Geometric Representation of Modified Forms of Seven Important Failure Criteria

Ranajay Bhowmick

***Abstract*—**Elastoplastic analysis of a structural system involves defining failure/yield criterion, flow rules and hardening rules. The failure/yield criterion defines the limit beyond which the material flows plastically and hardens/softens or remains perfectly plastic before ultimate collapse. The failure/yield criterion is represented geometrically in three/two dimensional Haigh-Westergaard stress-space to facilitate a better understanding of the behavior of the material. In the present study geometric representations in three and two-dimensional stress-space of a few important failure/yield criterion are presented. The criteria presented are the modified forms obtained due to the conditional solutions of the equation of stress invariants. A comparison of the failure/yield surfaces is also presented here to obtain the effectiveness of each of them and it has been found that for identical conditions the Rankine's criterion gives the largest values of limiting stresses.

***Keywords*—**Deviatoric plane, failure criteria, geometric representation, hydrostatic axis, modified form.

I. Introduction

NONLINEARITY can be of the geometry of a structural element or of the material of the element or both. As more quest is being made towards understanding the nonlinear behavior of the structural elements, nonlinear analysis has gained significance. Therefore, nonlinear analysis of structural elements, in the recent years, has become increasingly important for understanding the structural behavior in the post-elastic range and also for the determination of design parameters especially with the advent of the sophisticated user-friendly time saving computer programs. A complete progressive failure analysis of a structure up to failure helps us to determine its deformational characteristics and assess all the safety aspects. Since nonlinear behavior of a material consists of yielding, plastic flow and hardening of the material, carrying out nonlinear analysis of structures/structural elements requires defining failure criterion (brittle materials)/ yield criterion (ductile materials), flow rules and hardening rules. Failure/Yield criterion is the limit beyond which the material loses its elasticity and flows like a plastic material and hardens/softens before ultimate collapse. Failure/Yield criteria are represented in the three-dimensional stress-spaces as surfaces so as to improve the understanding of the criterion. In the present study some important failure/yield criteria are represented in the three-dimensional principal stress-space and their comparisons are observed. The failure/yield criteria represented are the modified forms obtained from the solution of S^3 problem of continuum mechanics for a definite condition [1].

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II. Geometric Representation of Stress-State

The geometric representation of stress-state at a point is very useful in the study of failure/yield criteria and the plasticity theory [2], [3]. Since the stress tensor σ_{ij} has six independent components, it is very much difficult to represent them using a six- dimensional stress-space and the simplest way is to consider the three principal stresses σ_1, σ_2 and σ_3 as coordinates and represent the stress-state at a point as a point in the three-dimensional stress-space. This stress-space is called the Haigh-Westergaard stress-space and every point in the space having coordinates σ_1, σ_2 and σ_3 represents a possible stress-space of a stressed body. If we consider a situation when $\sigma_1 = \sigma_2 = \sigma_3$, a line Ω is obtained that passes through the origin and makes $\cos^{-1}(1/\sqrt{3}) = 54°44'$ with each of the three reference axes and such a line is called the hydrostatic axis as every point on this line corresponds to a hydrostatic or spherical state of stress (Fig. 1). The plane that passes through the origin and is perpendicular to the line Ω' is called the π plane and has the equation $\sigma_1 + \sigma_2 + \sigma_3 = 0$. The planes that are parallel to the π plane but not containing the origin have the equation $\sigma_1 + \sigma_2 + \sigma_3 = C$ (where C is an arbitrary constant) and are called the deviatoric planes.

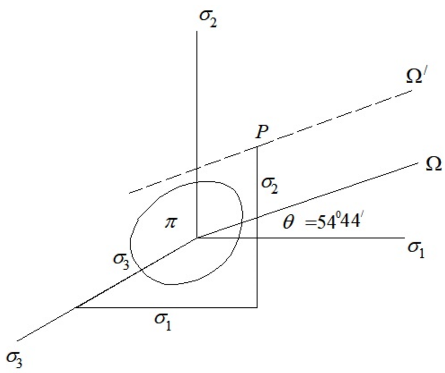


Fig. 1 Haigh-Westergaard Stress-Space

An arbitrary state of stress at a given point within a stressed body with stress components σ_1, σ_2 and σ_3 is represented by point P $(\sigma_1, \sigma_2, \sigma_3)$ in the Haigh-Westergaard stress-space (Fig. 2). The stress vector OP can be decomposed into two components – ON and NP. The component ON is in the direction of the unit vector $(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$ and represents the hydrostatic stress component (p, p, p). The component NP is perpendicular to ON (parallel to the π plane). Thus,

$$|ON| = \xi = OP.n = (\sigma_1, \sigma_1, \sigma_1).(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}) = \frac{1}{\sqrt{3}}(\sigma_1 + \sigma_2 + \sigma_3) = \frac{I_1}{\sqrt{3}} = \sqrt{3}p = \sqrt{3}\sigma_{oct}.$$

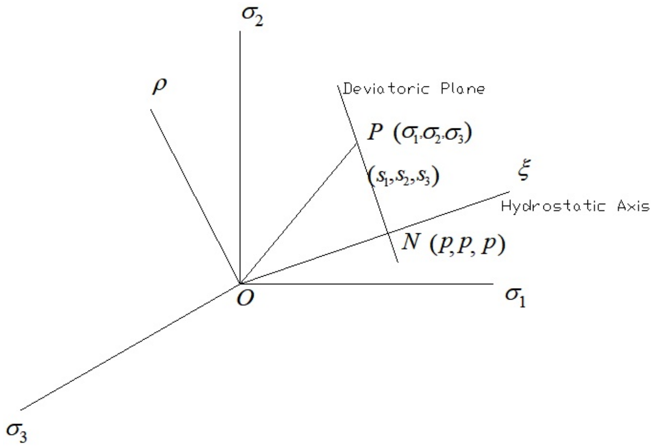


Fig. 2 Deviatoric Plane and Hydrostatic Axis

The vector NP represents the deviatoric component of the stress-state (s_1, s_2, s_3) and lies in the Meridian plane and perpendicular to the hydrostatic axis ξ and has the ρ value: $|NP| = \rho = \left(s_1^2 + s_2^2 + s_3^2\right)^{1/2} = \sqrt{2J_2}$.

The projections of the vector NP and the coordinate axes σ_i on a deviatoric plane are shown in Fig. 3. The axes $\sigma_1^/, \sigma_2^/$ and $\sigma_3^/$ are the projections of the axes σ_1, σ_2 and σ_3 respectively. The vector $N^/P^/$ is the projection of the vector NP on the deviatoric plane making an angle θ with the axis $\sigma_1^/$.

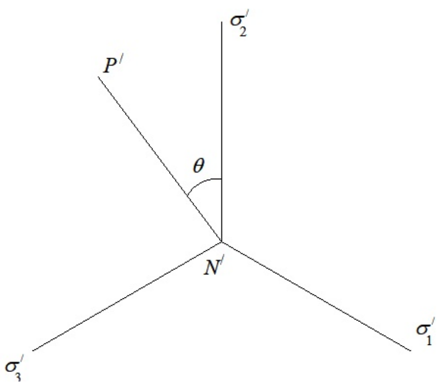


Fig. 3 Stress-State on a Deviatoric Plane

III. Geometric Representation of Failure/Yield Criterion

A failure/yield criterion is generally expressed as:

$$F(\sigma_1, \sigma_2, \sigma_3, k_i, n_i) = 0. \quad (1)$$

where, σ_1, σ_2 and σ_3 are the principal stresses, k_i are the material properties and n_i represent the principal directions with respect to the material directions x_1, x_2, x_3.

In case of isotropic materials, the failure criteria are independent of any material directions and can be expressed in the simple form as

$$F(\sigma_1, \sigma_2, \sigma_3, \sigma_Y) = 0. \quad (2)$$

Alternatively, (2) can be written in the form consisting the three principal invariants of stresses, due to their independence of material orientation, as:

$$F(I_1, I_2, I_3, \sigma_Y) = 0. \quad (3)$$

Moreover, since the principal stresses can be expressed in terms of either stress invariants I_1, I_2, I_3 or the Haigh – Westergaard coordinates ξ, ρ, θ (3) can also be expressed as:

$$F(\xi, \rho, \theta) = 0. \quad (4)$$

Equations (1)-(4) each represent a surface in the principal stress space known as the Haigh-Westergaard stress space and such a surface is referred to as the yield surface. The shape of a yield surface is best described by its cross-sectional shapes on deviatoric planes (ρ, θ) and its meridians on meridian planes (ξ, ρ) (Fig. 4). The cross-sections of a yield surface are the intersection curves between the yield surface and a deviatoric plane which is perpendicular to the hydrostatic axis ξ with $\xi = cons\tan t$. The meridians of a yield surface are the intersection curves between the surface and a meridian plane which contains the hydrostatic axis and with $\theta = cons\tan t$. The two extreme meridian planes corresponding to $\theta = -\frac{\pi}{6}$ *and* $\theta = \frac{\pi}{6}$ are called tensile meridian (ρ_t) and compressive meridian (ρ_t) respectively.

IV. Modification of Important Failure Criteria and Their Geometric Representations

Failure/Yield criteria, which are expressed using (1), can be expressed using conventional forms of invariants I_1, J_2, J_3 and even those conventional forms of invariants are not convenient for an explicit evaluation of principal stresses and thus present difficulties in arriving at compact descriptions of yield surfaces dependent on them. Researchers, however, have previously derived some alternative forms that allow much simplified descriptions of various failure/yield criteria. They had derived some explicit method for the evaluation of roots of the cubic equation of three stress invariants (linear, quadratic and cubic) by converting them into cubic equations consisting of deviatoric invariants and using a trigonometric identity [4], [5]. In [1], an approach had been formulated for obtaining explicit solutions for the cubic equation for a definite condition and thereby determining the principal stresses and the orientation of the principal planes. In that study solution of the cubic equation – $S^3 - I_2 \times S - I_3 = 0$ was obtained for $\sigma_{xx} + \sigma_{yy} + \sigma_{zz} = 0$ i.e. $I_1 = 0$ and the explicit solution obtained was obtained as –

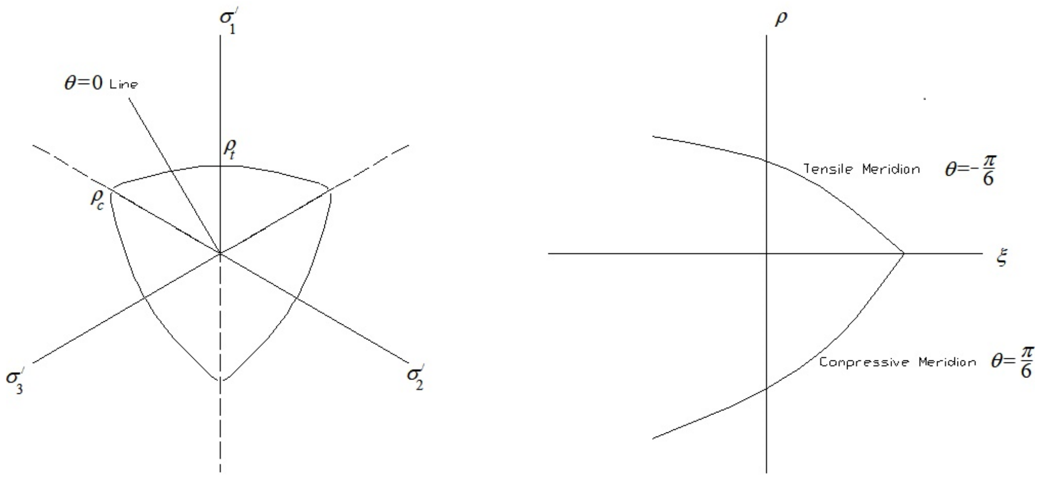
$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{Bmatrix} = \left(\frac{2}{\sqrt{3}}\right) \times \sqrt{I_2} \times \begin{Bmatrix} \sin\theta_1 \\ \sin\theta_2 \\ \sin\theta_3 \end{Bmatrix} \tag{5}$$

where,

$$\theta_1 = \left(\frac{1}{3}\right) \times \sin^{-1}(k) + \left(\frac{2\pi}{3}\right) = \theta + \left(\frac{2\pi}{3}\right)$$

$$\theta_2 = \left(\frac{1}{3}\right) \times \sin^{-1}(k) = \theta$$

$$\theta_3 = \left(\frac{1}{3}\right) \times \sin^{-1}(k) + \left(\frac{4\pi}{3}\right) = \theta - \left(\frac{2\pi}{3}\right) \text{ and } -\frac{\pi}{6} \le \theta \le \frac{\pi}{6}.$$



(a) (b)

Fig. 4 (a) Deviatoric Plane (b) Meridian Plane

Those solutions were then applied to various failure theories to obtain modified descriptions of those theories [6]. The failure theories that are described and discussed in this article are Rankine Criterion, St. Venant Criterion, Tresca Criterion, Von Mises Criterion, Nadai's Criterion, Mohr – Coulomb Criterion and Drucker Prager Criterion. The modified descriptions of these important failure/yield criteria are hereby geometrically represented to find out the effect of such modifications.

A. Rankine - Lame - Navier Criterion

The Rankine criterion states that the yield criterion is reached when the combined stress results in principal stresses that attain the ultimate strength (for brittle materials) or yield strength (for ductile materials) in the uniaxial state of stress [7]. In terms of the principal stresses, the Rankine criterion can be expressed as:

$$\sigma_1 \le \sigma_Y \ ; \sigma_2 \le \sigma_Y \ ; \sigma_3 \le \sigma_Y . \tag{6}$$

The modified form of the Rankine criterion can be expressed as:

$$\left(\frac{2}{\sqrt{3}}\right)\sqrt{I_2}\sin\left(\theta + \frac{2\pi}{3}\right) - \sigma_Y = 0 \text{ for } -\frac{\pi}{6} \le \theta \le \frac{\pi}{6}. \tag{7}$$

On π plane: The yield criterion is expressed as:

$$\rho = \sqrt{2I_2} = \left(\sqrt{\frac{3}{2}}\right)\frac{\sigma_Y}{\sin\left(\theta + \frac{2\pi}{3}\right)} \text{ for } -\frac{\pi}{6} \le \theta \le \frac{\pi}{6}. \tag{8}$$

$$\rho_t = \sqrt{\frac{3}{2}}\sigma_Y \ \ for\,\theta = -\frac{\pi}{6} \text{ and } \rho_c = \sqrt{6}\sigma_Y \ \ for\ \theta = \frac{\pi}{6}$$

The cross-section of the yield surface with a deviatoric plane is a regular triangle as shown in Fig. 5 (a).

On $\sigma_1 - \sigma_2$ space:

$$\sqrt{I_2} = \left(\frac{\sqrt{3}}{2}\right)\frac{\sigma_Y}{\sin\left(\theta + \frac{2\pi}{3}\right)}$$

$$I_2 = -(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1) = -\sigma_1\sigma_2 \;\; as \;\; \sigma_3 = 0$$

As, $\sigma_1 + \sigma_2 + \sigma_3 = 0$ and $\sigma_3 = 0$ on $\sigma_1 - \sigma_2$ space, we get:

$$\sigma_1 + \sigma_2 = 0 \Rightarrow \sigma_1 = -\sigma_2 .$$

$$\therefore I_2 = \sigma_1^2 = \sigma_2^2$$

$$\sigma_1 = \pm\left(\frac{\sqrt{3}}{2}\right)\frac{\sigma_Y}{\sin\left(\theta + \frac{2\pi}{3}\right)} \;\; and \;\; \sigma_2 = \mp\left(\frac{\sqrt{3}}{2}\right)\frac{\sigma_Y}{\sin\left(\theta + \frac{2\pi}{3}\right)} \quad (9)$$

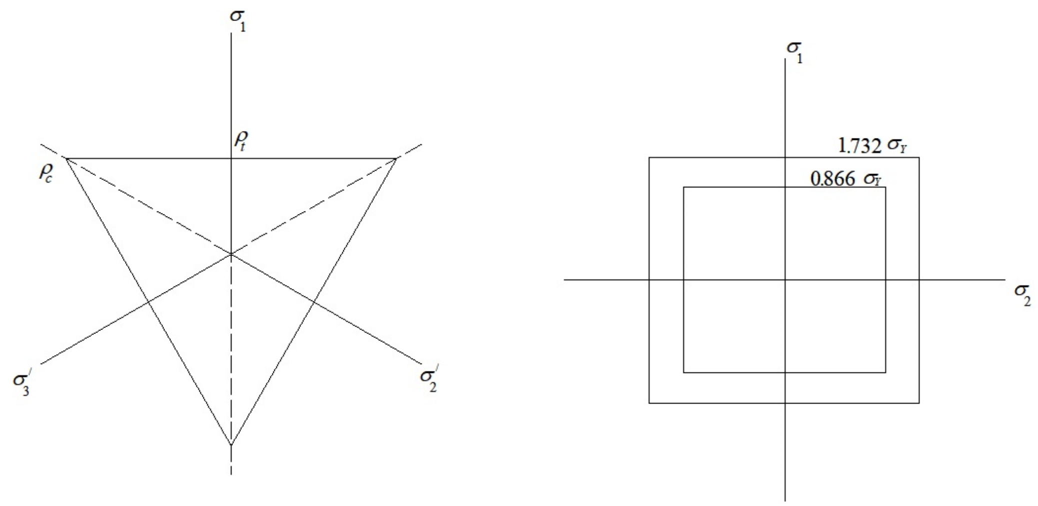
As Rankine's criterion is dependent on the θ value the σ_1 *and* σ_1 values for different θ values are given hereby. For,

$$\theta = 0^0 \rightarrow \sigma_1 = \pm\sigma_Y \;\; and \;\; \sigma_2 = \mp\sigma_Y$$

$$\theta = \frac{\pi}{6} \rightarrow \sigma_1 = \pm\sqrt{3}\sigma_Y \;\; and \;\; \sigma_2 = \mp\sqrt{3}\sigma_Y$$

$$\theta = -\frac{\pi}{6} \rightarrow \sigma_1 = \pm\frac{\sqrt{3}}{2}\sigma_Y \;\; and \;\; \sigma_2 = \mp\frac{\sqrt{3}}{2}\sigma_Y$$

The geometric representation of the Rankine criterion on a $\sigma_1 - \sigma_2$ plane is given in Fig. 5 (b).



(a) (b)

Fig. 5 (a) Rankine Criteria on Deviatoric Plane (b) Rankine Criteria on $\sigma_1 - \sigma_2$ Plane

B. St. Venant Criterion

The St. Venant criterion states that a material will fail under combined stress-state if the maximum unit linear strain (ε_{max}) exceeds the allowable unit linear strain assumed for uniaxial tension [8]. This theory does not conform to experimental data. As per this theory:

$$\varepsilon_{max} = \varepsilon_1 = \frac{1}{E} \times [\sigma_1 - \mu \times (\sigma_2 + \sigma_3)] \le \frac{\sigma_Y}{E} . \quad (10)$$

The St. Venant criterion can be expressed in modified form as:

$$\left(\sqrt{\frac{I_2}{3}}\right)(1+\mu)\left(\sqrt{3}\cos\theta - \sin\theta\right) - \sigma_Y = 0 \;\; \text{for} \;\; -\frac{\pi}{6} \le \theta \le \frac{\pi}{6} \quad (11)$$

On π plane, the yield criterion is expressed as:

$$\rho = \sqrt{2I_2} = \frac{\sqrt{6}\sigma_Y}{(1+\mu)\left(\sqrt{3}\cos\theta - \sin\theta\right)} \;\; \text{for} \;\; -\frac{\pi}{6} \le \theta \le \frac{\pi}{6} . \quad (12)$$

$$\rho_t = \left(\frac{\sqrt{3}}{\sqrt{2}\,(1+\mu)}\right)\sigma_Y \;\; for \;\; \theta = -\frac{\pi}{6} \;\; and \;\; \rho_c = \left(\frac{\sqrt{6}}{(1+\mu)}\right)\sigma_Y \;\; for \;\; \theta = \frac{\pi}{6}$$

The cross-section of the yield surface with a deviatoric plane is a regular triangle as shown in Fig. 6 (a).

On $\sigma_1 - \sigma_2$ space:

$$\sqrt{I_2} = \frac{\sqrt{3}\sigma_Y}{(1+\mu)\left(\sqrt{3}\cos\theta - \sin\theta\right)}$$

As previously, $I_2 = -(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1) = -\sigma_1\sigma_2$ *as* $\sigma_3 = 0$ and $I_2 = \sigma_1^2 = \sigma_2^2$

$$\sigma_1 = \pm\frac{\sqrt{3}\sigma_Y}{(1+\mu)\left(\sqrt{3}\cos\theta - \sin\theta\right)} \;\; and \;\; \sigma_2 = \mp\frac{\sqrt{3}\sigma_Y}{(1+\mu)\left(\sqrt{3}\cos\theta - \sin\theta\right)} \quad (13)$$

St. Venant's criterion is dependent also on the μ value except the θ value and the σ_1 *and* σ_1 values for different θ values are given hereby.

For,

$$\theta = 0^0 \;\&\; \mu = 0.5 \rightarrow \sigma_1 = \pm\frac{2}{3}\sigma_Y \quad and \quad \sigma_2 = \mp\frac{2}{3}\sigma_Y$$

$$\theta = 0^0 \;\&\; \mu = 0 \rightarrow \sigma_1 = \pm\sigma_Y \quad and \quad \sigma_2 = \mp\sigma_Y$$

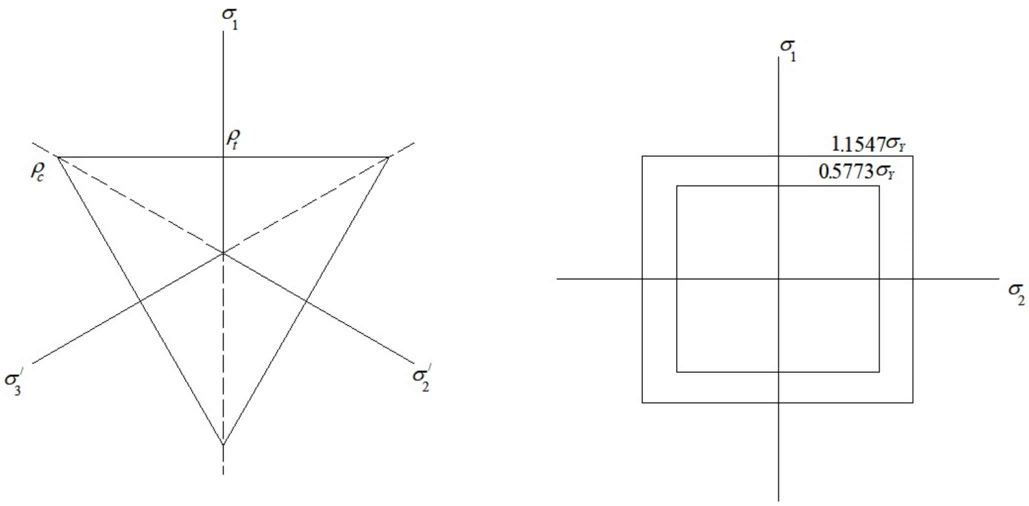
$$\theta = \frac{\pi}{6} \;\&\; \mu = 0.5 \rightarrow \sigma_1 = \pm\frac{2}{\sqrt{3}}\sigma_Y \quad and \quad \sigma_2 = \mp\frac{2}{\sqrt{3}}\sigma_Y$$

$$\theta = \frac{\pi}{6} \;\&\; \mu = 0 \rightarrow \sigma_1 = \pm\sqrt{3}\sigma_Y \quad and \quad \sigma_2 = \mp\sqrt{3}\sigma_Y$$

$$\theta = -\frac{\pi}{6} \;\&\; \mu = 0.5 \rightarrow \sigma_1 = \pm\frac{\sigma_Y}{\sqrt{3}} \quad and \quad \sigma_2 = \mp\frac{\sigma_Y}{\sqrt{3}}$$

$$\theta = -\frac{\pi}{6} \;\&\; \mu = 0 \rightarrow \sigma_1 = \pm\frac{\sqrt{3}}{2}\sigma_Y \quad and \quad \sigma_2 = \mp\frac{\sqrt{3}}{2}\sigma_Y$$

The geometric representation of the St. Venant criterion on a $\sigma_1 - \sigma_2$ plane is given in Fig. 6 (b).



(a) (b)

Fig. 6 (a) St. Venant Criteria on a Deviatoric Plane (b) St. Venant Criteria on a $\sigma_1 - \sigma_2$ Plane

C. Tresca Criterion

The Tresca criterion states that yielding of a material would occur when the maximum shearing stress at a point of the material reaches a critical value k [8]. In terms of principal stresses, we have

$$Max(\frac{1}{2}|\sigma_1 - \sigma_2|, \frac{1}{2}|\sigma_2 - \sigma_3|, \frac{1}{2}|\sigma_3 - \sigma_1|) = k\,.$$

From a uniaxial test, for $\sigma_2 = \sigma_3 = 0$ and $\sigma_1 = \sigma_Y$, we determine $k = \frac{\sigma_1}{2} = \frac{\sigma_Y}{2}$. And from a pure shear test, for $\sigma_1 = \tau$, $\sigma_2 = 0$ and $\sigma_3 = -\tau$, we determine $k = \tau$. Therefore, if the Tresca criterion is used, the tensile strength and the shear strength of a material are related by $\sigma_Y = 2\tau$. Therefore, the Tresca criterion can be expressed as:

$$\sigma_1 - \sigma_3 \le \sigma_Y\,. \tag{14}$$

The Tresca criterion can be expressed in its modified form as:

$$2\sqrt{I_2}\cos\theta - \sigma_Y = 0 \text{ for } -\frac{\pi}{6} \le \theta \le \frac{\pi}{6}\,. \tag{15}$$

On π plane, the yield criterion is expressed as:

$$\rho = \sqrt{2I_2} = \frac{\sigma_Y}{\sqrt{2}\cos\theta} \text{ for } -\frac{\pi}{6} \le \theta \le \frac{\pi}{6}\,. \tag{16}$$

$$\rho_t = \left(\sqrt{\frac{2}{3}}\right)\sigma_Y \quad for \;\; \theta = -\frac{\pi}{6} \quad and \quad \rho_c = \left(\sqrt{\frac{2}{3}}\right)\sigma_Y \quad for \;\; \theta = \frac{\pi}{6}$$

On a deviatoric plane Tresca criterion is a regular hexagon with six singular corners (Fig. 7 (a)).

On $\sigma_1 - \sigma_2$ space:

$$\sqrt{I_2} = \frac{\sigma_Y}{2\cos\theta}$$

As previously, $I_2 = -(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1) = -\sigma_1\sigma_2 \;\; as \;\; \sigma_3 = 0$ and $\therefore I_2 = \sigma_1^2 = \sigma_2^2$

$$\therefore \sigma_1 = \pm\left(\frac{\sigma_Y}{2\cos\theta}\right) \quad and \quad \sigma_2 = \mp\left(\frac{\sigma_Y}{2\cos\theta}\right) \tag{17}$$

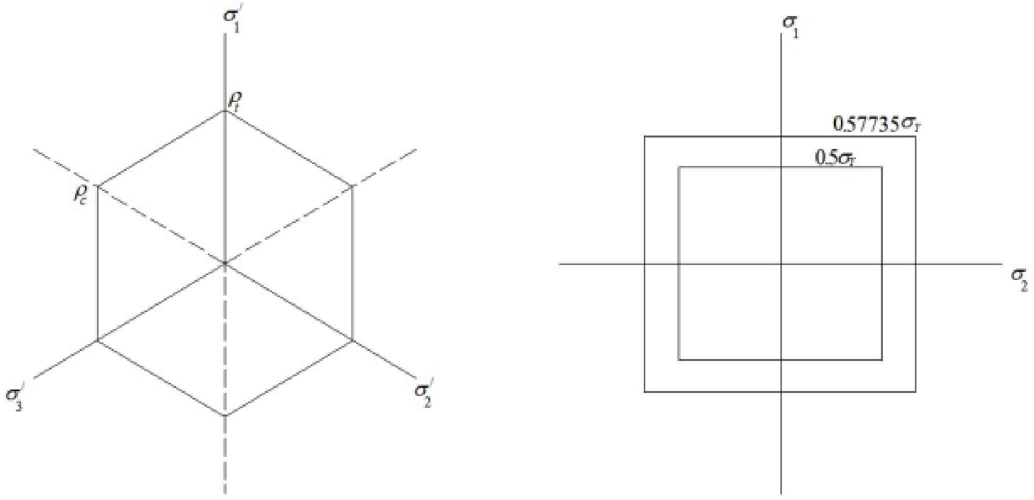
For,

$$\theta = 0^0 \rightarrow \sigma_1 = \pm 0.5\sigma_Y \quad and \quad \sigma_2 = \mp 0.5\sigma_Y$$

$$\theta = \frac{\pi}{6} \rightarrow \sigma_1 = \pm \frac{\sigma_Y}{\sqrt{3}} \quad and \quad \sigma_2 = \mp \frac{\sigma_Y}{\sqrt{3}}$$

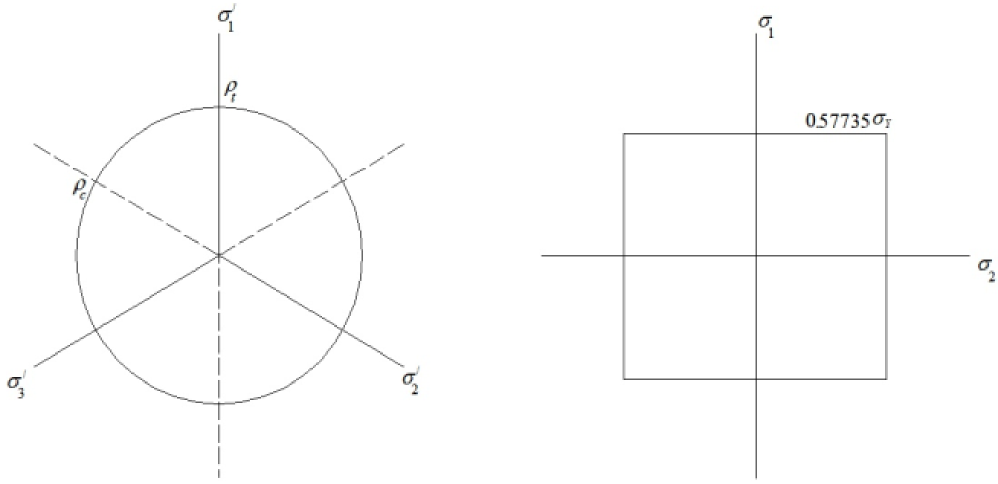
$$\theta = -\frac{\pi}{6} \rightarrow \sigma_1 = \pm \frac{\sigma_Y}{\sqrt{3}} \quad and \quad \sigma_2 = \mp \frac{\sigma_Y}{\sqrt{3}}$$

The geometric representation of the Tresca criterion on a $\sigma_1 - \sigma_2$ plane is given in Fig. 7 (b).



(a) (b)

Fig. 7 (a) Tresca Criteria on Deviatoric Plane (b) Tresca Criteria on $\sigma_1 - \sigma_2$ Plane



(a) (b)

Fig. 8 (a) Von Mises Criteria on Deviatoric Plane (b) Von Mises Criteria on $\sigma_1 - \sigma_2$ Plane

D.Von Mises Criterion

Von Mises criterion states that yielding of a material would occur when the distortional strain energy exceeds the strain energy under uniaxial tension [9]. Since the distortional/shear strain energy is proportional to the second invariant of the deviatoric stress tensor J_2, the criterion can be expressed as: $J_2 - k^2 = 0$.

From a uniaxial test, for $\sigma_2 = \sigma_3 = 0$ and $\sigma_1 = \sigma_Y$, we determine $k = \frac{\sigma_Y}{\sqrt{3}}$. And from a pure shear test, for $\sigma_1 = \tau$, $\sigma_2 = 0$ and $\sigma_3 = -\tau$, we determine $k = \tau$. Therefore, if the Tresca criterion is used, the tensile strength and the shear strength of a material are related by $\sigma_Y = \sqrt{3}\tau$. Therefore, the von Mises criterion can be expressed as:

$$J_2 - \frac{\sigma_Y^2}{3} = 0 \;\; => \;\; \sqrt{3J_2} - \sigma_Y = 0 \cdot \quad (18)$$

The von Mises criterion in modified form is expressed as:

$$\sqrt{3I_2} - \sigma_Y = 0 \text{ for } -\frac{\pi}{6} \le \theta \le \frac{\pi}{6}. \quad (19)$$

On π plane, the yield criterion is expressed as:

$$\rho = \sqrt{2I_2} = \left(\sqrt{\frac{2}{3}}\right)\sigma_Y \text{ for } -\frac{\pi}{6} \le \theta \le \frac{\pi}{6}. \quad (20)$$

$$\rho_t = \left(\sqrt{\frac{2}{3}}\right)\sigma_Y \quad for \;\; \theta = -\frac{\pi}{6} \quad and \quad \rho_c = \left(\sqrt{\frac{2}{3}}\right)\sigma_Y \quad for \;\; \theta = \frac{\pi}{6}$$

The cross-section of the yield surface with a deviatoric plane is a circle as shown in Fig. 8 (a).

On the $\sigma_1 - \sigma_2$ sub-space:

$$\sqrt{I_2} = \frac{\sigma_Y}{\sqrt{3}}$$

As previously, $I_2 = -(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1) = -\sigma_1\sigma_2 \;\; as \;\; \sigma_3 = 0$ and $I_2 = \sigma_1^2 = \sigma_2^2$

$$\therefore \sigma_1 = \pm\left(\frac{\sigma_Y}{\sqrt{3}}\right) \;\; and \;\; \sigma_2 = \mp\left(\frac{\sigma_Y}{\sqrt{3}}\right) \quad (21)$$

The geometric representation of the von Mises criterion on a $\sigma_1 - \sigma_2$ plane is given in Fig. 8 (b).

E. Nadai's Criterion

Nadai's criterion, a different form of von Mises criterion, states that a material failure will occur when the octahedral shear stress ($\tau_{OCT} = \sqrt{\frac{2}{3}J_2}$) reaches a critical value given as $\frac{\sqrt{2}}{3}\sigma_Y$ [10]. The criterion is expressed as:

$$\tau_{OCT} \le \frac{\sqrt{2}}{3} \times \sigma_Y. \quad (22)$$

The modified expression comes out to be the same as the previous one i.e., that of von Mises expression:

$$\sqrt{3I_2} - \sigma_Y = 0 \text{ for } -\frac{\pi}{6} \le \theta \le \frac{\pi}{6}. \quad (23)$$

On π plane, the yield criterion is expressed as:

$$\rho = \sqrt{2I_2} = \left(\sqrt{\frac{2}{3}}\right)\sigma_Y \text{ for } -\frac{\pi}{6} \le \theta \le \frac{\pi}{6}. \quad (24)$$

$$\rho_t = \left(\sqrt{\frac{2}{3}}\right)\sigma_Y \quad for \;\; \theta = -\frac{\pi}{6} \quad and \quad \rho_c = \left(\sqrt{\frac{2}{3}}\right)\sigma_Y \quad for \;\; \theta = \frac{\pi}{6}$$

The cross-section of the yield surface with a deviatoric plane is a circle (same as that of the von Mises criterion) as shown in Fig. 8 (a).

On the $\sigma_1 - \sigma_2$ sub-space:

$$\sqrt{I_2} = \frac{\sigma_Y}{\sqrt{3}}$$

As previously, $I_2 = -(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1) = -\sigma_1\sigma_2 \;\; as \;\; \sigma_3 = 0$ and $I_2 = \sigma_1^2 = \sigma_2^2$

$$\therefore \sigma_1 = \pm\left(\frac{\sigma_Y}{\sqrt{3}}\right) \;\; and \;\; \sigma_2 = \mp\left(\frac{\sigma_Y}{\sqrt{3}}\right) \quad (25)$$

The geometric representation of the Nadai's criterion on a $\sigma_1 - \sigma_2$ plane (same as that of the von Mises criterion) is given in Fig. 8 (b).

F. Mohr – Coulomb Criterion

The Mohr – Coulomb criterion is a generalization of the Tresca criterion. It considers the critical value of shearing stress on a plane to be a function of the normal stress acting on the same plane [11]. The Mohr – Coulomb criterion is expressed as:

$$\tau \le c - \sigma \tan\phi. \quad (26)$$

The Mohr – Coulomb criterion can also be expressed in modified form as:

$$\sqrt{I_2}\cos\theta - \left(\sqrt{\frac{I_2}{3}}\right)\sin\theta\sin\phi - c\cos\phi = 0 \text{ for } -\frac{\pi}{6} \le \theta \le \frac{\pi}{6} \quad (27)$$

On π plane, the yield criterion is expressed as:

$$\rho = \sqrt{2I_2} = \frac{\sqrt{6}c\cos\theta}{\left(\sqrt{3}\cos\theta - \sin\theta\sin\phi\right)} \text{ for } -\frac{\pi}{6} \le \theta \le \frac{\pi}{6}. \quad (28)$$

$$\rho_t = \frac{3\sqrt{2}}{3 + \sin\phi}c \quad for \;\; \theta = -\frac{\pi}{6} \quad and \quad \rho_c = \frac{3\sqrt{2}}{3 - \sin\phi}c \quad for \;\; \theta = \frac{\pi}{6}$$

The cross-section Çof the yield surface with a deviatoric plane is an irregular hexagon with curved edges as shown in Fig. 9 (a).

On $\sigma_1 - \sigma_2$ space:

$$\sqrt{I_2} = \frac{\sqrt{3}c\cos\theta}{\left(\sqrt{3}\cos\theta - \sin\theta\sin\phi\right)}$$

As previously, $I_2 = -(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1) = -\sigma_1\sigma_2 \;\; as \;\; \sigma_3 = 0$ and $I_2 = \sigma_1^2 = \sigma_2^2$

$$\sigma_1 = \pm\frac{\sqrt{3}c\cos\theta}{\left(\sqrt{3}\cos\theta - \sin\theta\sin\phi\right)} \;\; and \;\; \sigma_2 = \mp\frac{\sqrt{3}c\cos\theta}{\left(\sqrt{3}\cos\theta - \sin\theta\sin\phi\right)} \quad (29)$$

It can be seen that the σ_1 *and* σ_1 values of the Mohr – Coulomb criterion depend on both θ and ϕ values. The various σ_1 *and* σ_1 values for different θ and ϕ values are given as:

For,

$$\theta = 0^0 \ \& \ \phi = \frac{\pi}{6} \rightarrow \sigma_1 = \pm c \ \ and \ \ \sigma_2 = \mp c$$

$$\theta = \frac{\pi}{6} \ \& \ \phi = \frac{\pi}{6} \rightarrow \sigma_1 = \pm \frac{6}{5} c \ \ and \ \ \sigma_2 = \mp \frac{6}{5} c$$

$$\theta = -\frac{\pi}{6} \ \& \ \phi = -\frac{\pi}{6} \rightarrow \sigma_1 = \pm \frac{6}{7} c \ \ and \ \ \sigma_2 = \mp \frac{6}{7} c$$

The geometric representation of the Mohr - Coulomb criterion on a $\sigma_1 - \sigma_2$ plane is given in Fig. 9 (b).

G.Drucker - Prager Criterion

An approximation of the Coulomb law was expressed by Drucker – Prager as a simple modification of the von Mises criteria whereby a hydrostatic stress dependent first invariant I_1 was introduced in the von Mises equation [12].

$$\alpha \times I_1 + \sqrt{3 \times J_2} - \sigma_Y = 0 \,. \tag{30}$$

In the present case of $I_1 = 0$, the modified expression comes out to be as:

$$\sqrt{3I_2} - \sigma_Y = 0 \ \text{ for } -\frac{\pi}{6} \le \theta \le \frac{\pi}{6} \,. \tag{31}$$

The modified expression comes out to be the same as the previous one i.e. that of Von Mises expression.

On π plane, the yield criterion is expressed as:

$$\rho = \sqrt{2I_2} = \left(\sqrt{\frac{2}{3}}\right)\sigma_Y \ \text{ for } -\frac{\pi}{6} \le \theta \le \frac{\pi}{6} \,. \tag{32}$$

$$\rho_t = \left(\sqrt{\frac{2}{3}}\right)\sigma_Y \ \ for \ \ \theta = -\frac{\pi}{6} \ \ and \ \ \rho_c = \left(\sqrt{\frac{2}{3}}\right)\sigma_Y \ \ for \ \ \theta = \frac{\pi}{6}$$

The cross-section of the yield surface with a deviatoric plane is a circle (same as that of the von Mises criterion) as shown in Fig. 8 (a).

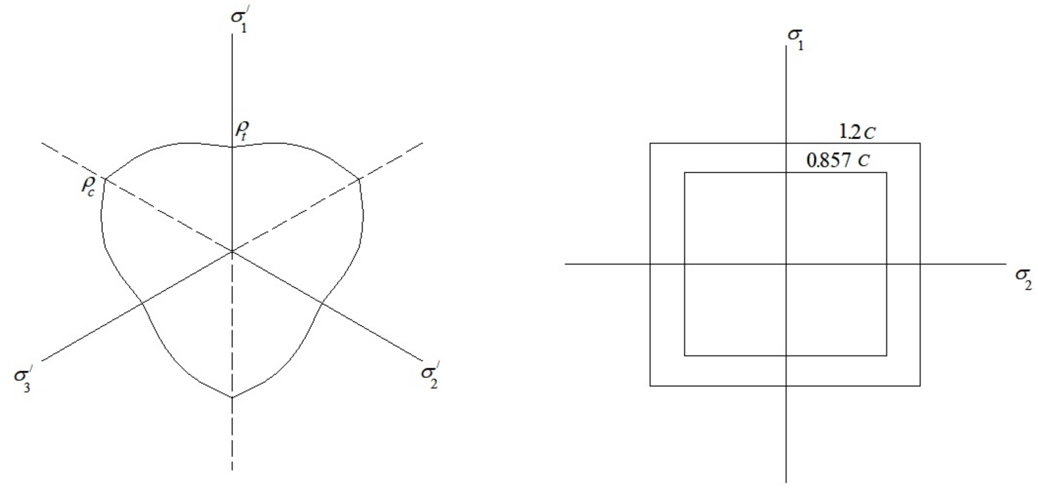
On the $\sigma_1 - \sigma_2$ sub-space:

$$\sqrt{I_2} = \frac{\sigma_Y}{\sqrt{3}}$$

As previously, $I_2 = -(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1) = -\sigma_1\sigma_2 \ \ as \ \sigma_3 = 0$ and $I_2 = \sigma_1^2 = \sigma_2^2$.

$$\therefore \sigma_1 = \pm\left(\frac{\sigma_Y}{\sqrt{3}}\right) \ \ and \ \ \sigma_2 = \mp\left(\frac{\sigma_Y}{\sqrt{3}}\right) \tag{33}$$

The geometric representation of the Drucker - Prager criterion on a $\sigma_1 - \sigma_2$ plane (same as that of the von Mises criterion) is given in Fig. 8 (b).



(a) (b)

Fig. 9 (a) Mohr – Coulomb Criteria on Deviatoric Plane (b) Mohr – Coulomb Criteria on $\sigma_1 - \sigma_2$ Plane

V. Discussions and Conclusion

In this article modified forms of several important failure/ yield criteria are represented geometrically in the $\sigma_1 - \sigma_2 - \sigma_3$ stress-space and the $\sigma_1 - \sigma_2$ stress-space to understand the characteristics of the failure surfaces. The compressive and tensile meridian values for each of the criterion are also indicated alongside. It is observed from the figures that in the $\sigma_1 - \sigma_2 - \sigma_3$ stress-space the geometry of von Mises, Nadai and Drucker - Prager criteria are of similar nature and circular in shape, whereas Tresca and Mohr – Coulomb criteria have similar hexagonal shape with Tresca's criterion having straight edges and the Mohr – Coulomb's criterion having curved edges. The Rankine's and St. Venant's criteria have triangular shape in the $\sigma_1 - \sigma_2 - \sigma_3$ space.

A comparison of the geometric shapes of the different criteria is presented in Fig. 10 for same σ_Y value. The shapes of various criteria except that of Mohr – Coulomb and their the tensile and compressive meridian values are as already discussed. In the case of St Venant's criterion the ρ_t *and* ρ_c values are obtained for $\mu = 0$ *and* $\mu = 0.5$ and it can be seen that for $\mu = 0$ the criterion matches with the Rankine's. Mohr - Coulomb's criterion is of circular shape and that is obtained for $\theta = 0^0, \phi = 0^0$ *and* $c = \frac{\sigma_Y}{2}$. For other values of θ *and* ϕ hexagonal shapes as described before can be obtained.

In this article the $\sigma_1 - \sigma_2$ stress-space plots of the various failure/yield criteria are also presented. The values of the limits of each criterion are given along with their descriptions and then subsequently indicated in Figs. 5-9 (b). The criteria are all of similar shape i.e. square.

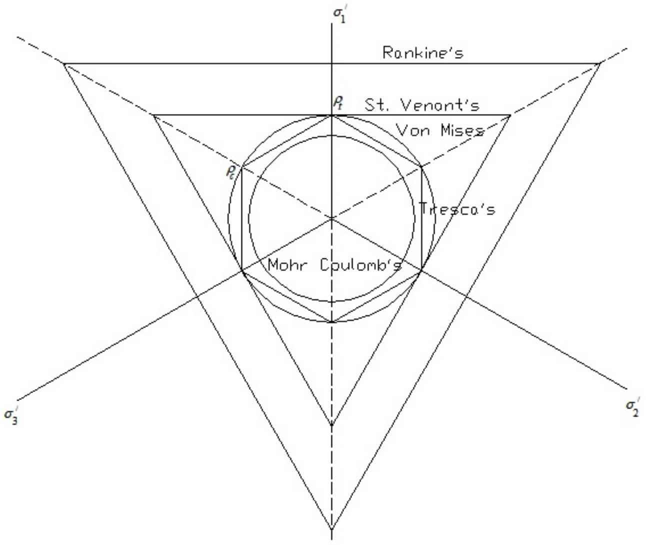


Fig. 10 Comparison of Different Failure Criteria

REFERENCES

[1] R. Bhowmick, "Solution of Cubic Equation of Stress Invariants for a Particular Condition", IRJET, vol. 07, pp. 4423 – 4425, 2020.

[2] H. M. Westergaard, "Theory of Elasticity and Plasticity", John Wiley & Sons, 1952.

[3] W. F. Chen and A. F. Saleeb, "Constitutive Equations for Engineering Materials", John Wiley & Sons, 1982.

[4] G. C. Nayak and O. C. Zienkiewicz, "Convenient forms of stress invariants for plasticity", ASCE, Journal of Structural Division, vol. 98, pp. 949 – 954, 1972.

[5] W. Lode 'Versucheueber den Einfluss der mitt lerenHauptspannung auf das Fliessen der Metalle Eisen Kupfen und Nickel', Z, Physik, vol. 36, pp. 913-39, 1926.

[6] R. Bhowmick, "Solution of S3 Problem of Deformation Mechanics for a Definite Condition and Resulting Modifications of Important Failure Theories", International Journal of Mechanical and Mechatronics Engineering, Vol 14 (12), pp. 508 – 511, 2020.

[7] W. J. M. Rankine, "A Manual of Applied Mechanics", Richard Griffin & Company, 1958.

[8] R. Chatterjee, "Mathematical Theory of Continuum Mechanics", Narosa Publishing House.

[9] R. von Mises, "Mechanics of Solid Bodies in the Plastically Deformed State", Nachr. d. Kgl. Ges. Wiss. Göttingen, Math.-phys. Klasse 4 (1913), pp. 582-592.

[10] A. Nadai, "Plastic Behaviour of Metals in the Strain – Hardening Range Part I", Journal of Applied Physics, vol. 8, pp. 205 – 213, 1937.

[11] S.P. Timoshenko and D.H. Young, "Elements of Strength of Materials", Affiliated East – West Press.

[12] D. C. Drucker and W. Prager, "Soil Mechanics and Plastic Analysis or Limit design", Quarterly of Applied Mechanics, vol. 10, pp. 157 – 165, 1952.