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Node Insertion in Coalescence Hidden-Variable Fractal Interpolation Surface

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Abstract—The Coalescence Hidden-variable Fractal Interpolation Surface (CHFIS) was built by combining interpolation data from the Iterated Function System (IFS). The interpolation data in a CHFIS comprise a row and/or column of uncertain values when a single point is entered. Alternatively, a row and/or column of additional points are placed in the given interpolation data to demonstrate the node added CHFIS. There are three techniques for inserting new points that correspond to the row and/or column of nodes inserted, and each method is further classified into four types based on the values of the inserted nodes. As a result, numerous forms of node insertion can be found in a CHFIS.

Keywords—Fractal, interpolation, iterated function system, coalescence, node insertion, knot insertion.

I. Introduction

ONE of the most important contributions in understanding nature's structures is the concept of fractals. An increasing number of research publications have shown the fractal character of numerous systems with various physical attributes ever since Mandelbrot [1], [2] coined the term "fractal". The study of fractals soared to new heights with the invention of Barnsley's Fractal Interpolation Function (FIF) [3]. Massopust [4] extended this design to a triangular simplex surface, creating a Fractal Interpolation Surface (FIS) with co-planar interpolation points on the border. Then, Geronimo and Hardin [5], Xie, and Sun [6], Dalla [7], Malysz [8], and others created numerous FIS constructions on various sorts of domains that gave self-affine attractors.

Most naturally formed objects, such as rocks, sea surfaces, clouds, and so on, are made up of both self-affine and non-affine components. In [9], Chand and Kapoor created a non-diagonal Iterated Function System (IFS) that creates both self-affine and non-self-affine FIS simultaneously based on free and constrained variables on a large collection of interpolation data. Coalescence Hidden-variable Fractal Interpolation Surface was born from the attractor corresponding to such IFS (CHFIS). The smoothness, stability and fractal dimension of such a CHFIS was investigated in [10], [11] and [12].

We assume that interpolation data are obtained from various districts of a location and that one or more of the districts is later subdivided into smaller districts. In this scenario, we must employ node insertion to the preceding data to use it with smaller districts. Similarly, the results of some tests may provide us with a tip as to where we should place nodes while performing approximation. This leads to the investigation of the node insertion problem in bivariate functions.

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Node insertion is described as the process of adding a new point to an existing set of interpolation data. The node insertion for Coalescence Hidden-variable Fractal Interpolation Functions (CHFIFs) was explored in [13]. If we introduce a single point in the bivariate situation, there are an unknown quantities in the row and/or column. As a result, the problem of new point inclusion in the Coalescence Hidden-variable Fractal Interpolation Surface is characterized by adding a row and/or column of new points. In the bivariate example, there are now several methods for inserting additional points. The impact of such new points on the related non-diagonal IFS and CHFIS is investigated in this work. Furthermore, the problem of node insertion is categorized into four types based on the values of inserted points for each method.

The organization of the paper is as follows: Section II provides an outline of how a CHFIS is constructed. Section III discusses the three possible methods for inserting new points into interpolation data. It is demonstrated that the non-diagonal IFS produced utilizing the new set of interpolation data gives rise to a new CHFIS for each mode of node insertion. Following the study of the three techniques of insertion of new points, Section IV considers several types of insertion based on the values of added nodes for each approach. So, in the bivariate instance, there are 12 different types of node insertion. Finally, Section V provides an example of a computer model of a few different forms of insertion.

II. Construction of CHFIS

A set of real parameters $\{t_{i,j}\}$ for $i = 0, 1, \ldots, N$ and $j = 0, 1, \ldots, M$ is introduced in a given interpolation data $\Lambda = \{(x_0, y_0, z_{0,0}), (x_1, y_0, z_{1,0}), \ldots, (x_0, y_1, z_{0,1}), \ldots, (x_N, y_M, z_{N,M})\}$ to form the generalized interpolation data $\Delta = \{(x_i, y_j, z_{i,j}, t_{i,j}) : i = 0, 1, \ldots, N$ and $j = 0, 1, \ldots, M\}$. From the interpolation data, the rectangle $S = I \times J = [x_0, x_N] \times [y_0, y_M]$ is subdivided into smaller rectangles $S_{n,m} = I_n \times J_m = [x_{n-1}, x_n] \times [y_{m-1}, y_m]$ for $n = 1, 2, \ldots, N$ and $m = 1, 2, \ldots, M$. For $n = 1, 2, \ldots, N$ and $m = 1, 2, \ldots, M$, the contractive homeomorphisms $L_n : I \to I_n$, $\tilde{L}_m : J \to J_m$ and the functions $F_{n,m} : S \times \mathbb{R}^2 \to D$, where D is a compact subset of $\mathbb{R}^2$, are defined by:

$$L_n(x) = x_{n-1} + \frac{x_n - x_{n-1}}{x_N - x_0}\ (x - x_0),$$
$$\tilde{L}_m(y) = y_{m-1} + \frac{y_m - y_{m-1}}{y_M - y_0}\ (y - y_0) \qquad (1)$$

and

$$F_{n,m}(x,y,z,t)= (\alpha_{n,m}\ z+\beta_{n,m}\ t+p_{n,m}(x,y),\\ \gamma_{n,m}\ t+q_{n,m}(x,y)). \quad (2)$$

In the above definition, $\alpha_{n,m}$ and $\gamma_{n,m}$ are called free variables and are chosen randomly but satisfying the condition: $|\alpha_{n,m}| < 1$ and $|\gamma_{n,m}| < 1$. The values of $\beta_{n,m}$ are chosen such that $|\beta_{n,m}|+|\gamma_{n,m}| < 1$ and hence are called constrained variables. The functions $p_{n,m}$ and $q_{n,m}$ in $F_{n,m}$ are selected such that the following join-up conditions are satisfied:

$$\begin{aligned} F_{n,m}(x_0,y_0,z_{0,0},t_{0,0}) &= (z_{n-1,m-1},t_{n-1,m-1})\\ F_{n,m}(x_N,y_0,z_{N,0},t_{N,0}) &= (z_{n,m-1},t_{n,m-1})\\ F_{n,m}(x_0,y_M,z_{0,M},t_{0,M}) &= (z_{n-1,m},t_{n-1,m})\\ F_{n,m}(x_N,y_M,z_{N,M},t_{N,M}) &= (z_{n,m},t_{n,m}). \end{aligned} \quad (3)$$

To ensure continuity, the functions $\tilde{F}_{n,m}$ are defined as:

$$\tilde{F}_{n,m}(x,y,z,t) = \begin{cases} F_{n+1,m}(x_0,y,z,t), & x=x_N,\ n=1,\dots,N-1,\ m=1,\dots,M \\ F_{n,m+1}(x,y_0,z,t), & y=y_M,\ n=1,\dots,N,\ m=1,\dots,M-1 \\ F_{n,m}(x,y,z,t), & \text{otherwise.} \end{cases} \quad (4)$$

It has been proved in [9] that if $\omega_{n,m}(x,y,z,t) = (L_n(x),\tilde{L}_m(y),\tilde{F}_{n,m}(x,y,z,t))$, the IFS defined by

$$\{S\times\mathbb{R}^2;\omega_{n,m}, n=1,2,\dots N \text{ and } m=1,2,\dots M\} \quad (5)$$

is hyperbolic with respect to a suitable metric equivalent to the Euclidean metric and there exists an attractor $G\subseteq\mathbb{R}^4$ satisfying $G=\bigcup_{n=1}^{N}\bigcup_{m=1}^{N}\omega_{n,m}(G)$ and G is graph of a continuous function $f : S\to\mathbb{R}^2$ such that $f(x_i,y_j) = (z_{i,j},t_{i,j})$ for $i=0,1,\dots,N$ and $j=0,1,\dots,M$. Hence the Coalescence Hidden variable Fractal Interpolation Surface (CHFIS) is defined as

Definition 1: The **Coalescence Hidden variable Fractal Interpolation Surface (CHFIS)** for the given interpolation data $\{(x_i,y_j,z_{i,j}) : i,j = 0,1,\dots,N\}$ is defined as the graph of projection f_1 of f on $\mathbb{R}^3$, where f_1 is the first component of the function $f=(f_1,f_2)$.

Remark 1: The function f_1 is called a Coalescence Hidden-variable Fractal Interpolation bivariate function as it exhibits both self-affine and non-self-affine nature. For the same interpolation data, the function $f_2(x,y)$ is a self-affine bivariate function.

III. Methods of Insertion of New Points

The problem of node insertion in the bivariate situation is characterized by introducing a row and/or column of additional points. In order to enter additional points, there are three options.

Method 1: The first method is inserting a point $\hat{x}$ between x_{k-1} and x_k. This gives a column of new points as $\Gamma_1 = \{(\hat{x},y_0,\hat{z}_{*,0},\hat{t}_{*,0}),\dots,(\hat{x},y_M,\hat{z}_{*,M},\hat{t}_{*,M})\}$ in the generalized interpolation data Δ. So, the new generalized interpolation data are

$$\widehat{\Delta}_1 = \Big\{(x_0,y_0,z_{0,0},t_{0,0}),\dots,(x_{k-1},y_0,z_{k-1,0},t_{k-1,0}), (\hat{x},y_0,\hat{z}_{*,0},\hat{t}_{*,0}),(x_k,y_0,z_{k,0},t_{k,0}),\dots\dots, (x_N,y_0,z_{N,0},t_{N,0}),\dots,(x_0,y_M,z_{0,M},t_{0,M}),\dots, (x_{k-1},y_M,z_{k-1,M},t_{k-1,M}),(\hat{x},y_M,\hat{z}_{*,M},\hat{t}_{*,M}), (x_k,y_M,z_{k,M},t_{k,M}),\dots,(x_N,y_M,z_{N,M},t_{N,M})\Big\}.$$

The interval I_k is split into two intervals $I_k^l = [x_{k-1},\hat{x}]$ and $I_k^r = [\hat{x},x_k]$ which in turn gives that the rectangles $S_{k,m}, m=1,2,\dots,M, m\neq l$ are broken into two rectangles, say $S_{k,m}^l = I_k^l\times J_m$ and $S_{k,m}^r = I_k^r\times J_m$. Define $L_k^l : I\to I_k^l$ and $L_k^r : I\to I_k^r$ as

$$\begin{aligned} L_k^l(x) &= x_{k-1}+\frac{\hat{x}-x_{k-1}}{x_N-x_0}\ (x-x_0)\\ L_k^r(x) &= \hat{x}+\frac{x_k-\hat{x}}{x_N-x_0}\ (x-x_0) \end{aligned} \quad (6)$$

For $m=1,2,\dots,M$, define $F_{k,m}^l : S\to S_{k,m}^l$ and $F_{k,m}^r : S\to S_{k,m}^r$ as

$$\begin{aligned} F_{k,m}^l(x,y) &= \big(\alpha_{k,m}^l z+\beta_{k,m}^l t+p_{k,m}^l(x,y),\\ &\qquad \gamma_{k,m}^l t+q_{k,m}^l(x,y)\big),\ m=1,2,\dots,M\\ F_{k,m}^r(x,y) &= \big(\alpha_{k,m}^r z+\beta_{k,m}^r t+p_{k,m}^r(x,y),\\ &\qquad \gamma_{k,m}^r t+q_{k,m}^r(x,y)\big),\ m=1,2,\dots,M \end{aligned} \quad (7)$$

where, $\alpha_{k,m}^l,\alpha_{k,m}^r,\gamma_{k,m}^l,\gamma_{k,m}^r$ are free variables whose absolute value is strictly less than one; $\beta_{k,m}^l$ and $\beta_{k,m}^r$ are constrained variables such that $|\beta_{k,m}^l|+|\gamma_{k,m}^l| < 1$ and $|\beta_{k,m}^r|+|\gamma_{k,m}^r| < 1$. The functions $p_{k,m}^l$, $p_{k,m}^r$, $q_{k,m}^l$ and $q_{k,m}^r$ are continuous functions selected such that the functions $F_{k,m}^l$ map the end points of the rectangle S to end points of the rectangle $S_{k,m}^l$ and $F_{k,m}^r$ map the end points of the rectangle S to end points of the rectangle $S_{k,m}^r$ i.e. $F_{k,m}^l$ and $F_{k,m}^r$ for $m=1,\dots,M$ satisfy the following conditions:

$$\begin{aligned} F_{k,m}^l(x_0,y_0,z_{0,0},t_{0,0}) &= (z_{k-1,m-1},t_{k-1,m-1})\\ F_{k,m}^r(x_0,y_0,z_{0,0},t_{0,0}) &= (\hat{z}_{*,m-1},\hat{t}_{*,m-1})\\ F_{k,m}^l(x_0,y_M,z_{0,M},t_{0,M}) &= (z_{k-1,m},t_{k-1,m})\\ F_{k,m}^r(x_0,y_M,z_{0,M},t_{0,M}) &= (\hat{z}_{*,m},\hat{t}_{*,m})\\ F_{k,m}^l(x_N,y_0,z_{N,0},t_{N,0}) &= (\hat{z}_{*,m-1},\hat{t}_{*,m-1})\\ F_{k,m}^r(x_N,y_0,z_{N,0},t_{N,0}) &= (z_{k,m-1},t_{k,m-1})\\ F_{k,m}^l(x_N,y_M,z_{N,M},t_{N,M}) &= (\hat{z}_{*,m},\hat{t}_{*,m})\\ F_{k,m}^r(x_N,y_M,z_{N,M},t_{N,M}) &= (z_{k,m},t_{k,m}) \end{aligned}$$

Theorem 1: Let $\widehat{\Delta}_1 = \Delta\bigcup\Gamma_1$. Then,

$$\Big\{S\times\mathbb{R}^2;\ \omega_{n,m},\ n=1,\dots,N, n\neq k;\ \omega_{k,m}^l\ \omega_{k,m}^r,\ m=1,\dots,M\Big\} \quad (8)$$

with $\omega^l_{k,m} = (L^l_k, \tilde{L}_m, F^l_{k,m})$ and $\omega^r_{k,m} = (L^r_k, \tilde{L}_m, F^r_{k,m})$ is a hyperbolic IFS on $S \times \mathbb{R}^2$ and there exists an attractor $\hat{A}$ that satisfies $\hat{A} = \bigcup_{\substack{n=1\\n\neq k}}^{N} \bigcup_{m=1}^{M} \omega_{n,m}(\hat{A}) \bigcup_{m=1}^{M} \bigcup_{p\in\{l,r\}} \omega^p_{k,m}(\hat{A})$. In addition, the aforementioned IFS's attractor is a graph of a continuous bivariate function that passes over the new generalized interpolation points $\widehat{\Delta}_1$.

Proof: We suppose $\alpha^p_{k,m} \leq \alpha_{k,m}$, $\gamma^p_{k,m} \leq \gamma_{k,m}$ and $\beta^p_{k,m} \leq \beta_{k,m}$ for all $m = 1, 2, \ldots, M$ and $p \in \{l, r\}$. Then the maps $\omega^l_{k,m}$ and $\omega^r_{k,m}$ are contraction maps for the same metric, by which $\omega_{n,m}$ are contraction maps. Alternatively, a metric could be defined as in [9] where, $\omega_{n,m}; n = 1, 2, \ldots N, n \neq k,\ \omega^l_{k,m}; \omega^r_{k,m}; m = 1, 2, \ldots, M$, are contraction maps. Then the IFS represented by (8) is hyperbolic and contains an attractor $\hat{A}$ that fulfills $\hat{A} = \bigcup_{\substack{n=1\\n\neq k}}^{N} \bigcup_{m=1}^{M} \omega_{n,m}(\hat{A}) \bigcup_{m=1}^{M} \omega^l_{k,m}(\hat{A}) \bigcup_{m=1}^{M} \omega^r_{k,m}(\hat{A})$.

We consider the metric space of functions $(\mathcal{G}, d_{\mathcal{G}})$ such that $\mathcal{G} = \Big\{ g \mid g : S \to \mathbb{R}^2$ is continuous, $g(x_0, y_0) = (z_{0,0}, t_{0,0}),\ g(x_N, y_0) = (z_{N,0}, t_{N,0}),$ $g(x_0, y_M) = (z_{0,M}, t_{0,M})$, and $g(x_N, y_M) = (z_{N,M}, t_{N,M}) \Big\}$ and $d_{\mathcal{G}}(g, \hat{g}) = \max_{(x,y)\in S} \Big(|g_1(x, y) - \hat{g}_1(x, y)|, |g_2(x, y) - \hat{g}_2(x, y)| \Big)$, for $g, \hat{g} \in \mathcal{G}$. Define Read-Bajraktarević operator on $(\mathcal{G}, d_{\mathcal{G}})$ as

$$
\begin{aligned}
\hat{T}(g)(x, y) &= F_{n,m}(L^{-1}_n(x), \tilde{L}^{-1}_m(y), g(L^{-1}_n(x), \tilde{L}^{-1}_m(y))),\\
&\quad (x, y) \in S_{n,m}, n = 1, 2, \ldots, N; n \neq k,\\
\hat{T}(g)(x, y) &= F^l_{k,m}(L^{l^{-1}}_k(x), \tilde{L}^{-1}_m(y), g(L^{l^{-1}}_k(x), \tilde{L}^{-1}_m(y))),\\
&\quad (x, y) \in S^l_{k,m}\\
\hat{T}(g)(x, y) &= F^r_{k,m}(L^{r^{-1}}_k(x), \tilde{L}^{-1}_m(y), g(L^{r^{-1}}_k(x), \tilde{L}^{-1}_m(y))),\\
&\quad (x, y) \in S^r_{k,m}
\end{aligned} \tag{9}
$$

for all $m = 1, 2, \ldots, M$. Following the lines of proof as in [9], it is straightforward to demonstrate that the bivariate Read-Bajraktarević operator defined by (9) is a contraction map and that a continuous bivariate function $\hat{f} : S \to \mathbb{R}^2$ exists which passes through the generalized interpolation points $\widehat{\Delta}_1$. In addition, distinctiveness provides $\hat{A}$ represents the graph of the function $\hat{f}$. ■

Method 2: The second method is inserting a point $\hat{y}$ between y_{l-1} and y_l which gives a row of new points as $\Gamma_2 = \{(x_0, \hat{y}, \hat{z}_{0,*}, \hat{t}_{0,*}), (x_1, \hat{y}, \hat{z}_{1,*}, \hat{t}_{1,*}), \ldots\ldots, (x_N, \hat{y}, \hat{z}_{N,*}, \hat{t}_{N,*})\}$ in the generalized interpolation data Δ. Here, the new generalized interpolation data are $\widehat{\Delta}_2 = \Big\{ (x_0, y_0, z_{0,0}, t_{0,0}), \ldots\ldots, (x_N, y_0, z_{N,0}, t_{N,0}), \ldots\ldots,$ $(x_0, y_{l-1}, z_{0,l-1}, t_{0,l-1}), \ldots\ldots, (x_N, y_{l-1}, z_{N,l-1}, t_{N,l-1}),$ $(x_0, \hat{y}, \hat{z}_{0,*}, \hat{t}_{0,*}), \ldots\ldots, (x_N, \hat{y}, \hat{z}_{N,*}, \hat{t}_{N,*}), (x_0, y_l, z_{0,l}, t_{0,l}),$ $\ldots\ldots, (x_N, y_l, z_{N,l}, t_{N,l}), \ldots\ldots, (x_0, y_M, z_{0,M}, t_{0,M}), \ldots,$ $(x_N, y_M, z_{N,M}, t_{N,M}) \Big\}$. The interval J_l is broken into two intervals $J^b_l = [y_{l-1}, \hat{y}]$ and $J^t_l = [\hat{y}, y_l]$ and the rectangles $S_{n,l}$ for $n = 1, 2, \ldots, N$ are split into two rectangles, say $S^b_{n,l} = I_n \times J^b_l$ and $S^t_{n,l} = I_n \times J^t_l$.

We define $\tilde{L}^b_l : J \to J^b_l$ and $\tilde{L}^t_l : J \to J^t_l$ are defined as

$$
\begin{aligned}
\tilde{L}^b_l(y) &= y_{l-1} + \frac{\hat{y} - y_{l-1}}{y_M - y_0}\ (y - y_0)\\
\tilde{L}^t_l(y) &= \hat{y} + \frac{y_l - \hat{y}}{y_M - y_0}\ (y - y_0)
\end{aligned} \tag{10}
$$

For $n = 1, 2, \ldots, N$, define $F^b_{n,l} : S \to S^b_{n,l}$ and $F^t_{n,l} : S \to S^t_{n,l}$ as

$$
\begin{aligned}
F^b_{n,l}(x, y) &= \big(\alpha^b_{n,l} z + \beta^b_{n,l} t + p^b_{n,l}(x, y),\\
&\quad \gamma^b_{n,l} t + q^b_{n,l}(x, y)\big),\\
&\quad n = 1, 2, \ldots, N,\ n \neq k\\
F^t_{n,l}(x, y) &= \big(\alpha^t_{n,l} z + \beta^t_{n,l} t + p^t_{n,l}(x, y),\\
&\quad \gamma^t_{n,l} t + q^t_{n,l}(x, y)\big),\\
&\quad n = 1, 2, \ldots, N,\ n \neq k
\end{aligned} \tag{11}
$$

where, $\alpha^b_{n,l}, \alpha^t_{n,l}, \gamma^b_{n,l}, \gamma^t_{n,l}$ are free variables whose absolute value is strictly less than one; $\beta^b_{n,l}$ and $\beta^t_{n,l}$ are constrained variables such that $|\beta^b_{n,l}| + |\gamma^b_{n,l}| < 1$ and $|\beta^t_{n,l}| + |\gamma^t_{n,l}| < 1$. The functions $p^b_{n,l}$, $p^t_{n,l}$, $q^b_{n,l}$ and $q^t_{n,l}$ are continuous functions selected such that the functions $F^b_{n,l}$ map the end points of the rectangle S to end points of the rectangle $S^b_{n,l}$ and $F^t_{n,l}$ map the end points of the rectangle S to end points of the rectangle $S^t_{n,l}$ i.e. $F^b_{n,l}$ and $F^b_{n,l}$ for $n = 1, \ldots, N$ satisfy the following conditions:

$$
\begin{aligned}
F^b_{n,l}(x_0, y_0, z_{0,0}, t_{0,0}) &= (z_{n-1,l-1}, t_{n-1,l-1})\\
F^t_{n,l}(x_0, y_0, z_{0,0}, t_{0,0}) &= (\hat{z}_{n-1,*}, \hat{t}_{n-1,*})\\
F^b_{n,l}(x_0, y_M, z_{0,M}, t_{0,M}) &= (\hat{z}_{n-1,*}, \hat{t}_{n-1,*})\\
F^t_{n,l}(x_0, y_M, z_{0,M}, t_{0,M}) &= (z_{n-1,l}, t_{n-1,l})\\
F^b_{n,l}(x_N, y_0, z_{N,0}, t_{N,0}) &= (z_{n,l-1}, t_{n,l-1})\\
F^t_{n,l}(x_N, y_0, z_{N,0}, t_{N,0}) &= (\hat{z}_{n,*}, \hat{t}_{n,*})\\
F^b_{n,l}(x_N, y_M, z_{N,M}, t_{N,M}) &= (\hat{z}_{n,*}, \hat{t}_{n,*})\\
F^t_{n,l}(x_N, y_M, z_{N,M}, t_{N,M}) &= (z_{n,l}, t_{n,l})
\end{aligned}
$$

Theorem 2: Let $\widehat{\Delta}_2 = \Delta \bigcup \Gamma_2$. Then,

$$
\Big\{ S \times \mathbb{R}^2;\ \omega_{n,m};\ m = 1, \ldots, M, m \neq l,\quad \omega^b_{n,l};\ \omega^t_{n,l};\ n = 1, \ldots, N, \Big\} \tag{12}
$$

with $\omega^b_{n,l} = (L_n, \tilde{L}^b_l, F^b_{n,l})$ and $\omega^t_{n,l} = (L_n, \tilde{L}^t_l, F^t_{n,l})$ is a hyperbolic IFS on $S \times \mathbb{R}^2$ and there exists an attractor $\hat{A}$ that satisfies $\hat{A} = \bigcup_{\substack{m=1\\m\neq l}}^{M} \bigcup_{n=1}^{N} \omega_{n,m}(\hat{A}) \bigcup_{n=1}^{N} \bigcup_{q\in\{b,t\}} \omega^q_{n,l}(\hat{A})$. Furthermore, the attractor of the aforementioned IFS is a graph of a continuous bivariate function that passes through the new generalized interpolation points. $\widehat{\Delta}_2$.

Proof: Suppose $\alpha^q_{n,l} \leq \alpha_{n,l}$, $\gamma^q_{n,l} \leq \gamma_{n,l}$ and $\beta^q_{n,l} \leq \beta_{n,l}$ for all $m = 1, 2, \ldots, M$ and $q \in \{b, t\}$. Then the maps $\omega^b_{n,l}$ and $\omega^t_{n,l}$ are contraction maps for the same metric by which $\omega_{n,m}$ are contraction maps. Or else, a metric could be defined as in [9] where, $\omega_{n,m}; m = 1, \ldots, M, m \neq l, \omega^b_{n,l}; \omega^t_{n,l};\ n = 1, 2, \ldots N$ are contraction maps. Then, the IFS given by (12)is hyperbolic and contains an attractor $\hat{A}$ which fulfills $\hat{A} = \bigcup_{\substack{m=1\\m\neq l}}^{N} \bigcup_{n=1}^{N} \omega_{n,m}(\hat{A}) \bigcup_{n=1}^{N} \omega^b_{n,l}(\hat{A}) \bigcup_{n=1}^{N} \omega^t_{n,l}(\hat{A})$.

We consider the metric space of continuous functions $(\mathcal{G}, d_{\mathcal{G}})$ and define Read-Bajraktarević operator on $(\mathcal{G}, d_{\mathcal{G}})$ as

$$\begin{aligned}
&\hat{T}(g)(x,y)\\
&= F_{n,m}(L_n^{-1}(x), \tilde{L}_m^{-1}(y), g(L_n^{-1}(x), \tilde{L}_m^{-1}(y))),\\
&\ \text{for } (x,y) \in S_{n,m}, m = 1,2,\ldots,M; m \neq l\\
&\hat{T}(g)(x,y)\\
&= F^b_{n,l}(L_n^{-1}(x), \tilde{L}_l^{b^{-1}}(y), g(L_n^{-1}(x), \tilde{L}_l^{b^{-1}}(y))),\\
&\ \text{for } (x,y) \in S^b_{n,l}\\
&\hat{T}(g)(x,y)\\
&= F^t_{n,l}(L_n^{-1}(x), \tilde{L}_l^{t^{-1}}(y), g(L_n^{-1}(x), \tilde{L}_l^{t^{-1}}(y))),\\
&\ \text{for } (x,y) \in S^t_{n,l}
\end{aligned} \tag{13}$$

for $n = 1, 2, \ldots, N$. Just as in case (i), the bivariate Read-Bajraktarević operator defined by (13) is a contraction map and there exists a continuous bivariate function $\hat{f} : S \to \mathbb{R}^2$ which passes through the new generalized interpolation points $\widehat{\Delta}_2$. Additionally, uniqueness indicates that $\hat{A}$ is graph of the function $\hat{f}$. ■

Method 3: The third kind is inserting the point $(\hat{x}, \hat{y})$ in the given interpolation data where, $x_{k-1} < \hat{x} < x_k$ and $y_{l-1} < \hat{y} < y_l$. Here, a row and a column of new points given by $\Gamma = \{(\hat{x}, y_0, \hat{z}_{*,0}, \hat{t}_{*,0}), \ldots, (\hat{x}, y_M, \hat{z}_{*,M}, \hat{t}_{*,M}), (x_0, \hat{y}, \hat{z}_{0,*}, \hat{t}_{0,*}), \ldots, (x_N, \hat{y}, \hat{z}_{N,*}, \hat{t}_{N,*}), (\hat{x}, \hat{y}, \hat{z}, \hat{t})\}$ is inserted in the generalized interpolation data Δ. It is easy to see that the third case is nothing but combination of Method 1 and Method 2.

Theorem 3: Let $\widehat{\Delta} = \Delta \bigcup \Gamma$. Then,

$$\begin{Bmatrix}
S \times \mathbb{R}^2;\ \omega_{n,m},\ n = 1,\ldots,N,\ m = 1,\ldots,M,\ n \neq k,\ m \neq l;\\
\omega^p_{n,l}, n = 1,\ldots,N,\ p \in \{b,t\}, n \neq k;\\
\omega^A_{k,m}\ m = 1,\ldots,M, m \neq l, p \in \{l,r\};\\
\omega^p_{k,l}, p \in \{(l,b),(r,b),(l,t),(r,t)\}
\end{Bmatrix} \tag{14}$$

with $\omega^p_{n,m} = (L^p_n, \tilde{L}^p_m, F^p_{n,m})$ is a hyperbolic IFS on $S \times \mathbb{R}^2$ and there exists an attractor $\hat{A}$ that satisfies

$$\hat{A} = \left(\bigcup_{\substack{n=1\\n\neq k}}^{N} \bigcup_{\substack{m=1\\m\neq l}}^{M} \omega_{n,m}(\hat{A}) \bigcup_{\substack{n=1\\n\neq k}}^{N} \bigcup_{p\in\{b,t\}} \omega^p_{n,l}(\hat{A}) \bigcup_{\substack{m=1\\m\neq l}}^{M} \bigcup_{p\in\{l,r\}} \omega^p_{k,m}(\hat{A}) \bigcup_{p\in\{(l,b),(r,b),(l,t),(r,t)\}} \omega^p_{k,l}(\hat{A}) \right).$$

In addition, the attractor of the aforementioned IFS is graph of a continuous bivariate function that passes across the generalized interpolation points $\widehat{\Delta}$.

Proof: Let $\Gamma_1 = \{(\hat{x}, y_0, \hat{z}_{*,0}, \hat{t}_{*,0}), \ldots, (\hat{x}, y_M, \hat{z}_{*,M}, \hat{t}_{*,M})\}$ and $\Gamma_3 = \{(x_0, \hat{y}, \hat{z}_{0,*}, \hat{t}_{0,*}), \ldots, (\hat{x}, \hat{y}, \hat{z}, \hat{t}), \ldots (x_N, \hat{y}, \hat{z}_{N,*}, \hat{t}_{N,*})\} = \Gamma_2 \bigcup (\hat{x}, \hat{y}, \hat{z}, \hat{t})$. Then, Theorem 1 is applied on $\widehat{\Delta}_1 = \Delta \bigcup \Gamma_1$ followed by Theorem 2 on $\widehat{\Delta} = \widehat{\Delta}_1 \bigcup \Gamma_3$. So, there exists a continuous bivariate function $\hat{f} : S \to \mathbb{R}^2$ passing through the new generalized interpolation points $\widehat{\Delta}$. ■

Remark 2: In the above Theorem 3, we suppose $\Gamma_2 = \{(x_0, \hat{y}, \hat{z}_{0,*}, \hat{t}_{0,*}), \ldots, (x_N, \hat{y}, \hat{z}_{N,*}, \hat{t}_{N,*})\}$ and $\Gamma_3 = \{(\hat{x}, y_0, \hat{z}_{*,0}, \hat{t}_{*,0}), \ldots, (\hat{x}, \hat{y}, \hat{z}, \hat{t}) \ldots (\hat{x}, y_M, \hat{z}_{*,M}, \hat{t}_{*,M})\} = \Gamma_1 \bigcup (\hat{x}, \hat{y}, \hat{z}, \hat{t})$. In this case, Theorem 2 is applied first on $\widehat{\Delta}_2 = \Delta \bigcup \Gamma_2$ followed by Theorem 1 on $\widehat{\Delta} = \widehat{\Delta}_2 \bigcup \Gamma_3$. Again, it is obtained that there exist a continuous function $\hat{f} : S \to \mathbb{R}^2$ passing through the new generalized interpolation points $\widehat{\Delta}$.

IV. Different Kinds of Node Insertion

Let us now describe different types of insertion according to the values of inserted nodes for each method of insertion. We suppose $\hat{f} = (\hat{f}_1, \hat{f}_2)$ is a component-wise expression of the function $\hat{f}$. The graph of $\hat{f}_1$ then becomes a CHFIS that passes through the interpolation data $\widehat{\Lambda}$.

Method 1: In this method, a column of new points are inserted in the given interpolation data.

- If $\hat{t_{*,m}} = f_2(\hat{x}, y_m)$ for all $m = 0,1,\ldots,M$ and $\hat{t} = f_2(\hat{x}, \hat{y})$ but $\hat{z_{*,m}} \neq f_1(\hat{x}, y_m)$ for some $m = 1,2,\ldots,M$ or $\hat{z} \neq f_1(\hat{x}, \hat{y})$ then it is called **C-Node-Knot** insertion problem.
- If $\hat{z_{*,m}} = f_1(\hat{x}, y_m)$ for all $m = 0,1,\ldots,M$ and $\hat{z} = f_1(\hat{x}, \hat{y})$ but $\hat{t_{*,m}} \neq f_2(\hat{x}, y_m)$ for some $m = 1,2,\ldots,M$ or $\hat{t} \neq f_2(\hat{x}, \hat{y})$ then it is called **C-Knot-Node** insertion problem.
- If $\hat{z_{*,m}} = f_1(\hat{x}, y_m)$, $\hat{t_{*,m}} = f_2(\hat{x}, y_m)$ for all $m = 0,1,\ldots,M$, $\hat{z} = f_1(\hat{x}, \hat{y})$ and $\hat{t} = f_2(\hat{x}, \hat{y})$ then it is called **C-Knot-Knot** insertion problem.
- If $\hat{z_{*,m}} \neq f_1(\hat{x}, y_m)$ and $\hat{t_{*,m}} \neq f_2(\hat{x}, y_m)$ for some $m = 0,1,\ldots,M$ then it is called **C-Node-Node** insertion problem.

Method 2: In this method, a row of new points are inserted in the given interpolation data.

- If $\hat{t_{n,*}} = f_2(x_n, \hat{y})$ for all $n = 0,1,\ldots,N$ and $\hat{t} = f_2(\hat{x}, \hat{y})$ but $\hat{z_{n,*}} \neq f_1(x_n, \hat{y})$ for some $n = 1,2,\ldots,N$ or $\hat{z} \neq f_1(\hat{x}, \hat{y})$ then it is called **R-Node-Knot** insertion problem.
- If $\hat{z_{n,*}} = f_1(x_n, \hat{y})$ for all $n = 0,1,\ldots,N$ and $\hat{z} = f_1(\hat{x}, \hat{y})$ but $\hat{t_{n,*}} \neq f_2(x_n, \hat{y})$ for some $n = 1,2,\ldots,N$ or $\hat{t} \neq f_2(\hat{x}, \hat{y})$ then it is called **R-Knot-Node** insertion problem.
- If $\hat{z_{n,*}} = f_1(x_n, \hat{y})$, $\hat{t_{n,*}} = f_2(x_n, \hat{y})$ for all $n = 0,1,\ldots,N$, $\hat{z} = f_1(\hat{x}, \hat{y})$ and $\hat{t} = f_2(\hat{x}, \hat{y})$ then it is called **R-Knot-Knot** insertion problem.
- If $\hat{z_{n,*}} \neq f_1(x_n, \hat{y})$ and $\hat{t_{n,*}} \neq f_2(x_n, \hat{y})$ for some $n = 0,1,\ldots,N$ then it is called **R-Node-Node** insertion problem.

Method 3: In this method, both row and column of new points are inserted in the given interpolation data.

- If $\hat{t_{n,*}} = f_2(x_n, \hat{y})$ for all $n = 0,1,\ldots,N$, $\hat{t_{*,m}} = f_2(\hat{x}, y_m)$ for all $m = 0,1,\ldots,M$ and $\hat{t} = f_2(\hat{x}, \hat{y})$ but $\hat{z_{n,*}} \neq f_1(x_n, \hat{y})$ for some $n = 1,2,\ldots,N$ or $\hat{z_{*,m}} \neq f_1(\hat{x}, y_m)$ for some

TABLE I
VALUE OF $z_{n,m}$ AT (x_n, y_m) IN A SAMPLE INTERPOLATION DATA

y_m/x_n	**0**	**8**	**22**	**42**	**50**
0	23	20	20	31	18
15	36	26	24	24	23
35	26	26	26	26	25
40	32	29	36	36	30

TABLE II Value of $t_{n,m}$ at (x_n, y_m) in generalized interpolation data

y_m/x_n	**0**	**8**	**22**	**42**	**50**
0	20	60	37	57	45
15	29	22	16	38	68
35	19	58	36	63	71
40	69	82	44	35	15

$m = 1, 2, \dots, M$ or $\hat{z} \neq f_1(\hat{x}, \hat{y})$ then it is called **RC-Node-Knot** insertion problem.

- If $\hat{z_{n,*}} = f_1(x_n, \hat{y})$ for all $n = 0, 1, \dots, N$, $\hat{z_{*,m}} = f_1(\hat{x}, y_m)$ for all $m = 0, 1, \dots, M$ and $\hat{z} = f_1(\hat{x}, \hat{y})$ but $\hat{t_{n,*}} \neq f_2(x_n, \hat{y})$ for some $n = 1, 2, \dots, N$ or $\hat{t_{*,m}} \neq f_2(\hat{x}, y_m)$ for some $m = 1, 2, \dots, M$ or $\hat{t} \neq f_2(\hat{x}, \hat{y})$ then it is called **RC-Knot-Node** insertion problem.
- If $\hat{z_{n,*}} = f_1(x_n, \hat{y})$, $\hat{t_{n,*}} = f_2(x_n, \hat{y})$ for all $n = 0, 1, \dots, N$, $\hat{z_{*,m}} = f_1(\hat{x}, y_m)$, $\hat{t_{*,m}} = f_2(\hat{x}, y_m)$ for all $m = 0, 1, \dots, M$, $\hat{z} = f_1(\hat{x}, \hat{y})$ and $\hat{t} = f_2(\hat{x}, \hat{y})$ then it is called **RC-Knot-Knot** insertion problem.
- If $\hat{z_{n,*}} \neq f_1(x_n, \hat{y})$ and $\hat{t_{n,*}} \neq f_2(x_n, \hat{y})$ for some $n = 0, 1, \dots, N$, $\hat{z_{*,m}} \neq f_1(\hat{x}, y_m)$ and $\hat{t_{*,m}} \neq f_2(\hat{x}, y_m)$ for some $m = 0, 1, \dots, M$ or $\hat{z} \neq f_1(\hat{x}, \hat{y})$ and $\hat{t} \neq f_2(\hat{x}, \hat{y})$ then it is called **RC-Node-Node** insertion problem.

V. EXAMPLES

Let $\Delta = \{(x_i, y_j, z_{i,j}, t_{i,j}) : i = 0, 1, \dots, N$ and $j = 0, 1, \dots, M\}$ where $z_{i,j}$ is given by Table I and $t_{i,j}$ is given by Table II be a sample generalized interpolation data. Fig. 1 is created with $\alpha_{n,m} = 0.3$, $\beta_{n,m} = 0.2$ and $\gamma_{n,m} = 0.5$. Figs. 2 and 3 are simulations of CHFIS generated corresponding to insertion of set of nodes $\Gamma_1 = \{(30, 0, 12, 23), (30, 15, 25, 56), (30, 35, 31, 12), (30, 50, 40, 76)\}$ and $\Gamma_2 = \{(0, 20, 22, 12), (8, 20, 43, 45), (22, 20, 67, 76), (42, 20, 12, 21), (50, 20, 55, 55)\}$ in the Δ respectively. Fig. 4 is obtained by inserting Γ_1, Γ_2 and $(30, 20, 55, 99)$ in the Δ.

VI. CONCLUSION

The impact of node insertion on the associated non-diagonal IFS and CHFIS is investigated in this paper. In bivariate case, there are the three different ways of inserting new points such as inserting a row of new points, inserting a column of new points and inserting both row and column of new points. For each of these methods, it is proved that the new non-diagonal IFS constructed using the new set of interpolation data give

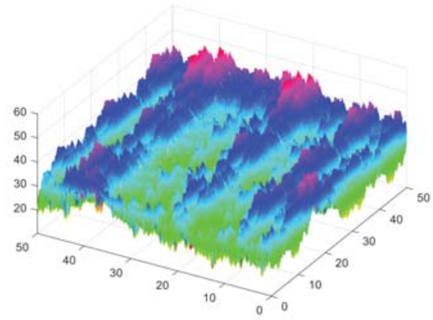

Fig. 1 Original Surface

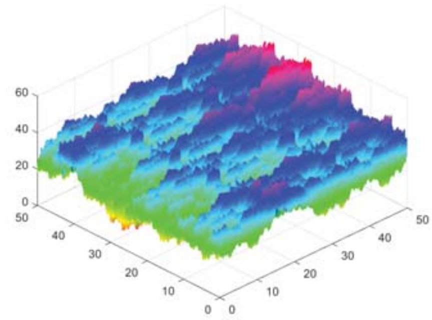

Fig. 2 Insertion of nodes Γ_1

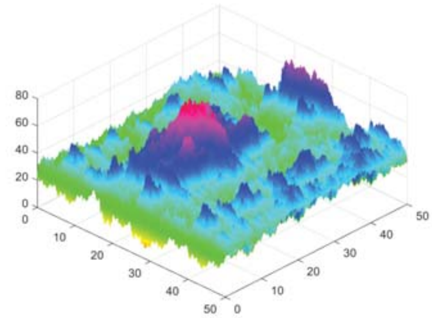

Fig. 3 Insertion of nodes Γ_2

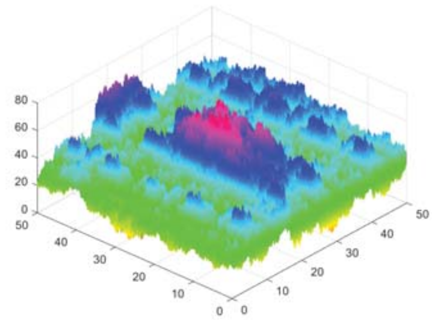

Fig. 4 Insertion of nodes $\Gamma_1 \bigcup \Gamma_2$

rise to a new CHFIS. Further, for each mode of insertion of nodes, the problem is further classified into four types of insertion according to inserted nodes' values. So, there are 12 kinds of node insertion in bivariate case as opposed to only 4 cases in single variable. These 12 cases also indicate whether the new set of points are inserted along a row or a column or in both. In future, the effect of insertion of nodes in smoothness and fractal dimension will be studied.

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