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An Insurer's Investment Model with Reinsurance Strategy under the Modified Constant Elasticity of Variance Process

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Abstract**—**One of the problems facing most insurance companies is how best the burden of paying claims to its policy holders can be managed whenever need arises. Hence there is need for the insurer to buy a reinsurance contract in order to reduce risk which will enable the insurer to share the financial burden with the reinsurer. In this paper, the insurer's and reinsurer's strategy is investigated under the modified constant elasticity of variance (M-CEV) process and proportional administrative charges. The insurer considered investment in one risky asset and one risk free asset where the risky asset is modeled based on the M-CEV process which is an extension of constant elasticity of variance (CEV) process. Next, a nonlinear partial differential equation in the form of Hamilton Jacobi Bellman equation is obtained by dynamic programming approach. Using power transformation technique and variable change, the explicit solutions of the optimal investment strategy and optimal reinsurance strategy are obtained. Finally, some numerical simulations of some sensitive parameters were obtained and discussed in details where we observed that the modification factor only affects the optimal investment strategy and not the reinsurance strategy for an insurer with exponential utility function.

Keywords**—**Reinsurance strategy, Hamilton Jacobi Bellman equation, power transformation, M-CEV process, exponential utility.

I. Introduction

THE M-CEV model is an improved stochastic volatility model (SVM) which is an extension of the CEV model. It was first developed by [9]. There are a number of volatility processes used in modeling the price processes of the risky assets. They include Ornstein-Uhlenbeck Process [1]-[3], jump diffusion [4], the CEV model [5], [6], Heston's volatility [7], [8], M-CEV model [9]-[12], and many more. More importantly, there are some attributes of the M-CEV process that makes it quite attractive; this includes the ability to take into consideration the volatility smile effects of the stock price, its probability can touch zero unlike the Geometric Brownian Motion whose probability is always positive and advance analytically tractable strategies. There are some literatures who used the M-CEV process to model the price of the risky assets; they include [9], who used the M-CEV process in developing a consistent and hedging pricing process by introducing a modification factor and showed that there is no equivalent risk or neutral pricing measure hence, the conventional risk neutral pricing methodology fails. Also, the benchmark method was used to set up a consistent pricing and hedging framework and showed that nonnegative price process and benchmark duplicate the contingent claim. Reference [10] determined a close form solution of the optimal investment strategy in terms of confluent hyper-geometric functions by using the Laplace transformation method and application of algorithmic tradition where the risky asset price was modelled by the M-CEV process. Reference [11] studied the price of the risky asset modelled M-CEV and Ornstein-Uhlenbeck processes; they showed that when there is no correlation between the Brownian motions, the insurer's investment plan is less compared to when the Brownian motions correlate. In [12], the optimal investment strategy with return of premium clause was studied for a pension plan member whose risky asset price followed the M-CEV process.

The insurance companies like other financial institutions are generally seen as risk-transfer institutions. They offer insurance to policy holders and this insurance is seen as risk transfer tools but can as well be used to prevent risk [13]. Since we cannot underestimate the merit of insurance in our day-to-day activities and the rapid increase in the study of portfolio management and reinsurance strategies, numerous authors, such as [14]-[24], have worked on this area of research. Different from other financial institutions such as pension system, banks, and many more, the insurer and the reinsurer are faced with double risks that exist both in the insurance market and the stock market. To reduce the risk of claims, the insurer can buy reinsurance contracts from the reinsurer thereby transferring some agreed percentage of the risk of claims to the reinsurer since the reinsurer is more risk-seeking than the insurer. Hence both the insurer and the reinsurer are faced with the option of finding a reliable investment plan in order to maximize profit; this has led to the study of optimal investment strategy.

A lot of work has been carried out on this area. They include [14] and [15] who studied an insurer optimal investment strategy under exponential utility function with the jump-diffusion process. Reference [16] studied an optimal investment strategy which followed the Hull and White SVM and also

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obtained a reinsurance strategy to maximize expected utility function. Reference [17] draws the attention of many researchers to the study of risk management with insurance. In [18], the authors were the first to propose a model on how insurance can be used as a risk prevention tool. References [19] and [20] give early contributions to insurance and reinsurance. In [21], the investment strategy for an insurer with stochastic premium was investigated under CARA utility; the premiums paid to the insurance companies were assumed to be stochastic and the solution was obtained using Legendre transformation and dual theory. Reference [22] studied the optimal reinsurance and investment problem of the maximum expected two exponential utility function whose claim process are modelled by Brownian motion with drift. References [23] and [24] considered the optimal reinsurance and investment problem of maximizing the expected power utility function and also obtained an optimal Excess-of-Loss reinsurance and investment by maximizing the exponential utility. Very recently, [22] and [25] studied the time-consistent investment and reinsurance strategies for insurers under mean variance utility, stochastic interest rate and stochastic volatility. In their work, the stock market price was modelled by Heston stochastic volatility and the interest rate follows the Vasicek model. References [26] and [27] discussed the problem of ruin probability minimization, the ruin probability for the insurer, and maximizing the exponential utility function. Reference [28] focused on the optimal investment problem with consumption. Reference [29] investigated the optimal reinsurance and investment strategy under the CEV model with fractional power utility function; in their work, they observed that the greater the value of the reinsurer's safety loading, the smaller the optimal reinsurance policy and to maintain a stable income, the insurer would prefer buying less reinsurance.

In this work, we intend to investigate the optimal investment and reinsurance strategy for an insurer with exponential utility function under the M-CEV process and proportional administrative charges. Furthermore, we will use the power transformation method to obtain the close form solutions of the optimal investment and reinsurance strategies; we also give some numerical simulations to explain our results. The difference between our work, [29] and [30] is that their utility functions are power and logarithm utility respectively while ours is exponential utility and also our risky asset is modelled using the M-CEV instead of the CEV model in their work.

II. Preliminaries

A. The Surplus Process

In this subsection, we formulate the surplus process of the insurer. In insurance, the surplus process is the process of accumulation of wealth. To derive the surplus process, we need the claim process. Following the framework of [26] and [31] we model the claim process $\mathcal{K}(t)$ which follows the Brownian motion with drift as

$$d\mathcal{K}(t) = \mathscr{k}_0 dt - \mathscr{k}_1 d\mathcal{H}_0(t), \tag{1}$$

where $\mathscr{k}_0$and $\mathscr{k}_1$are positive constant, and $\mathcal{H}_0(t)$ is a standard Brownian motion defined on the complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$. According to the expected value principle [32], the premium rate of an insurer is $\mathscr{k} = (1+\vartheta)\mathscr{k}_0$ where $\vartheta > 0$ is the safety loading of the insurer. In this paper, we assume a classical Cramer-Lundberg model for surplus process similar to [32] as

$$\mathcal{M}(t) = l_0 + \mathscr{k}t - \mathcal{K}(t) \geq 0, \tag{2}$$

where $\mathcal{M}(t)$and l_0 are the insurers capital at time t and initial capital $\mathcal{M}(0) = l_0$, respectively. According to (1), the surplus process for the insurer is given as

$$d\mathcal{M}(t) = \mathscr{k}dt - d\mathcal{K}(t) = \mathscr{k}_0 \mathfrak{H} dt + \mathscr{k}_1 d\mathcal{H}_0(t). \tag{3}$$

Furthermore, we assume that the insurer can afford to buy reinsurance contract to reduce risk [16].

Suppose the reinsurance premium rate at time t is given by $\mathscr{k}_2 = (1+\Theta)\mathscr{k}_0$, where Θ is the safety loading of the reinsurer and satisfies the condition $\Theta > \mathfrak{H} > 0$ and $\mathcal{N}(t)$ is the surplus process associated with the reinsurance of the insurer then from [16], the differential form of the surplus process is given thus;

$$d\mathcal{N}(t) = \mathscr{k}dt - \big(1 - \mathfrak{T}(t)\big)d\mathcal{K}(t) - \mathscr{k}_2 \mathfrak{T}(t)dt, \tag{4}$$

$$d\mathcal{N}(t) = \big(\mathfrak{H} - \Theta\mathfrak{T}(t)\big)\mathscr{k}_0 dt + \mathscr{k}_1\big(1 - \mathfrak{T}(t)\big)d\mathcal{H}_0(t). \tag{5}$$

where $\mathfrak{T}(t)$ is proportional reinsurance strategy at time *t*.

B. The Market Model

Suppose we have a portfolio with a risk free asset and a risky assets in a financial market which is continuously open over an interval $t \in [0, T]$ where T the expiration date of the policy. Let $\{\mathcal{H}_0(t), \mathcal{H}_1(t) : t \geq 0\}$ be two standard Brownian motion defined on a complete probability space $(\Omega, F, \mathcal{P})$, where Ω is a real space, $\mathcal{P}$ is a probability measure and F is the filtration which stands for the information generated by the two Brownian motions.

Let $\mathcal{D}_0(t)$ denote the price of the risk free asset at time t and from [6], [8], [29], the model is given as

$$\begin{cases} \frac{d\mathcal{D}_0(t)}{\mathcal{D}_0(t)} = \mathfrak{K}(t)dt \\ \mathcal{D}_0(0) = d_0 > 0 \end{cases}, \tag{6}$$

$\mathfrak{K}$ is the risk interest rate process.

Let $\mathcal{D}_1(t)$ be the price process of the risky asset modelled by the M-CEV process. The price process is described by the stochastic differential equation at $t \geq 0$ as follows

$$\begin{cases} \frac{d\mathcal{D}_1(t)}{\mathcal{D}_1(t)} = \Big(a + \kappa \mathscr{b}^2 \mathcal{D}_1^{2\beta}(t)\Big)dt + \mathscr{b}\mathcal{D}_1^{\beta}(t)\mathcal{H}_1(t). \\ \mathcal{D}_1(0) = 1 \end{cases}, \tag{7}$$

where a is the rate of appreciation of the risky asset, $\mathscr{b}$ is instantaneous volatility of the risky asset, $\kappa > 0$ is the modification factor and $\beta < 0$ is the elasticity parameter of the stock market price, see [9]-[12] for details. If $\kappa = 0$, the model in (7) reduces to that of a CEV model, also, if $\kappa = 0$, and $\beta = 0$, the model in (7) reduces to that of geometric Brownian motion.

III. MAIN RESULTS

A. Optimization Problem

Let ζ be the OIS and suppose the utility attained by an insurer from a given state $\mathcal{v}$ at time t as

$$\mathcal{A}_\zeta(t, d_1, \mathcal{v}) = E_\zeta\left[U(\mathcal{V}(t)) \mid \mathcal{D}_1(t) = d_1, \mathcal{V}(t) = \mathcal{v} \right],$$

where t is the time, $\mathfrak{K}(t)$ is the risk free interest rate and $\mathcal{v}$ is the wealth. The objective here is to determine the OIS and the optimal value function of the investor given as π^* and $\mathcal{A}(t, d_1, \mathcal{v}) = \sup_\pi \mathcal{A}_\zeta(t, d_1, \mathcal{v})$, respectively such that $\mathcal{A}_{\zeta^*}(t, d_1, \mathcal{v}) = \mathcal{A}(t, d_1, \mathcal{v})$.

Let $\mathcal{V}(t)$ be the insurer's wealth at time t, $\aleph$ is the proportional administrative charges on the insurer's investment, $\zeta(t)$ is the insurer's wealth invested on risky asset at time t and $\mathcal{V}(t) - \zeta(t)$ is the amount invested in the risk free asset. Therefore, the corresponding differential form of the fund size is given as:

$$\mathrm{d}\mathcal{V}(t) = \begin{pmatrix} \zeta(t)\frac{d\mathcal{D}_1(t)}{\mathcal{D}_1(t)} + \left(\mathcal{V}(t) - \zeta(t)\right)\frac{d\mathcal{D}_0(t)}{\mathcal{D}_0(t)} \\ +d\mathcal{N}(t) - \aleph\mathcal{V}(t)dt \end{pmatrix}. \tag{8}$$

Substituting (5)-(7) into (8), we have

$$\mathrm{d}\mathcal{V}(t) = \begin{pmatrix} \begin{pmatrix} (\mathfrak{K} - \aleph)\mathcal{V}(t) \\ +\zeta(t)\left(a + \kappa\mathcal{b}^2\mathcal{D}_1^{2\beta}(t) - \mathfrak{K}(t)\right) \\ +\left(\mathfrak{H} - \Theta\mathfrak{T}(t)\right)\mathcal{k}_0 \end{pmatrix} dt \\ +\mathcal{k}_1(1 - \mathfrak{T}(t))d\mathcal{H}_0(t) + \zeta(t)\mathcal{b}\mathcal{D}_1^\beta(t)d\mathcal{H}_1(t) \end{pmatrix}. \tag{9}$$

Applying the Ito's lemma and maximum principle, the Hamilton Jacobi Bellman (HJB) equation which is a nonlinear partial differential equation (PDE) associated associated with (9) is obtained by maximizing $\mathcal{A}_{\zeta^*}(t, d_1, \mathcal{v})$ subject to the insurer's wealth in (9) as follows

$$\left\{\begin{matrix} \mathcal{A}_t + \left(a + \kappa\mathcal{b}^2 d_1^{2\beta}(t)\right) d_1\mathcal{A}_{d_1} \\ +\begin{bmatrix} (\mathfrak{K} - \aleph)\mathcal{v} + \zeta\left(a + \kappa\mathcal{b}^2 d_1^{2\beta}(t) - \mathfrak{K}\right) \\ +\left(\mathfrak{H} - \Theta\mathfrak{T}(t)\right)\mathcal{k}_0 \end{bmatrix}\mathcal{A}_\mathcal{v} \\ +\frac{1}{2}\mathcal{b}^2 d_1^{2\beta+2}\mathcal{A}_{d_1d_1} + \frac{1}{2}\begin{pmatrix} \mathcal{k}_1^2\left(1 - \mathfrak{T}(t)\right)^2 \\ +\zeta^2\mathcal{b}^2 d_1^{2\beta} \end{pmatrix}\mathcal{A}_{\mathcal{v}\mathcal{v}} \\ +\zeta\mathcal{b}^2 d_1^{2\beta+1}\mathcal{A}_{d_1\mathcal{v}} \end{matrix}\right\} = 0. \tag{10}$$

The first order maximizing condition of (10) is obtained by differentiating (10) with respect to ζ *and* $\mathfrak{T}$ as follows

$$\zeta^*(t) = -\frac{\left(a + \kappa\mathcal{b}^2 d_1^{2\beta}(t) - \mathfrak{K}\right)\mathcal{A}_\mathcal{v} + \mathcal{b}^2 d_1^{2\beta+1}\mathcal{A}_{d_1\mathcal{v}}}{\mathcal{b}^2 d_1^{2\beta}\mathcal{A}_{\mathcal{v}\mathcal{v}}}, \tag{11}$$

$$\mathfrak{T}^*(t) = 1 + \frac{\Theta\mathcal{k}_0}{\mathcal{k}_1^2}\frac{\mathcal{A}_\mathcal{v}}{\mathcal{A}_{\mathcal{v}\mathcal{v}}}. \tag{12}$$

Putting (11) and (12) into (10), we have

$$\left\{\begin{matrix} \mathcal{A}_t + d_1\left(a + \kappa\mathcal{b}^2 d_1^{2\beta}(t)\right)\mathcal{A}_{d_1}\mathcal{A}_\mathcal{v} \\ +\begin{bmatrix} (\mathfrak{K} - \aleph)\mathcal{v} \\ +(\mathfrak{H} - \Theta)\mathcal{k}_0 \end{bmatrix} + \frac{1}{2}\mathcal{b}^2 d_1^{2\beta+2}\left[\mathcal{A}_{d_1d_1} - \frac{\mathcal{A}_{d_1\mathcal{v}}^2}{\mathcal{A}_{\mathcal{v}\mathcal{v}}}\right] \\ -\frac{1}{2}\left[\frac{(a - \mathfrak{K})^2}{\mathcal{b}^2 d_1^{2\beta}} + \kappa^2\mathcal{b}^2 d_1^{2\beta} - \frac{\Theta\mathcal{k}_0}{\mathcal{k}_1^2}\right]\frac{\mathcal{A}_\mathcal{v}^2}{\mathcal{A}_{\mathcal{v}\mathcal{v}}} \\ -\left(a + \kappa\mathcal{b}^2 d_1^{2\beta}(t) - \mathfrak{K}\right)d_1\frac{\mathcal{A}_\mathcal{v}\mathcal{A}_{d_1\mathcal{v}}}{\mathcal{A}_{\mathcal{v}\mathcal{v}}} \end{matrix}\right\} = 0 \tag{13}$$

B. Optimal Investment Strategy and Proportional Reinsurance Strategy

Here, we consider an insurer with utility function exhibiting constant absolute risk aversion (CARA) different from the one in [20]. Our aim is to solve (13) under the CARA utility for the optimal value function and proceed to find the optimal investment strategy and proportional reinsurance strategy. We choose the exponential utility function similar to the one in [6].

We assume that the member takes an exponential utility

$$U(\mathcal{v}) = -\frac{1}{\eta}e^{\mathcal{v}\eta},\ \eta > 0. \tag{14}$$

The absolute risk averse of an insurer with the utility in (14) is constant. Hence, we conjecture a solution to (14) with the form below similar to [6]:

$$\begin{cases} \mathcal{A}(t, d_1, \mathcal{v}) = -\frac{1}{\eta}exp - \eta[x(t)(\mathcal{v} - \mathcal{y}(t)) + z(t, d_1)] \\ x(T) = 1, \mathcal{y}(T) = 0, z(T, d_1) = 0 \end{cases} \tag{15}$$

Differentiating (15) with respect to $t, d_1, \mathcal{v}$

$$\left.\begin{matrix} \mathcal{A}_t = -\eta\mathcal{A}[x_t(\mathcal{v} - \mathcal{y}(t)) - x\mathcal{y}_t + z_t], \\ \mathcal{A}_{d_1} = -\eta\mathcal{A}z_{d_1}, \mathcal{A}_\mathcal{v} = -\eta x\mathcal{A}, \mathcal{A}_{d_1\mathcal{v}} = \eta^2 x z_{d_1}\mathcal{A} \\ \mathcal{A}_{\mathcal{v}\mathcal{v}} = \eta^2 x^2\mathcal{A}, \mathcal{A}_{d_1d_1} = \left(\eta^2 z_{d_1}^2 - \eta z_{d_1d_1}\right)\mathcal{A}, \end{matrix}\right\} \tag{16}$$

Substituting (16) into (13), we have

$$\begin{cases} [x_t + x(\mathfrak{K} - \aleph)]\mathcal{v} \\ +\left[-\frac{x_t}{x}\mathcal{y} - \mathcal{y}_t + (\mathfrak{H} - \Theta)\mathcal{k}_0\right]x \\ +z_t + \mathfrak{K}d_1 z_{d_1} + \frac{1}{2}\mathcal{b}^2 d_1^{2\beta+2} z_{d_1} = 0 \\ +\frac{1}{2\eta}\left(\frac{(a - \mathfrak{K})^2}{\mathcal{b}^2 d_1^{2\beta}} + \kappa^2\mathcal{b}^2 d_1^{2\beta} - \frac{\Theta\mathcal{k}_0}{\mathcal{k}_1^2}\right) \end{cases} \tag{17}$$

Simplifying (17), we have

$$\begin{cases} x_t + x(\mathfrak{K} - \aleph) = 0 \\ x(T) = 1 \end{cases}, \tag{18}$$

$$\begin{cases} \mathcal{y}_t - (\mathfrak{K} - \aleph)\mathcal{y} - (\mathfrak{H} - \Theta)\mathcal{k}_0 = 0 \\ \mathcal{y}(T) = 0 \end{cases} \tag{19}$$

$$\begin{cases} z_t + \mathfrak{R} d_1 z_{d_1} + \frac{1}{2}\mathscr{b}^2 d_1^{2\beta+2} z_{d_1} \\ + \frac{1}{2\eta}\left(\frac{(a-\mathfrak{R})^2}{\mathscr{b}^2 d_1^{2\beta}} + \kappa^2 \mathscr{b}^2 d_1^{2\beta} - \frac{\Theta \mathscr{k}_0}{\mathscr{k}_1^2}\right) = 0 \\ z(T, d_1) = 0 \end{cases} \tag{20}$$

Solving (18) and (19), we have

$$x(t) = e^{(\mathfrak{R}-\aleph)(T-t)} \tag{21}$$

$$\mathscr{y}(t) = \frac{(\mathfrak{H}-\Theta)\mathscr{k}_0\left(e^{-(\mathfrak{R}-\aleph)(T-t)}-1\right)}{\mathfrak{R}} \tag{22}$$

Next, we assume a solution to (20) in the form

$$\begin{cases} z(t, d_1) = \mathcal{H}(t) + \mathcal{J}(t) d_1^{-2\beta} \\ \mathcal{H}(T) = 0, \mathcal{J}(T) = 0, \end{cases} \tag{23}$$

$$\begin{cases} z_t = \mathcal{H}_t + \mathcal{J}_t d_1^{-2\beta}, z_{d_1} = -2\beta \mathcal{J} d_1^{-2\beta-1}, \\ z_{d_1 d_1} = 2\beta(2\beta+1)\mathcal{J} d_1^{-2\beta-2} \end{cases} \tag{24}$$

Substituting (24) in (20) we have

$$\begin{cases} \mathcal{H}_t + \beta(2\beta+1)\mathcal{J}\mathscr{b}^2 \\ + d_1^{-2\beta}\left(\mathcal{J}_t - 2\mathfrak{R}\beta\mathcal{J} + \frac{(a-\mathfrak{R})^2}{2\eta \mathscr{b}^2} - \frac{\Theta \mathscr{k}_0}{2\eta \mathscr{k}_1^2} + \frac{\kappa^2 \mathscr{b}^2 d_1^{4\beta}}{2\eta}\right) = 0 \\ \mathcal{H}(T) = 0, \mathcal{J}(T) = 0 \end{cases} \tag{25}$$

Splitting (25) into two parts, we have

$$\begin{cases} \mathcal{H}_t + \beta(2\beta+1)\mathcal{J}\mathscr{b}^2 = 0 \\ \mathcal{H}(T) = 0 \end{cases} \tag{26}$$

$$\begin{cases} \mathcal{J}_t - 2\mathfrak{R}\beta\mathcal{J} + \frac{(a-\mathfrak{R})^2}{2\eta \mathscr{b}^2} - \frac{\Theta \mathscr{k}_0}{2\eta \mathscr{k}_1^2} + \frac{\kappa^2 \mathscr{b}^2 d_1^{4\beta}}{2\eta} = 0 \\ \mathcal{J}(T) = 0 \end{cases} \tag{27}$$

Solving (26) and (27), we have

$$\mathcal{J}(t) = \frac{1}{4\mathfrak{R}\beta\eta}\left(\frac{(a-\mathfrak{R})^2}{\mathscr{b}^2} - \frac{\Theta \mathscr{k}_0}{\mathscr{k}_1^2} + \kappa^2 \mathscr{b}^2 d_1^{4\beta}\right)\left[1 - e^{2\mathfrak{R}\beta(t-T)}\right] \tag{28}$$

$$\mathcal{H}(t) = \left\{\frac{(2\beta+1)}{4\mathfrak{R}\eta}\begin{pmatrix} \frac{(a-\mathfrak{R})^2}{\mathscr{b}^2} - \frac{\Theta \mathscr{k}_0}{\mathscr{k}_1^2} \\ + \kappa^2 \mathscr{b}^2 d_1^{4\beta} \end{pmatrix}\right\}\begin{bmatrix} \frac{1}{2\mathfrak{R}\beta}\left(e^{2\mathfrak{R}\beta(t-T)} - 1\right) \\ + (T-t) \end{bmatrix} \tag{29}$$

Substituting (28) and (29) into (20), we have

$$z(t, d_1)) = \begin{cases} \left\{\frac{(2\beta+1)}{4\mathfrak{R}\eta}\begin{pmatrix} \frac{(a-\mathfrak{R})^2}{\mathscr{b}^2} - \frac{\Theta \mathscr{k}_0}{\mathscr{k}_1^2} \\ + \kappa^2 \mathscr{b}^2 d_1^{4\beta} \end{pmatrix}\right\}\begin{bmatrix} \frac{1}{2\mathfrak{R}\beta}\begin{pmatrix} e^{2\mathfrak{R}\beta(t-T)} \\ -1 \end{pmatrix} \\ + (T-t) \end{bmatrix} \\ + \frac{d_1^{-2\beta}}{4\mathfrak{R}\beta\eta}\begin{pmatrix} \frac{(a-\mathfrak{R})^2}{\mathscr{b}^2} - \frac{\Theta \mathscr{k}_0}{\mathscr{k}_1^2} \\ + \kappa^2 \mathscr{b}^2 d_1^{4\beta} \end{pmatrix}\left[1 - e^{2\mathfrak{R}\beta(t-T)}\right] \end{cases} \tag{30}$$

Result1. The optimal value function $\mathcal{A}(t, d_1, \mathscr{v})$ is given as

$$\mathcal{A}(t, d_1, \mathscr{v}) = \begin{cases} -\frac{1}{\eta} exp\begin{pmatrix} -\eta\begin{bmatrix} e^{(\mathfrak{R}-\aleph)(T-t)} \\ \times\left(\mathscr{v} - \frac{(\mathfrak{H}-\Theta)\mathscr{k}_0\left(e^{-(\mathfrak{R}-\aleph)(T-t)}-1\right)}{\mathfrak{R}}\right) \end{bmatrix} \\ + \left\{\frac{(2\beta+1)}{4\mathfrak{R}\eta}\begin{pmatrix} \frac{(a-\mathfrak{R})^2}{\mathscr{b}^2} - \frac{\Theta \mathscr{k}_0}{\mathscr{k}_1^2} \\ + \kappa^2 \mathscr{b}^2 d_1^{4\beta} \end{pmatrix}\right\}\begin{bmatrix} \frac{1}{2\mathfrak{R}\beta}\left(e^{2\mathfrak{R}\beta(t-T)} - 1\right) \\ + (T-t) \end{bmatrix} \\ + \frac{d_1^{-2\beta}}{4\mathfrak{R}\beta\eta}\left(\frac{(a-\mathfrak{R})^2}{\mathscr{b}^2} - \frac{\Theta \mathscr{k}_0}{\mathscr{k}_1^2} + \kappa^2 \mathscr{b}^2 d_1^{4\beta}\right)\left[1 - e^{2\mathfrak{R}\beta(t-T)}\right] \end{pmatrix} \end{cases} \tag{31}$$

Proof. By substituting (21), (22) and (30) into (15), we obtain the result 1.

Result2. The optimal investment strategy with administrative charges is given as

$$\zeta^*(t) = \begin{cases} \frac{(a-\mathfrak{R})e^{(\mathfrak{R}-\aleph)(t-T)}}{\eta \mathscr{b}^2 d_1^{2\beta}}\left[1 + \frac{(a-\mathfrak{R})}{2\mathfrak{R}}\left(1 - e^{2\mathfrak{R}\beta(t-T)}\right)\right] \\ + \frac{\kappa e^{(\mathfrak{R}-\aleph)(t-T)}}{\eta}\left[1 + \frac{\kappa \mathscr{b}^2 d_1^{2\beta}}{2\mathfrak{R}}\left(1 - e^{2\mathfrak{R}\beta(t-T)}\right)\right] \\ - \frac{\Theta \mathscr{k}_0}{2\mathfrak{R}\eta d_1^{2\beta} \mathscr{k}_1^2}\left(1 - e^{2\mathfrak{R}\beta(t-T)}\right) \end{cases} \tag{32}$$

Proof. By substituting (16) into (11), we have

$$\zeta^*(t) = \frac{1}{x\eta \mathscr{b}^2 d_1^{2\beta}}\left(a + \kappa \mathscr{b}^2 d_1^{2\beta} - \mathfrak{R}\right) - \frac{d_1 z_{d_1}}{x}, \tag{33}$$

From (24) and (30), we have

$$z_{d_1} = \frac{-d_1^{-2\beta-1}}{2\mathfrak{R}\eta}\left(\frac{(a-\mathfrak{R})^2}{\mathscr{b}^2} - \frac{\Theta \mathscr{k}_0}{\mathscr{k}_1^2} + \kappa^2 \mathscr{b}^2 d_1^{4\beta}\right)\left[1 - e^{2\mathfrak{R}\beta(t-T)}\right] \tag{34}$$

By substituting (21) and (34) into (33), (30) is proved.

Result3. The proportional reinsurance strategy with administrative charges is given as

$$\mathfrak{T}^*(t) = 1 - \frac{\Theta \mathscr{k}_0}{\eta \mathscr{k}_1^2} e^{(\mathfrak{R}-\aleph)(t-T)} \tag{35}$$

Proof. By substituting (16) into (12), we have

$$\mathfrak{T}^*(t) = 1 - \frac{\Theta \mathscr{k}_0}{\mathscr{k}_1^2}\frac{1}{x\eta}. \tag{36}$$

By substituting (21) into (36), (35) is proved.

Remark1. If the modification factor $\kappa = 0$, the optimal investment strategy in (32) reduces to a case where the price process of the risky asset is modelled by CEV model and is given thus;

$$\zeta^*(t) = \begin{cases} \frac{(a-\mathfrak{R})e^{(\mathfrak{R}-\aleph)(t-T)}}{\eta \mathscr{b}^2 d_1^{2\beta}}\left[1 + \frac{(a-\mathfrak{R})}{2\mathfrak{R}}\left(1 - e^{2\mathfrak{R}\beta(t-T)}\right)\right] \\ - \frac{\Theta \mathscr{k}_0}{2\mathfrak{R}\eta d_1^{2\beta} \mathscr{k}_1^2}\left(1 - e^{2\mathfrak{R}\beta(t-T)}\right) \end{cases} \tag{37}$$

Remark2. We observed that the modification factor does not affect the proportional reinsurance strategy.

IV. Numerical Simulations

In this section, the numerical simulations for an insurer's and reinsurer's strategy are presented with parameters on the models. The following basic parameters adapted from [29] and [30] are used in the analysis unless otherwise stated; $a = 0.5, \eta = 0.3, T = 20, \mathfrak{K} = 0.3, \Theta = 2.0, \mathcal{k}_0 = 1.5, , \mathcal{k}_1 = 1.0\ d_1 = 10, \beta = -1, \aleph = 0.1, \eta = 0.1, \mathcal{b} = 0.15, \kappa = 0.01\ and\ t = 0{:}5{:}20.$

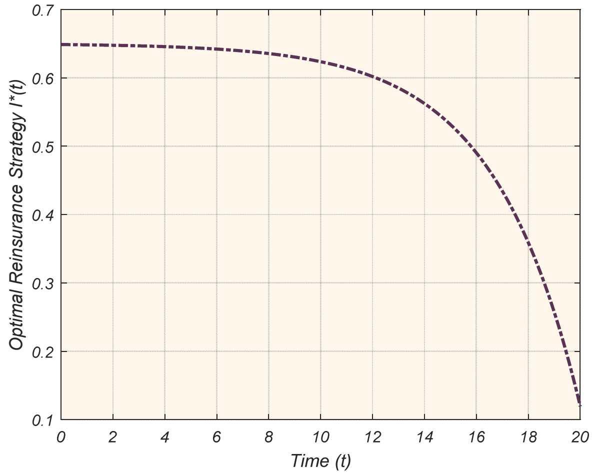

Fig. 1 The Evolution of $\mathfrak{T}^*(t)$ with time

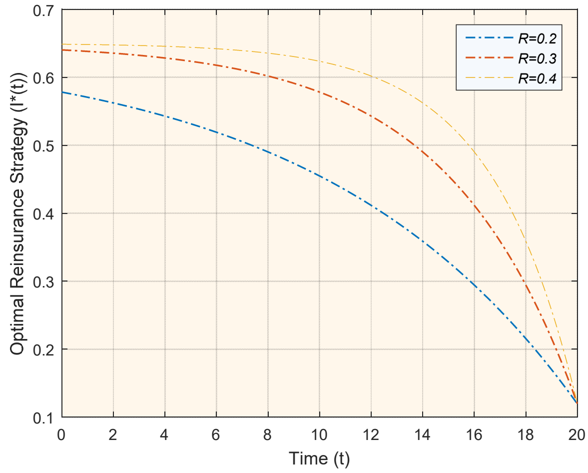

Fig. 2 The impact of different risk free interest rate ($\mathfrak{K}$) on $\mathfrak{T}^*(t)$

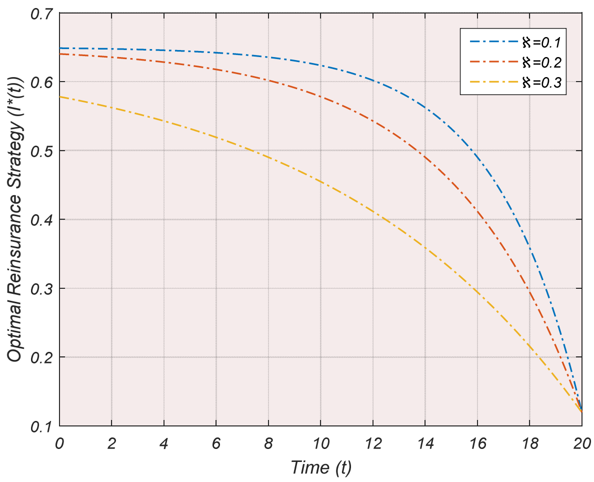

Fig. 3 The impact of different administrative charges ($\aleph$) on $\mathfrak{T}^*(t)$

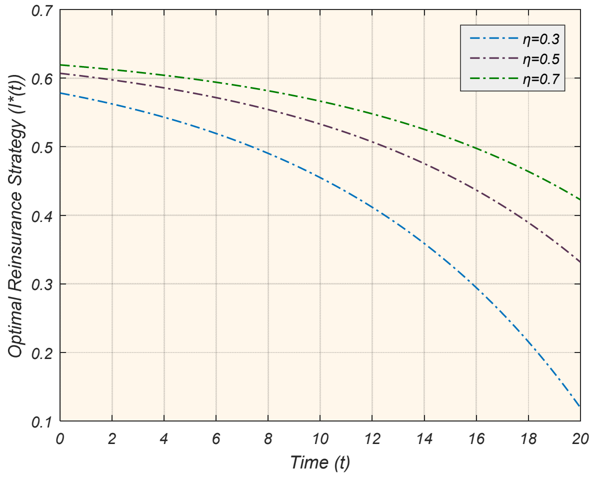

Fig. 4 The impact of risk averse coefficient (η) on $\mathfrak{T}^*(t)$

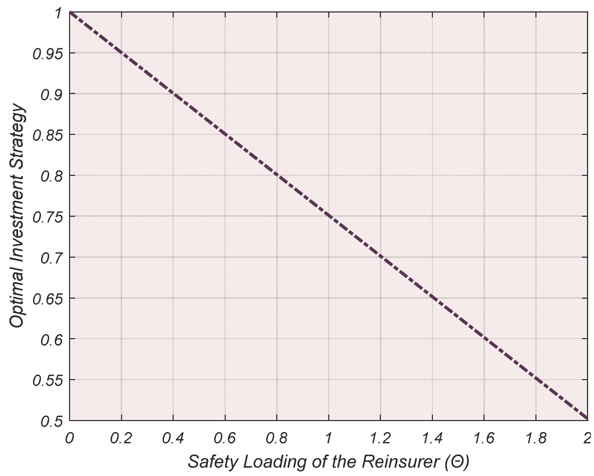

Fig. 5 The impact of reinsurer's safety loading on $\mathfrak{T}^*(t)$

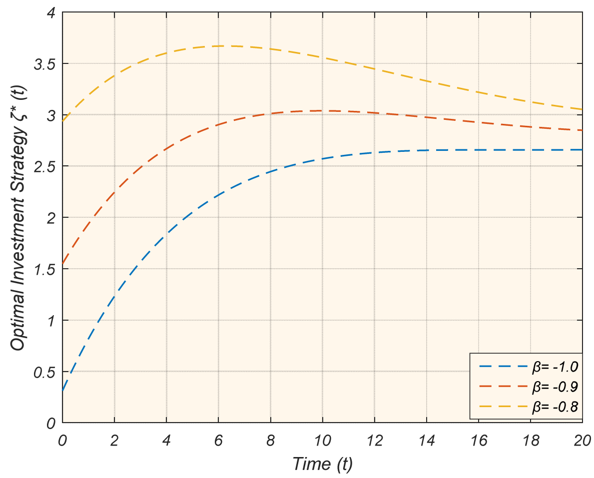

Fig. 6 The effect of elasticity parameters (β) on $\zeta^*(t)$

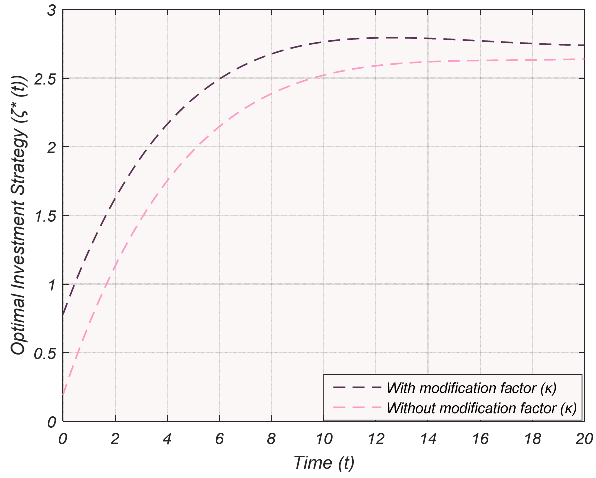

Fig. 7 The impact of modification factor (κ) on $\zeta^*(t)$

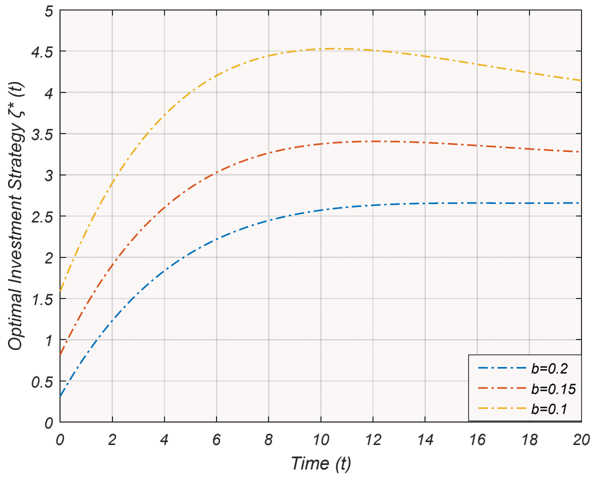

Fig. 8 The impact of instantaneous volatility ($\mathscr{b}$) on $\zeta^*(t)$

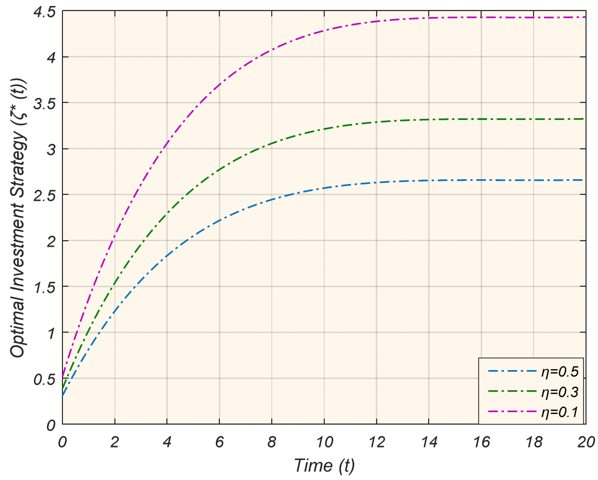

Fig. 9 The impact of risk averse coefficient (η) on $\zeta^*(t)$

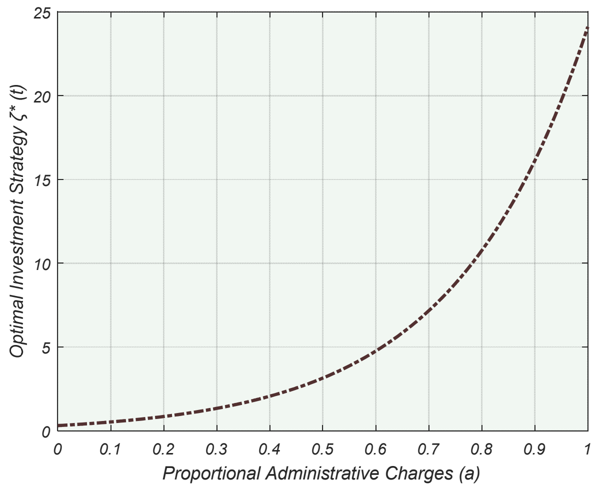

Fig. 10 The effect of administrative charges ($\aleph$) on $\zeta^*(t)$

V. Discussion

In this section, we discuss the effect of some sensitive parameters on the insurer's and reinsurer's investment plan. Fig. 1 shows the relationship between the reinsurer's investment strategy with time where we observe from the graph that the reinsurer's strategy is a decreasing function of time. This is so because the reinsurer cannot afford to fail during investment since it has the obligation of paying a certain proportion to the insurer in case of eventuality. Fig. 2 shows the effect of risk-free interest rate on the reinsurance strategy. It is observed that the reinsurer will invest more in risk free asset where there is higher and attractive interest rate and vice versa. Fig. 3 shows the effect of the proportional administrative charges on the reinsurer's investment plan where we observed that the both the administrative charges and the insurer's investment plan is inversely proportional to each other. The implication here is that the reinsurer will be attracted to investors with lesser proportion of administrative charges. Fig. 4 shows the effect of risk averse coefficient of the reinsurer where we observed that the insurer's investment plan increases as the risk averse coefficient of the insurer decreases. Hence, a reinsurer with low risk averse coefficient would love to invest more in the risky asset with little or no fear of making loss during the investment. Fig. 5 shows a linear relationship between the optimal reinsurance strategy and the reinsurer's safety loading. We observed that the optimal reinsurance strategy decreases as the safety loading of the reinsurer increases. The implication of this is that bigger values of the safety loading lead to smaller values of the optimal reinsurer strategy. Therefore, for a reinsurer to maintain a stable income, it is required that the reinsurer buys fewer insurance policies.

In Fig. 6, the relationship between the insurer's investment strategy and the elasticity parameter is presented. The graph shows that as the elasticity parameter become highly negative, the insurer's investment proportion in the risky asset decreases; this is due to the fact that when β is highly negative, the market tends to be highly volatile hence may bring discouragement and fear in the mind of the insurer from investing larger proportion of his wealth in the risky asset. Fig. 7 shows the effect of the modification factor in the MCEV model on the insurer's investment plan; it is observed that the modification factor helps the insurer to have a good knowledge and understanding about the true state of the behavior and the fluctuations in the market prices. Therefore, with the presence of modification factor, the member is more cautious and this may lead to reduction in risky investment.

Figs. 8 and 9 show the effect of instantaneous volatility of the risky asset and risk averse coefficient of the insurer where we observed that the insurer's investment plan increases as the instantaneous volatilities of the risky asset and risk averse coefficient of the insurer decrease. Therefore, an insurer with lower risk averse coefficient will like to invest more in the risky asset with little or no fear of making loss during the investment. Also, an insurer will want to do little with highly volatile investment for the fear of losing their wealth in the investment process. Fig. 10 shows the effect of the proportional administrative charges on the insurer's investment plan where we observed that the both the administrative charges and the insurer's investment plan is directly proportional to each other. Therefore, higher proportion of investment in risky asset attracts higher administrative fee due to the risk involvement in managing such businesses.

VI. Conclusion

In this paper, the insurer's and reinsurer's strategies were studied under the M-CEV model and proportional administrative charges. The insurer considered investment in one risky asset and one risk free asset where the risky asset is modeled based on the M-CEV process which is an extension of CEV. Next, a nonlinear PDE in the form of HJB equation was obtained using dynamic programming approach. Using power transformation technique and variable change, the explicit solutions of the optimal investment strategy and optimal reinsurance strategy were obtained. Finally, some numerical simulations of some sensitive parameters were obtained and discussed in details where we observed that the modification factor only affects the optimal investment strategy and not the reinsurance strategy for an insurer with exponential utility function.

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