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Localising Gauss's Law and the Electric Charge Induction on a Conducting Sphere

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***Abstract*—**Space debris has numerous manifestations including ferro-metalize and non-ferrous. The electric field will induce negative charges to split from positive charges inside the space debris. In this research, we focus only on conducting materials. The assumption is that the electric charge density of a conducting surface is proportional to the electric field on that surface due to Gauss's law. We are trying to find the induced charge density from an external electric field perpendicular to a conducting spherical surface. An object is a sphere on which the external electric field is not uniform. The electric field is, therefore, considered locally. The localised spherical surface is a tangent plane so the Gaussian surface is a very small cylinder and every point on a spherical surface has its own cylinder. The electric field from a circular electrode has been calculated in near-field and far-field approximation and shown Explanation Touchless manoeuvring space debris orbit properties. The electric charge density calculation from a near-field and far-field approximation is done.

***Keywords*—**Near-field approximation, far-field approximation, localized Gauss's law, electric charge density.

I. Introduction

When a conducting material is located in a region filled with an external electric field, the negative and positive charges are induced to separate from each other because the negative charges move to the opposite direction of an external electric field while the positive ones move in the same direction as the electric field [1], [2], [5], [6]. An electric field isometrically radiated from one electric charge can be easily computed from Gauss's law. If there are many electric charges, the net electric field will be a superposition of electric fields from those electric charges since an electric field is a vector. The separation of both positive and negative charges results in the zero electric field inside due to the superposition of the external and internal electric field.

A theorem that explains the relation between electric charge distribution and an electric field is the Gauss flux theorem. Because the electric field is released from an electric charge in every direction equally, an electric field can be measured on an element of a surface, and its product is called an electric flux [6]. Gauss's theorem explained that the total electric flux around a closed surface called a Gaussian surface is directly proportional to the total electric charge contained in a Gaussian surface.

The Gaussian surfaces can be chosen such that it is easiest to find the total electric flux. Gaussian surfaces usually have a high symmetry, for example, a sphere, a cylinder, and a cube. The induced charge density on a conducting surface depends on the external electric field. The induced electric charge orients in such a way that the net electric field inside a conducting material is zero and it is staying only on a surface. Due to a non-uniform electric field, charge density is necessarily computed point by point. When a point on a spherical surface is closely zoomed in, the neighbourhood of that point is approximately flat. The Gaussian surface containing that point is a cylinder in which only an external electric field perpendicular to the conducting surface is used to find the charge density. This idea comes from an electric field from an infinite electrode plate. The electric field is written as [1], [2]:

$$E_n\hat{n} = \frac{\sigma}{\epsilon_0}\hat{n} \tag{1}$$

When n is a unit vector perpendicular to the electrode and is a surface charge density. The total electric field on a conducting surface can be computed from an image charge. However, we approximate that the image charge is located on a solid line from the centre of a circular electrode. However, the crucial condition for this approximation is that the conducting sphere has to be near the central axis of an electrode or be very far away from an electrode. Otherwise, this approximation cannot be applied.

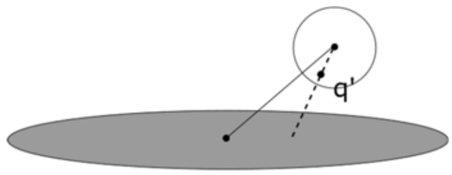

Fig. 1 An image charge is on a dashed line instead of a solid line

First part will be the introduction to the idea of localized Gauss's law which will be applied from the usual Gauss's law because the surface is not flat. Then, we show the form of the induced electric charge density. Next part will be the detail of the calculation process by using an image charge method. We will find the electric charge density induced from an external electric field generated from a circular electrode. We used a technique to compute the electric charge density on the conducting sphere under some restricted conditions. It begins with the electric potential and then finding the position and value of an image charge. Finally, we can calculate the electric field and electric charge density in the spherical coordinate.

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II. Localized Gauss's Law

The object is supposed to be a sphere whose surface is locally flat. We consider only the normal component of an electric field on that surface. The Gaussian surface is a cylinder covering electric charges at that point.

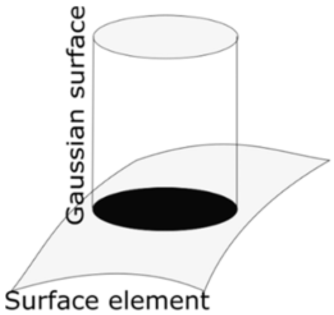

Fig. 2 The localised Gaussian surface from a surface element of a sphere covering electric charges in the black region

A. Normal Component of an Electric Field

Let E be the external electric field which can be in any direction. Its normal component is written as

$$E_n = \vec{E} \cdot \hat{n} \tag{2}$$

The unit vector n is perpendicular to the surface of a sphere whose direction varies on each location on a sphere. Let a point O be the centre of the sphere and an external electric field is projected to the direction of n.

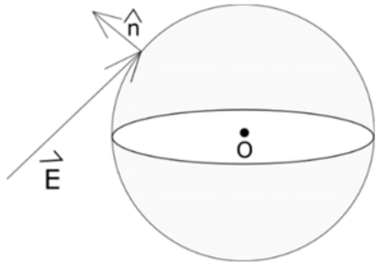

Fig. 3 The external electric field towards at a point with a unit vector n on a sphere O

The external electric field in a three-dimensional space generalised coordinate is written as [3], [4]:

$$\vec{E} = \sum_{i=1}^{3} E_i \hat{e}_i \tag{3}$$

When $\hat{e}_i$ is a unit vector of a generalised coordinate and E_i is the i component of an electric field. The normal component is written as

$$E_n = \vec{E} \cdot \hat{n} = \sum_{i=1}^{3} E_i \hat{e}_i \cdot \hat{n} \tag{4}$$

B. Electric Charge Density

According to the Gauss's law, the electric charge density on an element of a surface is obtained [1], [2]:

$$\sigma' = \epsilon_0 E_n \tag{5}$$

The closed surface is a cylinder above the surface because the electric field inside a conductor is completely zero. However, if the object is made of the other type of materials, we will have to compute the electric flux inside the object.

III. Image Charge Method

We suppose that the electric charge density on a circular electrode is σ. ρ, ρ_q is the radial component of an observed position and an image charge position respectively. z, z_q is the vertical component of an observed position and an image charge position respectively.

A. Electric Potential

The total electric potential is zero at the surface of the conducting sphere, the general term of total electric potential is obtained [1], [2].

$$V_{total} = \int \frac{k\sigma dA'}{\sqrt{\rho^2+{\rho'}^2-2\rho\rho' cos\alpha' + z^2}} + \frac{kq'}{r'} \tag{6}$$

When $r' = \sqrt{\rho^2 + \rho_q^2 - 2\rho\rho_q cos\theta_{cm} + \left(z - z_q\right)^2}$ is the relative distance from an image charge to an observed position, θ_{cm} is the angle between the radial component of the center of mass and the radial component of an arbitrary observed position ρ. The multipole expansion of the near-field and far-field electric potential approximation up to the second order is

$$V_{near} = \mathrm{k}\sigma\left[2\pi\left(\sqrt{z^2+a^2} - \mathrm{z}\right) - \frac{\pi\rho a^2}{2(\mathrm{a}^2+\mathrm{z}^2)^{\frac{3}{2}}}\right] + \frac{\mathrm{kq}'}{\mathrm{r}'} \tag{7}$$

$$V_{far} = \frac{\mathrm{k}\sigma\pi a^2}{\sqrt{\rho^2+z^2}} + \frac{\mathrm{kq}'}{\mathrm{r}'} \tag{8}$$

We need the position of an image charge and the value of an electric charge q'. This part is a little bit tricky to calculate both physical quantities.

B. The Position of an Image Charge

The position of an image charge is supposed to be on the line between the centre of the conducting sphere and the centre of a circular electrode. It will be a good fit if the conducting sphere is near the central axis of the circular electrode as shown in Fig. 4.

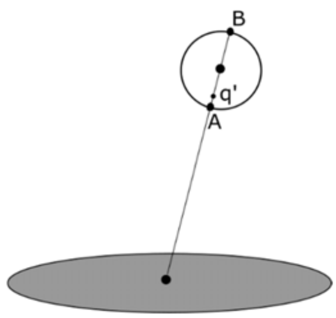

Fig. 4 An Image Charge q' on a Line between the Centre of the Conducting Sphere and the Centre of a Circular Electrode

The position labelled by A and B can be written in terms of centre of mass and the radius of the conducting sphere which is written in (9)-(12):

$$z_A = z_{cm} - r \times \frac{z_{cm}}{\sqrt{z_{cm}^2+\rho_{cm}^2}} \tag{9}$$

$$z_B = z_{cm} + r \times \frac{z_{cm}}{\sqrt{z_{cm}^2+\rho_{cm}^2}} \tag{10}$$

$$\rho_A = \rho_{cm} - r \times \frac{\rho_{cm}}{\sqrt{z_{cm}^2+\rho_{cm}^2}} \tag{11}$$

$$\rho_B = \rho_{cm} + r \times \frac{\rho_{cm}}{\sqrt{z_{cm}^2+\rho_{cm}^2}} \tag{12}$$

We use the Dirichlet condition to be a boundary condition of a conducting sphere. The electric potential of the surface of the conducting sphere is only zero. The two points of an intersection A and B between the diameter of the conducting sphere and the surface are applied to find the location of an image charge. For near-field approximation, the equations are

$$0 = \mathrm{k}\sigma\left[2\pi\left(\sqrt{z_A^2+a^2} - \mathrm{z}_A\right) - \frac{\pi\rho_A a^2}{2(\mathrm{a}^2+\mathrm{z}_A^2)^{\frac{3}{2}}}\right] + \frac{\mathrm{kq}'}{\mathrm{r}'} \tag{13}$$

$$0 = \mathrm{k}\sigma\left[2\pi\left(\sqrt{z_B^2+a^2} - \mathrm{z}_B\right) - \frac{\pi\rho_B a^2}{2(\mathrm{a}^2+\mathrm{z}_B^2)^{\frac{3}{2}}}\right] + \frac{\mathrm{kq}'}{\mathrm{r}'} \tag{14}$$

From (13) and (14), we can find an electric charge q'. The value of an image of an electric charge is defined as

$$q' = \frac{(C_1+C_2)\sqrt{(\rho_A-\rho_q)^2+(z_A-z_q)^2}\,(2r-\sqrt{(\rho_A-\rho_q)^2+(z_A-z_q)^2})}{2\left(r-\sqrt{(\rho_A-\rho_q)^2+(z_A-z_q)^2}\right)} \tag{15}$$

From this form, q' is a negative charge as it should be. The position z_q and ρ_q are able to write in terms of the center of mass since all points are on the same line passing through the center of mass. They can be written as

$$\rho_q = \frac{\rho_{cm}}{z_{cm}} \times z_q \tag{16}$$

$$z_q = \frac{\frac{\rho_A\rho_{cm}}{z_{cm}} + z_A + \sqrt{\left(\frac{\rho_A\rho_{cm}}{z_{cm}}+z_A\right)^2 - \left(\frac{\rho_{cm}^2}{z_{cm}^2}+1\right)\left(\rho_A^2+z_A^2-\frac{4r^2(2C_1+C_2)^2}{(3C_1+C_2)^2}\right)}}{\frac{\rho_{cm}^2}{z_{cm}^2}+1} \tag{17}$$

When the constant C_1 and C_2 are defined as in (18) and (19), respectively

$$C_1 = -\sigma\left[2\pi\left(\sqrt{z_A^2+a^2} - z_A\right) - \frac{\pi\rho_A^2 a^2}{2(z_A^2+a^2)^{\frac{3}{2}}}\right] \tag{18}$$

$$C_2 = -\sigma\left[2\pi\left(\sqrt{z_B^2+a^2} - z_B\right) - \frac{\pi\rho_B^2 a^2}{2(z_B^2+a^2)^{\frac{3}{2}}}\right] \tag{19}$$

and r is a radius of the conducting sphere. For far-field approximation, the equations are

$$0 = \frac{\mathrm{k}\sigma\pi a^2}{\sqrt{\rho_A^2+z_A^2}} + \frac{\mathrm{kq}'}{\mathrm{r}'} \tag{20}$$

$$0 = \frac{\mathrm{k}\sigma\pi a^2}{\sqrt{\rho_B^2+z_B^2}} + \frac{\mathrm{kq}'}{\mathrm{r}'} \tag{21}$$

Solving the equations, we find that the position and the value of an image charge are written in the form as

$$\rho_q = \frac{\rho_{cm}}{z_{cm}} \times z_q \tag{22}$$

$$z_q = \frac{\frac{\rho_A\rho_{cm}}{z_{cm}} + z_A + \sqrt{\left(\frac{\rho_A\rho_{cm}}{z_{cm}}+z_A\right)^2 - \left(\frac{\rho_{cm}^2}{z_{cm}^2}+1\right)\left(\rho_A^2+z_A^2-\frac{4r^2(2C_1+C_2)^2}{(3C_1+C_2)^2}\right)}}{\frac{\rho_{cm}^2}{z_{cm}^2}+1} \tag{23}$$

$$q' = \left(\frac{\sigma\pi a^2}{\sqrt{\rho_B^2+z_B^2}} - \frac{\sigma\pi a^2}{\sqrt{\rho_A^2+z_A^2}}\right)\frac{(2r-\sqrt{(\rho_A-\rho_q)^2+(z_A-z_q)^2})}{2\left(r-\sqrt{(\rho_A-\rho_q)^2+(z_A-z_q)^2}\right)} \tag{24}$$

C. Electric Field

The electric field can be calculated in near-field and far-field approximation defined from the negative of the gradient of the total electric potential.

$$\vec{E} = -\vec{\nabla}V \tag{25}$$

The electric field in the cylindrical coordinate is written as

$$\vec{E}_{near} = k\left[\frac{\pi\sigma\rho a^2}{(a^2+z^2)^{\frac{3}{2}}} + \frac{q'(\rho-\rho_q)}{\left((\rho-\rho_q)^2+(z-z_q)^2\right)^{\frac{3}{2}}}\right]\hat{\rho} + k\left[\sigma\left(1 - \frac{z}{\sqrt{z^2+a^2}} + \frac{3\pi\rho^2a^2z}{2(a^2+z^2)^{\frac{3}{2}}}\right) + \frac{q'(z-z_q)}{\left((\rho-\rho_q)^2+(z-z_q)^2\right)^{\frac{3}{2}}}\right]\hat{k} \tag{26}$$

$$\vec{E}_{far} = k\left[\frac{\pi\sigma\rho a^2}{(a^2+z^2)^{\frac{3}{2}}} + \frac{q'(\rho-\rho_q)}{\left((\rho-\rho_q)^2+(z-z_q)^2\right)^{\frac{3}{2}}}\right]\hat{\rho} + k\left[\frac{\pi\sigma z a^2}{(a^2+z^2)^{\frac{3}{2}}} + \frac{q'(z-z_q)}{\left((\rho-\rho_q)^2+(z-z_q)^2\right)^{\frac{3}{2}}}\right]\hat{k} \tag{27}$$

When near-field means when the object is moving in the region of the circular electrode. Otherwise, it will be a far-field approximation. The approximation is performed only up to the second order. Now we have the general form of the electric field for near-field and far-field approximation. The electric charge density on the surface of the conducting sphere is obtained.

$$\sigma'_{near} = \epsilon_0 k\left[\frac{\pi\sigma\rho a^2}{(a^2+z^2)^{\frac{3}{2}}} + \frac{q'(\rho-\rho_q)}{\left((\rho-\rho_q)^2+(z-z_q)^2\right)^{\frac{3}{2}}}\right]\hat{\rho}\cdot\hat{n} +$$

$$k\left[\sigma\left(1-\frac{z}{\sqrt{z^2+a^2}}+\frac{3\pi\rho^2a^2z}{2(a^2+z^2)^{\frac{3}{2}}}\right)+\frac{q'(z-z_q)}{\left((\rho-\rho_q)^2+(z-z_q)^2\right)^{\frac{3}{2}}}\right]\hat{k}\cdot\hat{n} \quad (28)$$

$$\sigma'_{far}=\epsilon_0 k\left[\frac{\pi\sigma\rho a^2}{(a^2+z^2)^{\frac{3}{2}}}+\frac{q'(\rho-\rho_q)}{\left((\rho-\rho_q)^2+(z-z_q)^2\right)^{\frac{3}{2}}}\right]\hat{\rho}\cdot\hat{n}+$$
$$k\left[\frac{\pi\sigma z a^2}{(a^2+z^2)^{\frac{3}{2}}}+\frac{q'(z-z_q)}{\left((\rho-\rho_q)^2+(z-z_q)^2\right)^{\frac{3}{2}}}\right]\hat{k}\cdot\hat{n} \quad (29)$$

On the conducting surface, it is more convenient for the electric charge density to be written in terms of centre of mass with the spherical coordinates.

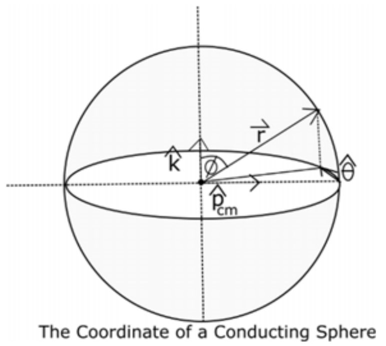

Fig. 5 The Spherical Coordinate of a Conducting Sphere O

The set of parameters in the cylindrical coordinate transforming into the spherical coordinate is written as:

$$\rho cos\theta_{cm}-\rho_{cm}=rsin\phi cos\theta \quad (30)$$

$$\rho sin\theta_{cm}=rsin\phi sin\theta \quad (31)$$

$$z-z_{cm}=rcos\phi \quad (32)$$

When θ and ϕ are azimuthal angle and polar angle of a conducting sphere respectively. According to the geometry of the conducting sphere, the dot product of unit vectors is written in (33) and (34):

$$\hat{k}\cdot\hat{n}=cos\phi \quad (33)$$

$$\hat{\rho}\cdot\hat{n}=sin\phi cos\theta \quad (34)$$

After the substitution of the parameters, we will get the electric charge density in terms of coordinates of the conducting sphere which is straightforward to calculate the electric force and predict the motion of the object under the electric force.

IV. Conclusion

The electric potential and electric field under the Dirichlet boundary condition can be written as a function of the position of the centre of mass and the conducting sphere coordinate which is much easier to compute. The value of an image charge varies on the position of a centre of mass of the conducting sphere as well as its position. With the result of electric field, it is straightforward to compute the electric charge density in the spherical coordinate.

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