \\
&+\int_0^1\frac{\partial f(a+t\eta_1(b,a),c+\eta_2(d,c),e+h\eta_3(f,e))}{\partial h} \\
&+\frac{\partial f(a+t\eta_1(b,a),c,e+h\eta_3(f,e))}{\partial h}dt\Big]dh \\
&+\frac{4}{\eta_1(b,a)\eta_2(d,c)}\int_0^1\int_0^1\int_0^1(1-2h) \\
&\frac{\partial f(a+t\eta_1(b,a),c+s\eta_2(d,c),e+h\eta_3(f,e))}{\partial h}dhdsdt.
\end{aligned}$$

Again by integration by parts with respect to h over the interval $[0,1]$, we obtain

$$\begin{aligned}
I_3 &= \frac{-1}{\eta_1(b,a)\eta_2(d,c)\eta_3(f,e)} \\
&\Big[f(a+\eta_1(b,a),c+\eta_2(d,c),e+\eta_3(f,e)) \\
&+f(a,c+\eta_2(d,c),e+\eta_3(f,e))+f(a+\eta_1(b,a),c,e+\eta_3(f,e)) \\
&+f(a,c,e+\eta_3(f,e))+f(a+\eta_1(b,a),c+\eta_2(d,c),e) \\
&+f(a,c+\eta_2(d,c),e)+f(a+\eta_1(b,a),c,e)+f(a,c,e)\Big] \\
&+\frac{2}{\eta_1(b,a)\eta_2(d,c)\eta_3(f,e)} \\
&\Big[\int_0^1\Big(f(a+\eta_1(b,a),c+\eta_2(d,c),e+h\eta_3(f,e)) \\
&+f(a,c+\eta_2(d,c),e+h\eta_3(f,e)) \\
&+f(a+\eta_1(b,a),c,e+h\eta_3(f,e)) \\
&+f(a,c,e+h\eta_3(f,e))\Big)dh \\
&+\int_0^1\Big(f(a+\eta_1(b,a),c+s\eta_2(d,c),e+\eta_3(f,e)) \\
&+f(a,c+s\eta_2(d,c),e+\eta_3(f,e)) \\
&+f(a+\eta_1(b,a),c+s\eta_2(d,c),e) \\
&+f(a,c+s\eta_2(d,c),e)\Big)ds \\
&+\int_0^1\Big(f(a+t\eta_1(b,a),c+\eta_2(d,c),e+\eta_3(f,e)) \\
&+f(a+t\eta_1(b,a),c,e+\eta_3(f,e)) \\
&+f(a+t\eta_1(b,a),c+\eta_2(d,c),e)+f(a+t\eta_1(b,a),c,e)\Big)dt\Big] \\
&\frac{-4}{\eta_1(b,a)\eta_2(d,c)\eta_3(f,e)} \\
&\Big[\int_0^1\int_0^1\Big(f(a+\eta_1(b,a),c+s\eta_2(d,c),e+h\eta_3(f,e)) \\
&+f(a,c+s\eta_2(d,c),e+h\eta_3(f,e))\Big)dsdh \\
&+\int_0^1\int_0^1\Big(f(a+t\eta_1(b,a),c+\eta_2(d,c),e+h\eta_3(f,e)) \\
&+f(a+t\eta_1(b,a),c,e+h\eta_3(f,e))\Big)dtdh \\
&+\int_0^1\int_0^1\Big(f(a+t\eta_1(b,a),c+s\eta_2(d,c),e+\eta_3(f,e)) \\
&+f(a+t\eta_1(b,a),c+s\eta_2(d,c),e)\Big)dtds\Big] \\
&+\frac{8}{\eta_1(b,a)\eta_2(d,c)\eta_3(f,e)} \\
&\int_0^1\int_0^1\int_0^1 f(a+t\eta_1(b,a),c+s\eta_2(d,c),e+h\eta_3(f,e))dtdsdh.
\end{aligned}$$

Multiplying both sides of the inequality above by $\frac{\eta_1(b,a)\eta_2(d,c)\eta_3(f,e)}{8}$ and utilizing the change of variables $x=a+t\eta_1(b,a)$, $y=c+s\eta_2(d,c)$ and $z=e+h\eta_3(f,e))$, we get the desired result. ■

Theorem 1. *Let $\Gamma\subseteq\mathbb{R}^3$ be an open invex set with respect to the mappings $\eta_1(.,.)$, $\eta_2(.,.)$ and $\eta_3(.,.)$ and let $f:\Gamma\to\mathbb{R}$ be a mapping having third partial derivatives and $\frac{\partial^3 f}{\partial t\partial s\partial h}\in L\Big([a,a+t\eta_1(b,a)]\times[c,c+s\eta_2(d,c)]\times[e,e+h\eta_3(f,e)]\Big)$ with $\eta_1(b,a)>0$, $\eta_2(d,c)>0$ and $\eta_3(f,e)>0$, where $a,b\in K_1$, $c,d\in K_2$ and $e,f\in K_3$. If $\left|\frac{\partial^3 f}{\partial t\partial s\partial h}\right|$ is a preinvex function on the co-ordinates on Γ, then one has the following inequality:*

$$\Big|\frac{1}{\eta_1(b,a)\eta_2(d,c)\eta_3(f,e)}\int_a^{a+\eta_1(b,a)}\int_c^{c+\eta_2(d,c)}\int_e^{e+\eta_3(f,e)} f(x,y,z)dzdydx - A + B - C\Big|$$
$$\leq \frac{\eta_1(b,a)\eta_2(d,c)\eta_3(f,e)}{64}\times\Big\{\Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(a,c,e)\Big| + \Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(a,d,e)\Big| + \Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(b,c,e)\Big| + \Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(b,d,e)\Big| + \Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(a,c,f)\Big| + \Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(a,d,f)\Big| + \Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(b,c,f)\Big| + \Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(b,d,f)\Big|\Big\},$$

where A, B and C are defined in Lemma 2.

Proof: Using Lemma 2, it follows that

$$\Big|\frac{1}{\eta_1(b,a)\eta_2(d,c)\eta_3(f,e)}\int_a^{a+\eta_1(b,a)}\int_c^{c+\eta_2(d,c)}\int_e^{e+\eta_3(f,e)} f(x,y,z)dzdydx - A + B - C\Big| \leq \frac{\eta_1(b,a)\eta_2(d,c)\eta_3(f,e)}{8}\int_0^1\int_0^1\int_0^1 |(1-2t)(1-2s)(1-2h)| \Big|\frac{\partial^3 f(a+t\eta_1(b,a), c+s\eta_2(d,c), e+h\eta_3(f,e))}{\partial t\partial s\partial h}\Big| dtdsdh. \tag{4}$$

Putting

$$J_1 = \int_0^1\int_0^1\int_0^1 |(1-2t)(1-2s)(1-2h)| \times \Big|\frac{\partial^3 f(a+t\eta_1(b,a), c+s\eta_2(d,c), e+h\eta_3(f,e))}{\partial t\partial s\partial h}\Big| dtdsdh.$$

Using the preinvexity property of $\Big|\frac{\partial^3 f}{\partial t\partial s\partial h}\Big|$ on the co-ordinates on Γ and utilizing the following facts:

$$\int_0^1 (1-t)|1-2t|dt = \int_0^{\frac{1}{2}}(1-t)(1-2t)dt - \int_{\frac{1}{2}}^1 (1-t)(1-2t)dt = \frac{1}{4}$$

and

$$\int_0^1 t|1-2t|dt = \int_0^{\frac{1}{2}} t(1-2t)dt - \int_{\frac{1}{2}}^1 t(1-2t)dt = \frac{1}{4},$$

it follows that

$$J_1 \leq \int_0^1\int_0^1 |(1-2s)(1-2h)| \times \Big\{\int_0^1 (1-t)|1-2t|\Big|\frac{\partial^3 f(a, c+s\eta_2(d,c), e+h\eta_3(f,e))}{\partial t\partial s\partial h}\Big|dt + \int_0^1 t|1-2t|\Big|\frac{\partial^3 f(b, c+s\eta_2(d,c), e+h\eta_3(f,e))}{\partial t\partial s\partial h}\Big|dt\Big\}dsdh$$
$$= \frac{1}{4}\int_0^1\int_0^1 |(1-2s)(1-2h)| \times \Big\{\Big|\frac{\partial^3 f(a, c+s\eta_2(d,c), e+h\eta_3(f,e))}{\partial t\partial s\partial h}\Big| + \Big|\frac{\partial^3 f(b, c+s\eta_2(d,c), e+h\eta_3(f,e))}{\partial t\partial s\partial h}\Big|\Big\}dsdh$$
$$\leq \frac{1}{4}\int_0^1 |(1-2h)| \times \Big\{\int_0^1 (1-s)|1-2s|\Big(\Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(a,c,e+h\eta_3(f,e))\Big| + \Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(b,c,e+h\eta_3(f,e))\Big|\Big)ds + \int_0^1 s|1-2s|\Big(\Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(a,d,e+h\eta_3(f,e))\Big| + \Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(b,d,e+h\eta_3(f,e))\Big|\Big)ds\Big\}dh$$
$$= \frac{1}{16}\int_0^1 |(1-2h)| \times \Big\{\Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(a,c,e+h\eta_3(f,e))\Big| + \Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(a,d,e+h\eta_3(f,e))\Big|] + \Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(b,c,e+h\eta_3(f,e))\Big| + \Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(b,d,e+h\eta_3(f,e))\Big|\Big\}dh$$
$$\leq \frac{1}{6}\int_0^1 |(1-2h)| \times \{(1-h)\Big[\Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(a,c,e)\Big| + \Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(a,d,e)\Big| + \Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(b,c,e)\Big| + \Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(b,d,e)\Big|\Big] + h\times\Big[\Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(a,c,f)\Big| + \Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(a,d,f)\Big| + \Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(b,c,f)\Big| + \Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(b,d,f)\Big|\Big]\Big\}dh$$
$$= \frac{1}{64}\times\Big\{\Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(a,c,e)\Big| + \Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(a,d,e)\Big| + \Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(b,c,e)\Big| + \Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(b,d,e)\Big| + \Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(a,c,f)\Big| + \Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(a,d,f)\Big| + \Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(b,c,f)\Big| + \Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(b,d,f)\Big|\Big\}.$$

Inserting J_1 in (4), we obtain the desired result. This completes the proof. ■

Theorem 2. *Let Γ be an open invex set with respect to the mappings $\eta_1(.,.)$, $\eta_2(.,.)$ and $\eta_3(.,.)$ and let $f:\Gamma\to\mathbb{R}$ be a mapping having third partial derivatives and $\frac{\partial^3 f}{\partial t\partial s\partial h}\in L\Big([a,a+t\eta_1(b,a)]\times[c,c+s\eta_2(d,c)]\times[e,e+h\eta_3(f,e)]\Big)$ with $\eta_1(b,a)>0$, $\eta_2(d,c)>0$ and $\eta_3(f,e)>0$, where $a,b\in K_1$, $c,d\in K_2$ and $e,f\in K_3$. If $\Big|\frac{\partial^3 f}{\partial t\partial s\partial h}\Big|^q$ is a preinvex function on the co-ordinates on $\Gamma, q>1$, then one has the following inequality:*

$$\Big|\frac{1}{\eta_1(b,a)\eta_2(d,c)\eta_3(f,e)}\int_a^{a+\eta_1(b,a)}\int_c^{c+\eta_2(d,c)}\int_e^{e+\eta_3(f,e)} f(x,y,z)dzdydx - A + B - C\Big|$$
$$\leq \frac{\eta_1(b,a)\eta_2(d,c)\eta_3(f,e)}{8}\times\Big(\frac{1}{(1+p)^3}\Big)^{\frac{1}{p}}$$
$$\times\Big\{\frac{1}{8}\Big(\Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(a,c,e)\Big|^q + \Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(a,c,f)\Big|^q + \Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(a,d,e)\Big|^q + \Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(a,d,f)\Big|^q + \Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(b,c,e)\Big|^q + \Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(b,c,f)\Big|^q + \Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(b,d,e)\Big|^q + \Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(b,d,f)\Big|^q\Big)\Big\}^{\frac{1}{q}}$$

where $\frac{1}{p}+\frac{1}{q}=1$ and A, B and C are defined in Lemma 6.

Proof: The well-known Holder integral inequality along with Lemma 6, imply that

$$\Big|\frac{1}{\eta_1(b,a)\eta_2(d,c)\eta_3(f,e)}\int_a^{a+\eta_1(b,a)}\int_c^{c+\eta_2(d,c)}\int_e^{e+\eta_3(f,e)} f(x,y,z)dzdydx - A + B - C\Big| \tag{5}$$
$$\leq \frac{\eta_1(b,a)\eta_2(d,c)\eta_3(f,e)}{8}\int_0^1\int_0^1\int_0^1 |(1-2t)(1-2s)(1-2h)| \Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(a+t\eta_1(b,a), c+s\eta_2(d,c), e+h\eta_3(f,e))\Big| dtdsdh$$
$$\leq \frac{\eta_1(b,a)\eta_2(d,c)\eta_3(f,e)}{8}\Big(\int_0^1\int_0^1\int_0^1 |(1-2t)(1-2s)(1-2h)|^p dtdsdh\Big)^{\frac{1}{p}} \times\Big(\int_0^1\int_0^1\int_0^1 \Big|\frac{\partial^3 f(a+t\eta_1(b,a), c+s\eta_2(d,c), e+h\eta_3(f,e))}{\partial t\partial s\partial h}\Big|^q dtdsdh^{\frac{1}{q}}.$$

Putting

$$J_1 = \int_0^1\int_0^1\int_0^1 \Big|\frac{\partial^3 f(a+t\eta_1(b,a), c+s\eta_2(d,c), e+h\eta_3(f,e))}{\partial t\partial s\partial h}\Big|^q dtdsdh.$$

Using the preinvexity property of $\Big|\frac{\partial^3 f}{\partial t\partial s\partial h}\Big|^q$, for $q>1$, on the co-ordinates Γ, we get

$$J_1 \leq \int_0^1\int_0^1\int_0^1 \Big\{(1-t)\Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(a, c+s\eta_2(d,c), e+h\eta_3(f,e))\Big|^q + t\Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(b, c+s\eta_2(d,c), e+h\eta_3(f,e))\Big|^q\Big\}dtdsdh$$
$$\leq \int_0^1\int_0^1\int_0^1 \Big\{(1-t)(1-s)\Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(a,c,e+h\eta_3(f,e))\Big|^q + (1-t)s\Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(a,d,e+h\eta_3(f,e))\Big|^q + t(1-s)\Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(b,c,e+h\eta_3(f,e))\Big|^q\} + ts\Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(b,d,e+h\eta_3(f,e))\Big|^q dt\}\Big\}dtdsdh$$
$$\leq \int_0^1\int_0^1\int_0^1 \Big\{(1-t)(1-s)(1-h)\Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(a,c,e)\Big|^q + (1-t)(1-s)h\Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(a,c,f)\Big|^q + (1-t)s(1-h)\Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(a,d,e)\Big|^q + (1-t)sh\Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(a,d,f)\Big|^q + t(1-s)(1-h)\Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(b,c,e)\Big|^q\} + t(1-s)h\Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(b,c,f)\Big|^q\} + ts(1-h)\Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(b,d,e)\Big|^q\} + tsh\Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(b,d,f)\Big|^q\}\Big\}dtdsdh$$
$$= \frac{1}{8}\Big(\Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(a,c,e)\Big|^q + \Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(a,c,f)\Big|^q + \Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(a,d,e)\Big|^q + \Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(a,d,f)\Big|^q + \Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(b,c,e)\Big|^q + \Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(b,c,f)\Big|^q + \Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(b,d,e)\Big|^q + \Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(b,d,f)\Big|^q\Big).$$

That is,

$$J_1 \leq \frac{1}{8}\Big(\Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(a,c,e)\Big|^q + \Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(a,c,f)\Big|^q + \Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(a,d,e)\Big|^q + \Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(a,d,f)\Big|^q + \Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(b,c,e)\Big|^q + \Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(b,c,f)\Big|^q \Big|\frac{\partial^3 f}{\quad}\Big|^q + \Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(b,d,f)\Big|^q\Big).$$

On the other hand, we have

$$\int_0^1\int_0^1\int_0^1 |1-2t|^p|1-2s|^p|1-2h|^p dtdsdh = \frac{1}{(p+1)^3}. \tag{6}$$

Inserting J_1 and (6) in (5), we obtain the desired result, as claimed. This completes the proof. ■

Theorem 3. *Let Γ be an open invex set with respect to the mappings $\eta_1(.,.)$, $\eta_2(.,.)$ and $\eta_3(.,.)$ and let $f:\Gamma\to\mathbb{R}$ be a mapping having third partial derivatives and $\frac{\partial^3 f}{\partial t\partial s\partial h}\in L\Big([a,a+t\eta_1(b,a)]\times[c,c+s\eta_2(d,c)]\times[e,e+h\eta_3(f,e)]\Big)$ with $\eta_1(b,a)>0$, $\eta_2(d,c)>0$ and $\eta_3(f,e)>0$, where $a,b\in K_1$, $c,d\in K_2$ and $e,f\in K_3$. If $\Big|\frac{\partial^3 f}{\partial t\partial s\partial h}\Big|$ is a preinvex function* *on the co-ordinates on* Γ *for some fixed* $q \geq 1$, *then we get the following inequality*

$$\Big|\frac{1}{\eta_1(b,a)\eta_2(d,c)\eta_3(f,e)}\int_a^{a+\eta_1(b,a)}\int_c^{c+\eta_2(d,c)}\int_e^{e+\eta_3(f,e)} f(x,y,z)dzdydx - A + B - C\Big|$$
$$\leq \frac{\eta_1(b,a)\eta_2(d,c)\eta_3(f,e)}{8}\times\left(\frac{1}{8}\right)^{1-\frac{1}{q}}$$
$$\times\Big(\frac{1}{64}\Big\{\Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(a,c,e)\Big|^q + \Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(a,c,f)\Big|^q + \Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(a,d,e)\Big|^q + \Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(a,d,f)\Big|^q$$
$$+ \Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(b,c,e)\Big|^q + \Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(b,c,f)\Big|^q + \Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(b,d,e)\Big|^q + \Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(b,d,f)\Big|^q\Big\}\Big)^{\frac{1}{q}}$$

where A, B *and* C *are defined by Lemma* 2.

Proof: The well-known power-mean integral inequality for triple integrals along with Lemma 2, conclude that

$$\Big|\frac{1}{\eta_1(b,a)\eta_2(d,c)\eta_3(f,e)}\int_a^{a+\eta_1(b,a)}\int_c^{c+\eta_2(d,c)}\int_e^{e+\eta_3(f,e)} f(x,y,z)dzdydx - A + B - C\Big| \tag{7}$$
$$\leq \frac{\eta_1(b,a)\eta_2(d,c)\eta_3(f,e)}{8}\int_0^1\int_0^1\int_0^1 |(1-2t)(1-2s)(1-2h)| \Big|\frac{\partial^3 f(a+t\eta_1(b,a), c+s\eta_2(d,c), e+h\eta_3(f,e))}{\partial t\partial s\partial h}\Big| dtdsdh$$
$$\leq \frac{\eta_1(b,a)\eta_2(d,c)\eta_3(f,e)}{8}\Big(\int_0^1\int_0^1\int_0^1 |(1-2t)(1-2s)(1-2h)|dtdsdh\Big)^{1-\frac{1}{q}}$$
$$\times\Big(\int_0^1\int_0^1\int_0^1 |(1-2t)(1-2s)(1-2h)| \Big|\frac{\partial^3 f(a+t\eta_1(b,a), c+s\eta_2(d,c), e+h\eta_3(f,e))}{\partial t\partial s\partial h}\Big|^q dtdsdh\Big)^{\frac{1}{q}}.$$

Taking

$$J_1 = \int_0^1\int_0^1\int_0^1 |(1-2t)(1-2s)(1-2h)| \Big|\frac{\partial^3 f(a+t\eta_1(b,a), c+s\eta_2(d,c), e+h\eta_3(f,e))}{\partial t\partial s\partial h}\Big|^q dtdsdh.$$

By the preinvexity property of $\Big|\frac{\partial^3 f}{\partial t\partial s\partial h}\Big|^q$ on the co-ordinates on Γ, it is noted

$$J_1 \leq \int_0^1\int_0^1\int_0^1 |(1-2t)(1-2s)(1-2h)|$$
$$\times\Big\{(1-t)\Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(a, c+s\eta_2(d,c), e+h\eta_3(f,e))\Big|^q + t\Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(b, c+s\eta_2(d,c), e+h\eta_3(f,e))\Big|^q\Big\}dtdsdh$$
$$\leq \int_0^1\int_0^1\int_0^1 |(1-2t)(1-2s)(1-2h)|$$
$$\times\Big\{(1-t)(1-s)\Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(a,c,e+h\eta_3(f,e))\Big|^q + (1-t)s\Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(a,d,e+h\eta_3(f,e))\Big|^q$$
$$+ t(1-s)\Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(b,c,e+h\eta_3(f,e))\Big|^q + ts\Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(b,d,e+h\eta_3(f,e))\Big|^q\Big\}dtdsdh$$
$$\leq \int_0^1\int_0^1\int_0^1 |(1-2t)(1-2s)(1-2h)|$$
$$\times\Big\{(1-t)(1-s)(1-h)\Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(a,c,e)\Big|^q + (1-t)(1-s)h\Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(a,c,f)\Big|^q$$
$$+ (1-t)s(1-h)\Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(a,d,e)\Big|^q + (1-t)sh\Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(a,d,f)\Big|^q$$
$$+ t(1-s)(1-h)\Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(b,c,e)\Big|^q + t(1-s)h\Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(b,c,f)\Big|^q$$
$$+ ts(1-h)\Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(b,d,e)\Big|^q + tsh\Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(b,d,f)\Big|^q\Big\}dtdsdh$$
$$= \frac{1}{64}\Big(\Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(a,c,e)\Big|^q + \Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(a,c,f)\Big|^q + \Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(a,d,e)\Big|^q$$
$$+ \Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(a,d,f)\Big|^q + \Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(b,c,e)\Big|^q + \Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(b,c,f)\Big|^q$$
$$+ \Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(b,d,e)\Big|^q + \Big|\frac{\partial^3 f}{\partial t\partial s\partial h}(b,d,f)\Big|^q\Big).$$

The last equality follows using the following fact:

$$\int_0^1\int_0^1\int_0^1 |(1-2t)(1-2s)(1-2h)|dtdsdh = \frac{1}{8}.$$

Writing J_1 in (7), we get the desired result. Thus, the proof is completed. ■

Remark 1. For $q = 1$, Theorem 3 reduce to Theorem 1. Thus, Theorem 3 is a generalization of Theorem 1.

Remark 2. Since $\frac{1}{8} < \frac{1}{(p+1)^{\frac{3}{p}}}$, therefore for $p > 1$, the obtained estimation in Theorem 3 is better than the derived estimation in Theorem 2.

Remark 3. In the obtained results, if we put $\eta_1(b,a) = b-a$, $\eta_2(d,c) = d-c$ and $\eta_3(f,e) = f-e$, then we obtain those results proved in [3]. This shows that the results of this paper are more general than those presented in [3].

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