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Modelling an Investment Portfolio with Mandatory and Voluntary Contributions under M-CEV Model

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Abstract**—**In this paper, the mandatory contribution, additional voluntary contribution (AVC) and administrative charges are merged together to determine the optimal investment strategy (OIS) for a pension plan member (PPM) in a defined contribution (DC) pension scheme under the modified constant elasticity of variance (M-CEV) model. We assume that the voluntary contribution is a stochastic process and a portfolio consisting of one risk free asset and one risky asset modeled by the M-CEV model is considered. Also, a stochastic differential equation consisting of PPM's monthly contributions, voluntary contributions and administrative charges is obtained. More so, an optimization problem in the form of Hamilton Jacobi Bellman equation which is a nonlinear partial differential equation is obtained. Using power transformation and change of variables method, an explicit solution of the OIS and the value function are obtained under constant absolute risk averse (CARA). Furthermore, numerical simulations on the impact of some sensitive parameters on OIS were discussed extensively. Finally, our result generalizes some existing result in the literature.

Keywords**—**DC pension fund, modified constant elasticity of variance, optimal investment strategies, voluntary contribution, administrative charges.

I. Introduction

THE concept of voluntary contributions, as contained in the Nigerian pension reform act 2004, gives PPMs the liberty to invest AVC into their retirement savings accounts (RSA) with the aim of increasing their terminal wealth at the time of retirement. Also, one interesting thing about the act is that it empowers PPMs to have access to a certain percentage of the accumulated voluntary contribution before retirement unlike the mandatory contributions which is only accessible after retirement or may be a certain percentage in the case of loss of job [1]. In this paper, the mandatory and AVC in a DC pension plan are combined together to study the OIS for PPM in the presence of administrative charges under the M-CEV. The voluntary contributions may be a fix amount or stochastic [22], [23].

The OIS is the proportion of the PPM's wealth invested in different assets with an expectation of obtaining maximal returns with minimal risk. The OIS in a DC pension plan under the CEV model have been studied by several authors under different assumptions such as [2]-[7]. References [8] and [9] studied OIS for a DC pension fund under affine interest rate which is a combination of Vasicek model and Cox Ingersoll model. Reference [10] studied OIS in the presence of a minimum guarantee. Recently, the OIS with return of premium clause have been studied by many authors, including [11]-[17].

The M-CEV process is an enhanced stochastic volatility model (SVM) and an extension of the CEV model. The model was developed and used first by [18]. The need to select a SVM that best fits the volatile nature of the financial market is very important in the study of OIS. There are different volatility models used in modelling the price of the risky assets; some of which include the CEV model, Heston's volatility, M-CEV model, jump diffusion process etc. Some attractive features of the M-CEV model include its ability to capture the volatility smile effects of the stock price, improve analytically tractable strategies and also its probability can touch zero unlike the Geometric Brownian Motion whose probability is always positive. Some authors have used the M-CEV process to model the price of the risky assets, including [18], where the M-CEV model was used in developing a consistent and hedging pricing process by introducing a modification factor and they showed that there is no equivalent risk or neutral pricing measure hence, the conventional risk neutral pricing methodology fails. Also, the benchmark method was used to set up a consistent pricing and hedging framework and showed that nonnegative price process and benchmark duplicate the contingent claim. In [19], the explicit solution of the OIS under M-CEV model was obtained in terms of confluent hyper-geometric functions by using the Laplace transformation method and application of algorithmic tradition. In [20], the price of the risky asset was modelled using M-CEV and Ornstein-Uhlenbeck processes. In their work, they showed that if there is no correlation between the Brownian motions, PPM's OIS are less than the PPM's OIS when the Brownian motions correlate. Also, [21] determined the explicit solution of the OIS with return of premium clause for PPM when the price of the risky asset follows the M-CEV model.

Researches in respect to AVC include [22] where the authors studied the impact of the AVC on the OIS under the CEV model. They observed that the OIS is a decreasing function of the AVC. Reference [23] studied the OIS under inflammatory market with minimum guarantee; in their work, the PPM contributes some extra fund to amortize the pension fund. References [8], [23], and [24] studied the effect of extra contribution on the OIS in a DC pension with stochastic salary under affine interest model when the extra contribution is

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constant and stochastic.

The main contribution in this work is that we merge AVC, mandatory contribution, and administrative fee to study the OIS under M-CEV model. We got our motivation from [20] and [22]. There are some differences between our work and them. In this paper, we used the M-CEV process similar to the one in [20] to model the risky asset different from the CEV process used in [22]. Also, we introduce the voluntary contributions and administrative charges which were not considered in [20] and [22].

II. Financial Market Model

We consider a complete and frictionless financial market which is open continuously for a given time interval $0\leq t \leq T$, where T is the investment period. We also consider portfolio comprising of one risk free asset and one risky asset where the price process of the risky asset follows the M-CEV model. Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a complete probability space where Ω represents a real space, $\mathcal{P}$ a probability measure and $\mathcal{F}$ represents the filtration and denotes the information generated by the Brownian motion $\{\mathcal{B}_0(t), \mathcal{B}_1(t)\}$ where $\mathcal{B}_0(t)$ and $\mathcal{B}_1(t)$ are correlated such that $d\mathcal{B}_0(t)d\mathcal{B}_1(t) = \rho dt, \rho\epsilon[-1,1]$.

Let $A_0(t)$ denote the price process of the risk-free asset whose model is represented thus

$$\frac{dA_0(t)}{A_0(t)} = rdt, \tag{1}$$

where $r > 0$ is the risk free interest of the risk free asset.

Let $\mathcal{W}(t)$ denote the price process of the risky asset which follows the M-CEV similar to the one in [20] and [21] whose model is represented thus

$$\frac{d\mathcal{W}(t)}{\mathcal{W}(t)} = \left(\upsilon + \kappa\xi^2\mathcal{W}^{2\beta}\right)dt + \xi\mathcal{W}^{\beta}\mathcal{B}_0(t). \tag{2}$$

where υ an expected instantaneous expected rate of return of the risky asset and satisfies the general condition $\upsilon - r > 0$, $\kappa > 0$, is the modification factor, ξ is the instantaneous volatility of the risky asset and $\beta < 0$ is the elasticity parameter of the risky asset. If $\kappa = 0$, the model in (2) reduces to that of a CEV model, also, if $\kappa = 0$, and $\beta = 0$, the model in (2) reduces to that of geometric Brownian motion.

In a DC pension system, members have the obligation to contribute a certain fraction of their earnings to their RSA monthly; secondly, from the Nigerian Pension Reform Act of 2004 [1], PPMs are at liberty to contribute AVC into their RSA. We assume that the voluntary contribution rate is stochastic; the contribution rate of the PPM can be modeled similar to [22] as thus

$$dC = cdt + c_0 d\mathcal{B}_1(t) \tag{3}$$

where c_0 and c represent the voluntary and mandatory contributions of the PPM.

III. Main Results

A. Optimization Problem

Let π represent the OIS of the PPM and we define the utility attained by the member from a given state η at time t as

$$\mathcal{H}_\pi(t, \mathcal{w}, \eta) = E_\tau\left[\, U\left(\mathcal{N}(T)\right) \mid \mathcal{W}(t) = \mathcal{w}, \mathcal{N}(t) = \eta \,\right] \tag{4}$$

where t, is the time, $\mathcal{w}$ the price of the risky asset and η is the wealth of the PPM. Next, we proceed to find the optimal value function $\mathcal{H}_{\pi^*}$ and OIS π^* given as

$$\mathcal{H}(t, \mathcal{w}, \eta) = \sup_{\tau} \mathcal{H}_\pi(t, \mathcal{w}, \eta) \quad \text{and} \quad \pi^* \tag{5}$$

Respectively such that

$$\mathcal{H}_{\pi^*}(t, \mathcal{w}, \eta) = \mathcal{H}(t, \mathcal{w}, \eta). \tag{6}$$

B. Formulation PPM's Wealth with Stochastic Rate of Voluntary Contribution

Let $\mathcal{N}(T)$ denote the PPM's wealth at $t \in [0, T]$, π the fraction of the PPM's wealth invested in the risky asset $\mathcal{W}(t)$ and $1 - \pi$, the fraction of investment in the risk free asset. If the pension fund administrator charges an administrative fee at a rate α, the dynamics of the pension wealth is given as

$$\begin{cases} d\mathcal{N}(t) = \begin{pmatrix} \pi\mathcal{N}(t)\frac{d\mathcal{W}(t)}{\mathcal{W}(t)} + (1-\pi)\mathcal{N}(t)\frac{dS_0(t)}{S_0(t)} \\ +dC - \alpha\mathcal{N}(t)dt \end{pmatrix} \\ \mathcal{N}(0) = \eta_0 \end{cases} \tag{7}$$

Substituting (1), (2) and (3) into (7), we have

$$\begin{cases} d\mathcal{N}(t) = \begin{bmatrix} \begin{pmatrix} \mathcal{N}(t)\left(\pi\left(\upsilon - r + \kappa\xi^2\mathcal{W}^{2\beta}\right) + r - \alpha\right) \\ +c_0 \end{pmatrix} dt \\ \pi\mathcal{N}(t)\xi\mathcal{W}^{\beta}\mathcal{B}_0(t) + c_0 d\mathcal{B}_1(t) \end{bmatrix} \\ \mathcal{N}(0) = \eta_0 \end{cases} \tag{8}$$

The HJB equation associated with (8) is

$$\begin{pmatrix} \mathcal{H}_t + \left(\upsilon\mathcal{w} + \kappa\xi^2\mathcal{w}^{2\beta+1}\right)\mathcal{H}_{\mathcal{w}} + \frac{1}{2}\xi^2\mathcal{w}^{2\beta+2}\mathcal{H}_{\mathcal{w}\mathcal{w}} \\ +\left((r-\alpha)\eta + c_0\right)\mathcal{H}_\eta + \frac{1}{2}c_0^2\mathcal{H}_{\eta\eta} + \rho c_0\xi\mathcal{w}^{\beta+1}\mathcal{H}_{\mathcal{w}\eta} + \\ \sup\begin{Bmatrix} \frac{1}{2}\left(\pi^2\eta^2\xi^2\mathcal{w}^{2\beta} + \pi\eta\rho c_0\xi\mathcal{w}^{\beta}\right)\mathcal{H}_{\eta\eta} \\ +\pi\eta\left(\upsilon - r + \kappa\xi^2\mathcal{W}^{2\beta}\right)\mathcal{H}_\eta + \pi\eta\xi^2\mathcal{w}^{2\beta+1}\mathcal{H}_{\mathcal{w}\eta} \end{Bmatrix} \end{pmatrix} = 0. \tag{9}$$

Differentiating (9) with respect to π, and solving for π, we obtain the first order maximizing condition

$$\pi^* = -\frac{\left[\rho c_0\xi\mathcal{w}^{\beta}\mathcal{H}_{\eta\eta} + \left(\upsilon - r + \kappa\xi^2\mathcal{W}^{2\beta}\right)\mathcal{H}_\eta + \xi^2\mathcal{w}^{2\beta+1}\mathcal{H}_{\mathcal{w}\eta}\right]}{\eta\xi^2\mathcal{w}^{2\beta}\mathcal{H}_{\eta\eta}}, \tag{10}$$

Substituting (10) into (9), we have

$$\begin{cases} \mathcal{H}_t + \left(\upsilon w + \kappa\xi^2 w^{2\beta+1}\right)\mathcal{H}_w \\ +\left((r-\alpha)\eta + c_0 - \frac{\rho c_0(\upsilon-r)}{2\xi w^\beta} - \frac{\kappa\rho c_0\xi w^\beta}{2}\right)\mathcal{H}_\eta + \frac{1}{2}c_0^2\mathcal{H}_{\eta\eta} \\ -\frac{1}{2}\left(\frac{(\upsilon-r)^2}{\xi^2 w^{2\beta}}+\kappa^2\xi^2 w^{2\beta}\right)\frac{\mathcal{H}_\eta^2}{\mathcal{H}_{\eta\eta}} + \frac{1}{2}\xi^2 w^{2\beta+2}\left(\mathcal{H}_{ww} - \frac{\mathcal{H}_{w\eta}^2}{\mathcal{H}_{\eta\eta}}\right) \\ \left((\upsilon-r)w+\kappa\xi^2 w^{2\beta+1}\right)\frac{\mathcal{H}_\eta\mathcal{H}_{w\eta}}{\mathcal{H}_{\eta\eta}}\end{cases} = 0. \quad (11)$$

where $\mathcal{H}(t,w,\eta)=U(\eta)$ and $U(\eta)$is the marginal utility of the PPM. Next, we solve (11) for $\mathcal{H}$, using power transformation and change of variable technique.

C. OIS for a PPM with CARA

We assume that the member takes an exponential utility

$$U(\eta) = -\frac{1}{\zeta}e^{-\zeta\eta}, \zeta > 0. \quad (12)$$

The absolute risk averse of a PPM with the utility in (12) is constant. Hence, we conjecture a solution to (12) similar to one in [22] as follows:

$$\begin{cases}\mathcal{H}(t,w,\eta) = -\frac{1}{\zeta}exp - \zeta\left[p(t)\left(\eta - b(t)\right) + q(t,w)\right] \\ q(T,w)=0, p(T)=1, b(T)=0\end{cases} \quad (13)$$

Differentiating (13) with respect to t, w, η

$$\left.\begin{array}{c}\mathcal{H}_t = -\zeta\mathcal{H}[p_t(\eta - b(t)) - p b_t + q_t], \\ \mathcal{H}_w = -\zeta\mathcal{H}q_w, \mathcal{H}_\eta = -\zeta p\mathcal{H}, \mathcal{H}_{w\eta} = \zeta^2 p q_w\mathcal{H} \\ \mathcal{H}_{\eta\eta} = \zeta^2 p^2\mathcal{H}, \mathcal{H}_{ww} = (\zeta^2 q_w^2 - \zeta q_{ww})\mathcal{H},\end{array}\right\} \quad (14)$$

Substituting (14) into (11), we have

$$\begin{cases}[p_t + p(r-\alpha)]\eta \\ +\left[c_0 - \frac{\rho c_0(\upsilon-r)}{2\xi w^\beta} - \frac{\kappa\rho c_0\xi w^\beta}{2} - \frac{p_t}{p}b - b_t - \frac{1}{2}c_0^2\zeta p\right]p \\ +q_t + rwq_w + \frac{1}{2}\xi^2 w^{2\beta+2}q_{ww} + \frac{1}{2\zeta}\left(\frac{(\upsilon-r)^2}{\xi^2 w^{2\beta}} + \kappa^2\xi^2 w^{2\beta}\right)\end{cases} = 0 \quad (15)$$

By simplifying (15), we have

$$\begin{cases}p_t + p(r-\alpha) = 0 \\ p(T) = 1\end{cases}, \quad (16)$$

$$\begin{cases}b_t - (r-\alpha)b - c_0 + \frac{\rho c_0(\upsilon-r)}{2\xi w^\beta} + \frac{\kappa\rho c_0\xi w^\beta}{2} + \frac{1}{2}c_0^2\zeta p = 0 \\ b(T) = 0\end{cases} \quad (17)$$

$$\begin{cases}q_t + rwq_w + \frac{1}{2}\xi^2 w^{2\beta+2}q_{ww} + \frac{1}{2\zeta}\left(\frac{(\upsilon-r)^2}{\xi^2 w^{2\beta}} + \kappa^2\xi^2 w^{2\beta}\right) = 0 \\ q(T,w) = 0\end{cases} \quad (18)$$

Solving (16) and (17), we have

$$p(t) = e^{(r-\alpha)(T-t)} \quad (19)$$

$$b(t) = \begin{pmatrix}\frac{c_0\left(e^{-r(T-t)}-1\right)}{r}\left(1 - \frac{\rho(\upsilon-r)}{2\xi w^\beta} - \frac{\kappa\rho\xi w^\beta}{2}\right) \\ + \frac{\zeta c_0^2 e^{r(T-t)}\left(e^{-\alpha(T-t)}-1\right)}{4r}\end{pmatrix} \quad (20)$$

Next, we conjecture a solution to (18) in the following form

$$\begin{cases}q(t,w) = \mathcal{E}(t) + \mathcal{F}(t)w^{-2\beta} \\ \mathcal{E}(T) = 0, \mathcal{F}(T) = 0,\end{cases} \quad (21)$$

$$\begin{cases}q_t = \mathcal{E}_t + \mathcal{F}_t w^{-2\beta}, q_w = -2\beta\mathcal{F}w^{-2\beta-1}, \\ q_{ww} = 2\beta(2\beta+1)\mathcal{F}w^{-2\beta-2}.\end{cases} \quad (22)$$

Substituting (22) in (18) we have

$$\begin{cases}\mathcal{E}_t + \beta(2\beta+1)\mathcal{F}\xi^2 \\ +w^{-2\beta}\left(\mathcal{F}_t - 2r\beta\mathcal{F} + \frac{(\upsilon-r)^2}{2\zeta\xi^2} + \kappa^2\xi^2 w^{4\beta}\right) = 0 \\ \mathcal{E}(T) = 0, \mathcal{F}(T) = 0\end{cases} \quad (23)$$

Decomposing (23) into two parts, we have

$$\begin{cases}\mathcal{E}_t + \beta(2\beta+1)\mathcal{F}\xi^2 = 0 \\ \mathcal{E}(T) = 0\end{cases} \quad (24)$$

$$\begin{cases}\mathcal{F}_t - 2r\beta\mathcal{F} + \frac{(\upsilon-r)^2}{2\zeta\xi^2} + \kappa^2\xi^2 w^{4\beta} = 0 \\ \mathcal{F}(T) = 0\end{cases} \quad (25)$$

Solving (24) and (25), we have

$$\mathcal{F}(t) = \frac{1}{4r\beta\zeta}\left(\frac{(\upsilon-r)^2}{\xi^2} + \kappa^2\xi^2 w^{4\beta}\right)\left[1 - e^{2r\beta(t-T)}\right] \quad (26)$$

$$\mathcal{E}(t) = \left\{\frac{(2\beta+1)}{4r\zeta}\left(\frac{(\upsilon-r)^2}{\xi^2} + \kappa^2\xi^2 w^{4\beta}\right)\left[\begin{array}{c}\frac{1}{2r\beta}\left(e^{2r\beta(t-T)} - 1\right) \\ +(T-t)\end{array}\right]\right. \quad (27)$$

Substituting (26) and (27) into (21), we have

$$q(t,w)) = \begin{cases}\left\{\frac{(2\beta+1)}{4r\zeta}\begin{pmatrix}\frac{(\upsilon-r)^2}{\xi^2} \\ +\kappa^2\xi^2 w^{4\beta}\end{pmatrix}\left[\begin{array}{c}\frac{1}{2r\beta}\left(e^{2r\beta(t-T)} - 1\right) \\ +(T-t)\end{array}\right]\right. \\ + \frac{w^{-2\beta}}{4r\beta\zeta}\left(\frac{(\upsilon-r)^2}{\xi^2 w^{2\beta}} + \kappa^2\xi^2 w^{2\beta}\right)\left[1 - e^{2r\beta(t-T)}\right]\end{cases} \quad (28)$$

Result1. The optimal value function $\mathcal{H}(t,w,\eta)$ is given as

$$\mathcal{H}(t,w,\eta) = \left\{-\frac{1}{\zeta}exp\left(\begin{array}{c}-\zeta\left[\begin{array}{c}e^{(r-\alpha)(T-t)} \\ \times\left(\eta - \begin{pmatrix}\frac{c_0\left(e^{-r(T-t)}-1\right)}{r}\left(1 - \frac{\rho(\upsilon-r)}{2\xi w^\beta} - \frac{\kappa\rho\xi w^\beta}{2}\right) \\ + \frac{\zeta c_0^2 e^{r(T-t)}\left(e^{-\alpha(T-t)}-1\right)}{4r}\end{pmatrix}\right)\end{array}\right] \\ + \begin{cases}\left\{\frac{(2\beta+1)}{4r\zeta}\begin{pmatrix}\frac{(\upsilon-r)^2}{\xi^2} \\ +\kappa^2\xi^2 w^{4\beta}\end{pmatrix}\left[\begin{array}{c}\frac{1}{2r\beta}\left(e^{2r\beta(t-T)} - 1\right) \\ +(T-t)\end{array}\right]\right. \\ + \frac{w^{-2\beta}}{4r\beta\zeta}\left(\frac{(\upsilon-r)^2}{\xi^2 w^{2\beta}} + \kappa^2\xi^2 w^{2\beta}\right)\left[1 - e^{2r\beta(t-T)}\right]\end{cases}\end{array}\right)\right. \quad (29)$$

Proof. By substituting (19), (20) and (28) into (13), we obtain the Result 1.

Result 2. The OIS with administrative charges stochastic voluntary contribution is given as

$$\pi^* = \begin{cases} \frac{(\upsilon-r)e^{(r-\alpha)(t-T)}}{\zeta\eta\xi^2 w^{2\beta}}\left[1+\frac{(\upsilon-r)}{2r}\left(1-e^{2r\beta(t-T)}\right)\right] \\ +\frac{\kappa e^{(r-\alpha)(t-T)}}{\zeta\eta}\left[1+\frac{\kappa\xi^2 w^{2\beta}}{2r}\left(1-e^{2r\beta(t-T)}\right)\right] \\ -\frac{\rho c_0}{\eta\xi w^{\beta}} \end{cases} \tag{30}$$

Proof. By substituting (14) into (10), we have

$$\pi^* = \frac{1}{p\zeta\eta}\left(\frac{(\upsilon-r)}{\xi^2 w^{2\beta}}+\kappa\right) - \frac{q_w}{p}\left(\frac{w}{\eta}\right) - \frac{\rho c_0}{\eta\xi w^{\beta}}, \tag{31}$$

From (19), (22) and (26), we have

$$p(t) = e^{(r-\alpha)(T-t)},$$
$$q_w = \frac{-w^{-2\beta-1}}{2r\zeta}\left(\frac{(\upsilon-r)^2}{\xi^2}+\kappa^2\xi^2 w^{4\beta}\right)\left[1-e^{2r\beta(t-T)}\right]$$

Substituting (19) and (32) into (31), (30) is proved.

Remark1. The OIS when there is no voluntary contribution i.e. $c_0 = 0$, is given as

$$\pi^* = \begin{cases} \frac{(\upsilon-r)e^{(r-\alpha)(t-T)}}{\zeta\eta\xi^2 w^{2\beta}}\left[1+\frac{(\upsilon-r)}{2r}\left(1-e^{2r\beta(t-T)}\right)\right] \\ +\frac{\kappa e^{(r-\alpha)(t-T)}}{\zeta\eta}\left[1+\frac{\kappa\xi^2 w^{2\beta}}{2r}\left(1-e^{2r\beta(t-T)}\right)\right] \end{cases}$$

Remark2. The OIS when the price of the risky asset follows the CEV model i.e. $\kappa = 0$, is given as

$$\pi^* = \left\{\frac{(\upsilon-r)e^{(r-\alpha)(t-T)}}{\zeta\eta\xi^2 w^{2\beta}}\left[1+\frac{(\upsilon-r)}{2r}\left(1-e^{2r\beta(t-T)}\right)\right] - \frac{\rho c_0}{\eta\xi w^{\beta}}\right.$$

Remark 3. The OIS when the price of the risky asset follows the CEV model i.e. $\kappa = 0$ and when there is no voluntary contribution i.e $c_0 = 0$, is given as

$$\pi^* = \left\{\frac{(\upsilon-r)e^{(r-\alpha)(t-T)}}{\zeta\eta\xi^2 w^{2\beta}}\left[1+\frac{(\upsilon-r)}{2r}\left(1-e^{2r\beta(t-T)}\right)\right]\right.$$

IV. Numerical Simulations

In this section, we use the MATLAB programming language to obtain some numerical simulations for the OIS. The following parameters are used in the simulations unless otherwise stated; $\upsilon = 0.05, \zeta = 0.1, T = 20, r = 0.02, \rho = -0.2, w = 2.5, \beta = -1, \alpha = 0.1, \eta = 1, \xi = 0.05, c_0 = 0.1$, e $\kappa = 0.01$ *and* $t = 0:5:20$.

V. Discussion

In this section, we discuss the impact of some sensitive parameters on the OIS. Fig. 1 shows the relationship between the OIS and the elasticity parameter β. It is observed that the OIS is a decreasing function of the β. This implies that when β is highly negative; there is an indication that the market is highly volatile hence the PPM will be discouraged from investing more in the risky asset. Fig. 2 shows the relationship between the OIS and the voluntary contribution. It is observed that the OIS with voluntary contribution is lesser compared to the one without voluntary contributions. The reason being that with the voluntary contributions, the overall pension wealth is increased hence the PPM will invest less in risky asset there by taking less risk.

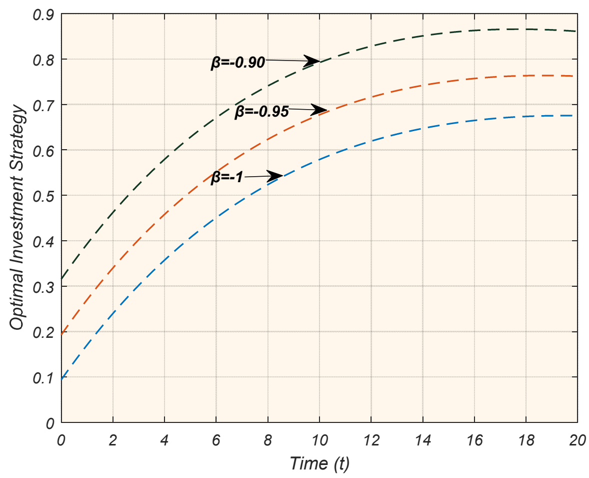

Fig. 1 The impact of different elasticity parameters (β) on π^*

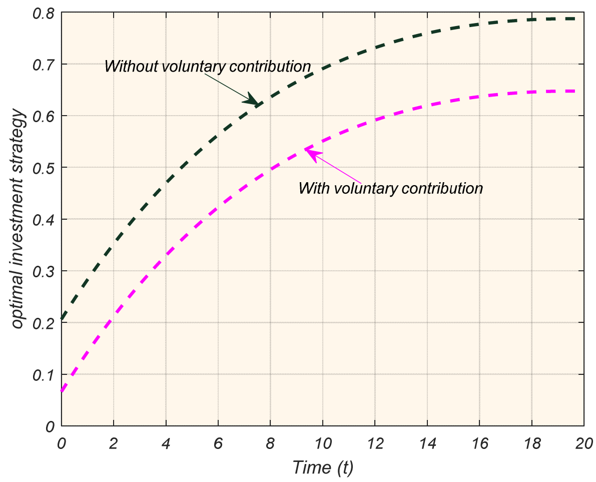

Fig. 2 The impact of voluntary contributions (c_0) on π^*

In Fig. 3, we observed that the presence of the modification factor enables the PPM member to know more about the true situation of the financial market. Hence with the presence of modification factor reduces investment in the risky asset and vice versa. Fig. 4 shows the relationship between the OIS and the administrative charges imposed by the pension fund administrators. It is observed that PPM with higher investment in risky asset pay higher administrative fee due to the risk involvement in managing such businesses.

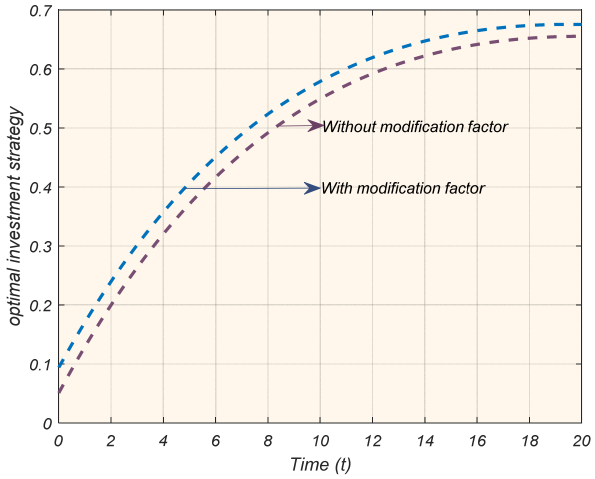

Fig. 3 The impact of modification factor (κ) on π^*

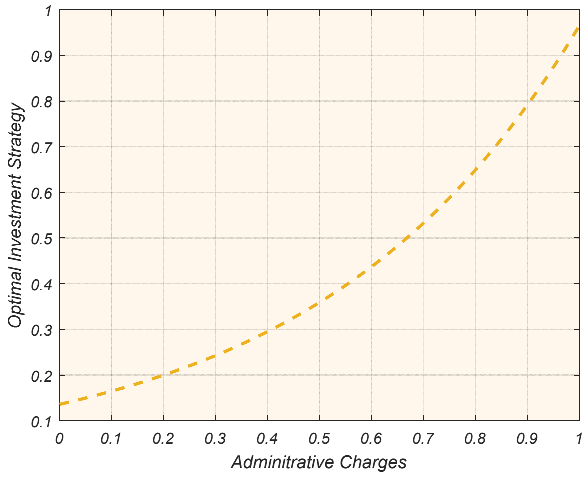

Fig. 4 The impact of administrative charges (α) on π^*

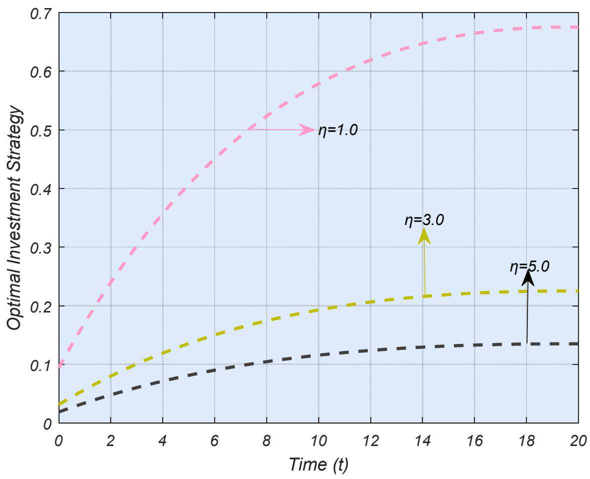

Fig. 5 The impact of initial fund size (η) on π^*

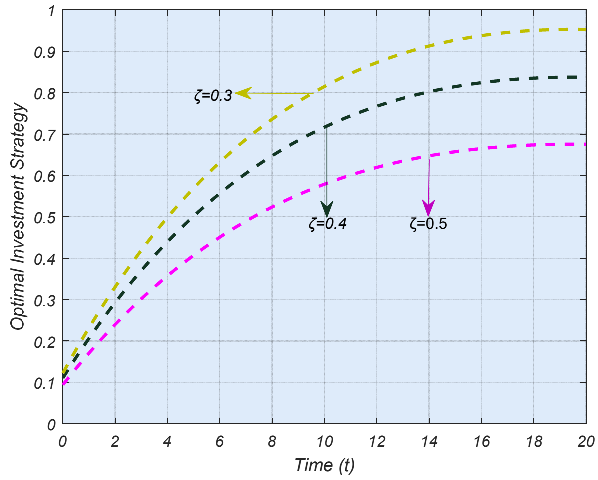

Fig. 6 The impact of risk averse coefficient (ζ) on π^*

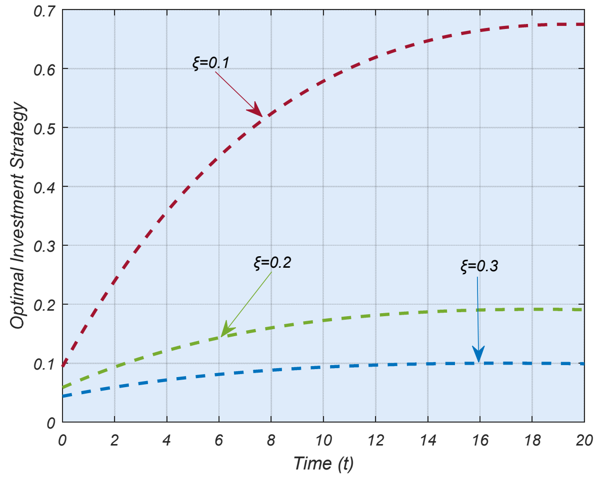

Fig. 7 The impact of instantaneous volatility of the risky asset (ξ) on π^*

Figs. 5-7 give the relationship between the optimal OIS and the initial fund size, risk aversion coefficient and instantaneous volatility of the risky asset. We observed that the OIS is a decreasing function of the initial fund size, risk averse coefficient and instantaneous volatility. This implies that when there are more funds in the PPM portfolio, the PPM will take less risk and may invest more in the risk-free asset and less in the risky asset. Similarly, PPM with high-risk averse coefficient will be scared to invest risky asset for the fear of losing what they have gathered already and vice versa. Finally, we observed that members will avoid highly volatile investment for the fear of losing what they had before.

VI. Conclusion

This paper studied the OIS for a PPM in a DC pension fund whose risky asset follows the MCEV model. We consider investments in a risk-free asset and a risky asset and also assumed the voluntary contribution to be stochastic. Also, we obtained a stochastic differential equation consisting of PPM's monthly contributions, voluntary contributions and administrative charges. More so, an optimization problem was obtained in the form Hamilton Jacobi Bellman equation. The power transformation and change of variables method was applied to solve for the explicit solution of the OIS and value function. Furthermore, numerical simulations of some sensitive parameters were discussed extensively. Finally, our result generalizes the results in [2] and [22].

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