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Solution of Two-Point Nonlinear Boundary Problems Using Taylor Series Approximation and the Ying Buzu Shu Algorithm

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***Abstract*—**One of the major challenges faced in solving initial and boundary problems is how to find approximate solutions with minimal deviation from the exact solution without so much rigor and complications. The Taylor series method provides a simple way of obtaining an infinite series which converges to the exact solution for initial value problems and this method of solution is somewhat limited for a two point boundary problem since the infinite series has to be truncated to include the boundary conditions. In this paper, the Ying Buzu Shu algorithm is used to solve a two point boundary nonlinear diffusion problem for the fourth and sixth order solution and compare their relative error and rate of convergence to the exact solution.

***Keywords*—**Ying Buzu Shu, nonlinear boundary problem, Taylor series algorithm, infinite series.

I. Introduction

MOST of the real life models derived in the field of mathematics, physics, chemistry, biology, engineering and other related fields often lead to nonlinear ordinary differential equations or nonlinear partial differential equations. Very often observed, most of these models are very difficult to solve for their exact solutions, hence this has necessitated the use of approximated technique to obtain approximated solutions. Some of these methods include; the Taylor series approximation method [1], Homotopy Perturbation method [2]-[8], Fourier spectral method [9], perturbation method [10], Gamma function method [11], He frequency formulation and the dimensional method [12]-[14], Adomian decomposition method [15], [16], differential transformation method [17], [18] etc.

In recent years, many researchers in the field of medicine, biochemistry, pharmacy and biotechnology have applied the aforementioned methods to solve enzymatic kinetics problems [19]. Enzymatic reactions are seen as being of importance in managing life processes. Enzymes are specific proteins stimulated by chemical reactions [25]. An excellent understanding of how enzymes work can be used in diagnosing pathological disorders, drugs workability and disease treatment based on the available data [20], [21].

One of the most used enzymatic kinetic models is a non-linear differential equation known as the Michaelis-Menten equation [22], [23]. This model has been solved by some researchers for its approximated solutions including homotopy perturbation method [24], [25], Fourier spectral method [9] etc.

Very recently, [26] used the ancient Chinese algorithm also known as the Ying Buzu algorithm to obtain fourth order approximated solution for the Michaelis-Menten equation. In this paper, we modify the boundary conditions of the Michaelis-Menten equation and apply the Taylor series and Ying Buzu algorithms to obtain fourth and sixth order approximated solutions for the Michaelis-Menten equation and compare the two solutions based on some factors.

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II. Preliminaries

In this section, we present the Taylor series approximation method for solving a second order nonlinear differential equation used in [1], [26], as a prerequisite for studying the Ying Buzu Shu algorithm.

We consider the nonlinear differential below:

$$\frac{d^2v}{dx^2} + K(v) = 0, \tag{1}$$

$$v'(c) = a, \tag{2}$$

$$v(d) = b, \tag{3}$$

where v is a function of x, $v'(c)$ and $v(d) = b$ are the boundary conditions.

If

$$v(c) = \alpha, \tag{4}$$

The infinite Taylor series expansion can be used to express the exact solution of (1).

From [1], [26], the Taylor series expansion for k^{th} order derivative is given as

$$\begin{cases} v(x) = v(c) + v'(c)(x-c) + \frac{1}{2!}v''(c)(x-c)^2 \\ \qquad + \frac{1}{3!}v'''(c)(x-c)^3 \\ + \cdots + \frac{1}{(k-1)!}v^{(k-1)}(c)(x-c)^{k-1} + \frac{1}{k!}v^k(c)(x-c)^k \end{cases},$$

where

$$\begin{cases} v(c) = \alpha, v'(c) = a, v''(c) = -K(\alpha), \\ v'''(c) = -a\frac{\partial K(\alpha)}{\partial v}, \end{cases}.$$

III. Methodology and Mathematical Formulation of Kinetic Model

A. The Ying Buzu Algorithm

In this section, we discuss the Ying Buzu algorithm method. This method was used in [26]-[28] to solve some differential equations. Also, this method was applied in [29]-[32] for He's frequency formulation for nonlinear oscillators as well as Chun-Hui He algorithm for numerical simulation [33].

The following steps are involved in the application of Ying Buzu algorithm.

For a nonlinear differential equation similar to (1)-(4), let $v_1(c)$ and $v_2(c)$ be two initial guesses such that

$$\begin{cases} v_1(c) = \alpha_1, \\ v_2(c) = \alpha_2. \end{cases} \tag{5}$$

where α_1 and α_2 are given approximate values [26]. From (3) and (5),

$$v(d, \alpha_1) = \mathscr{b}_1, v(d, \alpha_2) = \mathscr{b}_2$$

Similar to [6]-[12], the Ying Buzu algorithm with initial guess can be updated as:

$$v(c)_{est} = \alpha_3 = \frac{\alpha_1(\mathscr{b} - \mathscr{b}_2) - \alpha_2(\mathscr{b} - \mathscr{b}_1)}{\mathscr{b}_1 - \mathscr{b}_2}. \tag{6}$$

The terminal value can be calculated as:

$$v(d, \alpha_3) = \mathscr{b}_3.$$

This process can be repeated as follows to obtain an approximate solution with a very small relative error.

$$v(d, \alpha_1) = \mathscr{b}_1, v(d, \alpha_3) = \mathscr{b}_3,$$
$$\alpha_4 = \frac{\alpha_1(\mathscr{b} - \mathscr{b}_3) - \alpha_3(\mathscr{b} - \mathscr{b}_1)}{\mathscr{b}_1 - \mathscr{b}_3}.$$
$$v(d, \alpha_4) = \mathscr{b}_4.$$

For a given small threshold, δ, $|\mathscr{b} - \mathscr{b}_4| \le \delta$, we obtain $v(c) = \mathscr{b}_4$ as an approximate solution.

B. The Kinetic Model

In [25], the following nonlinear boundary value problem represents the law of mass action of oxygen:

$$v''(x) + \mathscr{n} v''(x) v(x) = \mathcal{N} v(x) \tag{7}$$

with boundary conditions $v(0) = \mathcal{N}\alpha, v(1) = 1 + \alpha, v'(0) = 0$ where x is the dimension, $\mathcal{N}$ is the reaction diffusion parameter, v is the oxygen concentration and $\mathscr{n}$ the saturation parameter.

IV. Main Result

We consider here the kinetic model in (7) where $\mathcal{N} = 1, \mathscr{n} = 1$, the problem reduces to the problem solved in [26].

In this paper, we will consider a case where $\mathcal{N} = 2, \mathscr{n} = 1$, such that (7) reduces to

$$v''(x) = \frac{2v(x)}{1 + v(x)}, \tag{8}$$

with boundary conditions

$$v(0) = 2\alpha, v(1) = 1 + \alpha, v'(0) = 0 . \tag{9}$$

From (9),

$$v(0) = 2\alpha , \tag{10}$$

$$v'(0) = 0 , \tag{11}$$

$$v''(0) = \frac{4\alpha}{1 + 2\alpha}, \tag{12}$$

$$v'''(0) = 0 ,$$

$$v'^{v}(0) = \frac{8\alpha}{(1 + 2\alpha)^3}, \tag{13}$$

$$v^{v}(0) = 0 , \tag{14}$$

$$v^{v'}(0) = \frac{16\alpha(1 - 12\alpha)}{(1 + 2\alpha)^5}. \tag{15}$$

A. Taylor Series Solution

First, we proceed to obtain a second, fourth and sixth order Taylor series solution for (8) as

$$v(x) = v(0) + v'(0)x + \frac{1}{2!} v''(0)x^2 \tag{16}$$

Substituting (10)-(12) into (16), we have

$$v(x) = 2\alpha + \frac{2\alpha}{1 + 2\alpha} x^2 \tag{17}$$

Substituting $v(1) = 1 + \alpha$ into (17) and solve for α, we have $\alpha = 0.5$, hence, the second order Taylor series expansion for (8) is given as

$$v(x) = 1 + 0.5x^2 . \tag{18}$$

For the fourth order Taylor series, we have

$$v(x) = v(0) + v'(0)x + \frac{1}{2!} v''(0)x^2 + \frac{1}{3!} v'''(0)x^3 + \frac{1}{4!} v'^{v}(0)x^4 \tag{19}$$

Substituting (10)-(13) into (19), we have

$$v(x) = 2\alpha + \frac{2\alpha}{1 + 2\alpha} x^2 + \frac{\alpha}{3(1 + 2\alpha)^3} x^4 . \tag{20}$$

Substituting $v(1) = 1 + \alpha$ into (20) and solve for α, we have

$$\mathcal{W}(\alpha) = \alpha + \frac{2\alpha}{1 + 2\alpha} + \frac{\alpha}{3(1 + 2\alpha)^3} - 1 .$$

To find α, we assume the two initial solutions as

$$\alpha_1 = 0.5 \text{ and } \alpha_2 = 0.8,$$
$$\mathcal{W}_1(0.5) = 0.02083 \text{ and } \mathcal{W}_2(0.8) = 0.43056,$$

$$\alpha = \frac{\alpha_1 \mathcal{W}_2 - \alpha_2 \mathcal{W}_1}{\mathcal{W}_2 - \mathcal{W}_1} = \frac{0.5(0.43056) - 0.8(0.0208)}{0.43056 - 0.02083} = 0.4848\ ,$$

Therefore, the fourth order Taylor series expansion for (8) is

$$v(x) = 0.9696 + 0.4923x^2 + 0.0216x^4. \qquad (21)$$

For the sixth order Taylor series, we have

$$v(x) = v(0) + v'(0)x + \frac{1}{2!}v''(0)x^2 + \frac{1}{3!}v'''(0)x^3 + \frac{1}{4!}v^{iv}(0)x^4 + \frac{1}{5!}v^{v}(0)x^5 + \frac{1}{6!}v^{vi}(0)x^6\ . \ (22)$$

Substituting (9)-(15) into (22), we have

$$v(x) = 2\alpha + \frac{2\alpha}{1+2\alpha}x^2 + \frac{\alpha}{3(1+2\alpha)^3}x^4 + \frac{\alpha(1-12\alpha)}{45(1+2\alpha)^5}x^6. \qquad (23)$$

Substituting $v(1) = 1 + \alpha$ into (20) and solve for α, we have

$$\mathcal{W}(\alpha) = \alpha + \frac{2\alpha}{1+2\alpha} + \frac{\alpha}{3(1+2\alpha)^3} + \frac{\alpha(1-12\alpha)}{45(1+2\alpha)^5} - 1\ .$$

To find α, we assume the two initial solutions as

$$\alpha_1 = 0.5 \text{ and } \alpha_2 = 0.8,$$
$$\mathcal{W}_1(0.5) = 0.0191 \text{ and } \mathcal{W}_2(0.8) = 0.4293,$$
$$\alpha = \frac{\alpha_1 \mathcal{W}_2 - \alpha_2 \mathcal{W}_1}{\mathcal{W}_2 - \mathcal{W}_1} = \frac{0.5(0.4293) - 0.8(0.0191)}{0.4293 - 0.0191} = 0.4860,$$

Therefore, the sixth order Taylor series expansion for (8) is

$$v(x) = 0.9720 + 0.4929x^2 + 0.0211x^4 - 0.0017x^6 \ (24)$$

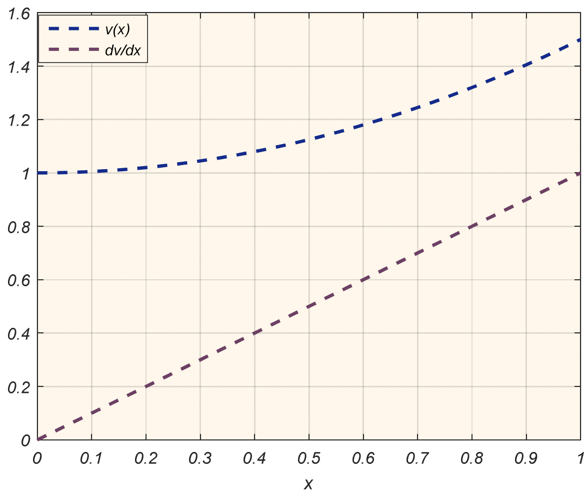

Fig. 1 The second order Taylor series solution

B. Ying Buzu Shu Algorithm

In this section, we will apply the Ying Buzu Shu algorithm to the fourth and sixth order Taylor series approximation to obtain a more accurate solution for (8) with a less relative error.

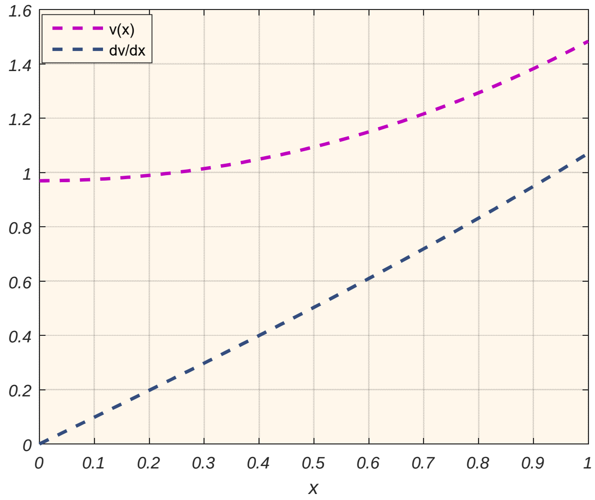

Fig. 2 The Fourth order Taylor series solution

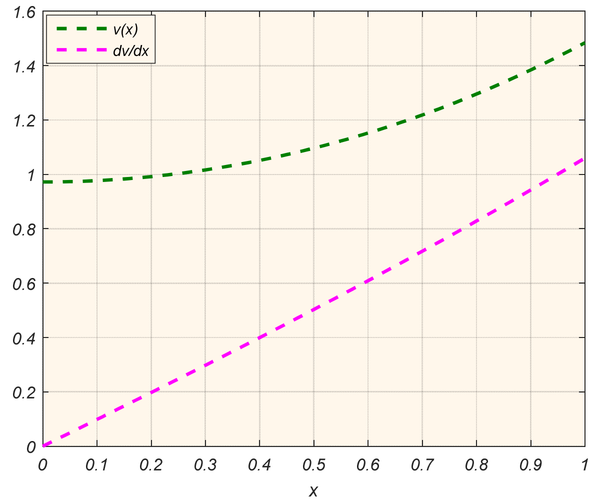

Fig. 3 The Sixth order Taylor series solution

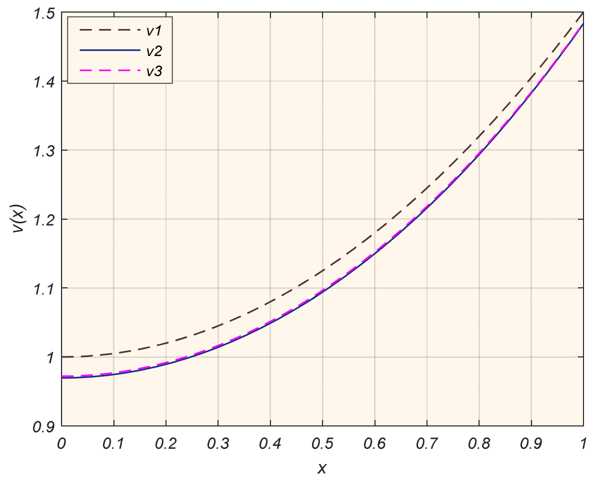

Fig. 4 Convergence of the second, fourth and sixth order Taylor series solutions

To apply the Ying Buzu Shu algorithm to the fourth order Taylor series expansion, we begin by choosing two initial guesses as:

$$\alpha_1 = v_1(0) = 0.5 \text{ and } \alpha_2 = v_2(0) = 1,\ \alpha = 0.4848 \quad (25)$$

Substituting (22) into the fourth order Taylor series expansion in (18) we have

$$v(x) = 2\alpha + \frac{2\alpha}{1+2\alpha}x^2 + \frac{\alpha}{3(1+2\alpha)^3}x^4$$
$$\mathscr{b}_1 = v_1 = v(1,0.5) = 1.5208$$
$$\mathscr{b}_2 = v_2 = v(1,1) = 2.6790$$
$$\mathscr{b} = 1 + \alpha = 1.4848$$

Using the Ying Buzu Shu algorithm in (6), we have

$$\alpha_3 = \begin{cases} \dfrac{\alpha_1(\mathscr{b}-\mathscr{b}_2)-\alpha_2(\mathscr{b}-\mathscr{b}_1)}{\mathscr{b}_1-\mathscr{b}_2} \\ = \dfrac{0.5\,(1.4848-2.6790)-1(1.4848-1.5208)}{1.5208-2.6790} = 0.4845 \end{cases}$$

$$v_3(x) = 0.969 + 0.4921x^2 + 0.0212x^4\,. \quad (26)$$

From (23), $v_3(1) = 1.4823$ and from (9): $v_3(1) = 1.4845$. Hence the relative error is 0.22%.

Next, we proceed with the algorithm to see if we can have a smaller relative error by using the following guesses

$$\alpha_1 = v_1(0) = 0.5,\ \alpha_3 = v_3(0) = 0.4845\ ,\ \alpha = 0.4848 \quad (27)$$

$$\mathscr{b}_1 = v_1 = v(1,0.5) = 1.5208,$$
$$\mathscr{b}_3 = v_3 = v(1,0.4845) = 1.4823,$$
$$\mathscr{b} = 1 + \alpha = 1.4848\,.$$

Using the Ying Buzu Shu algorithm in (6), we have

$$\alpha_4 = \frac{\alpha_1(\mathscr{b}-\mathscr{b}_3)-\alpha_3(\mathscr{b}-\mathscr{b}_1)}{\mathscr{b}_1-\mathscr{b}_3} = \frac{0.5\,(1.4848-1.4823)-0.4845(1.4848-1.5208)}{1.5208-1.4823} = 0.4855$$

$$v_4(x) = 0.971 + 0.4926x^2 + 0.0211x^4 \quad (28)$$

From (24), $v_4(1) = 1.4847$ and from (9), $v_4(1) = 1.4855$ Hence the relative error is 0.08%.

Repeating the algorithm for α_5 and v_5 with the following guesses $\alpha_1 = v_1(0) = 0.5$ and $\alpha_4 = v_4(0) = 0.4855$, $\alpha = 0.4848$, we obtain a relative error of 0.08% similar to the previous algorithm.

Next, we apply the Ying Buzu Shu algorithm to the sixth order Taylor series expansion, we begin by choosing two initial guesses as:

$$\alpha_1 = v_1(0) = 0.5 \text{ and } \alpha_2 = v_2(0) = 1,\ \alpha = 0.4860 \quad (29)$$

Substituting (25) into the sixth order Taylor series expansion in (20) we have

$$v(x) = 2\alpha + \frac{2\alpha}{1+2\alpha}x^2 + \frac{\alpha}{3(1+2\alpha)^3}x^4 + \frac{\alpha(1-12\alpha)}{45(1+2\alpha)^5}x^6\,,$$
$$\mathscr{b}_1 = v_1 = v(1,0.5) = 1.5191\,,$$
$$\mathscr{b}_2 = v_2 = v(1,1) = 2.6780\,,$$
$$\mathscr{b} = 1 + \alpha = 1.4860.$$

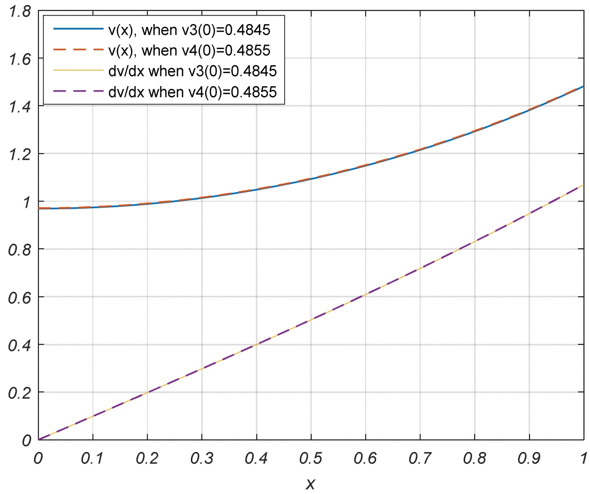

Fig. 5 The fourth order Ying Buzu Shu processes with different initial guesses

Using the Ying Buzu Shu algorithm in (6), we have

$$\alpha_3 = \begin{cases} \dfrac{\alpha_1(\mathscr{b}-\mathscr{b}_2)-\alpha_2(\mathscr{b}-\mathscr{b}_1)}{\mathscr{b}_1-\mathscr{b}_2} \\ = \dfrac{0.5\,(1.4860-2.6780)-1(1.4860-1.5191)}{1.5191-2.6780} = 0.4857 \end{cases}$$

$$v_3(x) = 0.9714 + 0.4927x^2 + 0.0211x^4 - 0.0018x^6. \quad (30)$$

From (30), $v_3(1) = 1.4835$ and from (9), $v_3(1) = 1.4857$, hence the relative error is 0.22%

Next, we proceed with the algorithm to see if we can have a smaller relative error by using the following guesses

$$\alpha_1 = v_1(0) = 0.5,\ \alpha_3 = v_3(0) = 0.4857,\ \alpha = 0.4860, \quad (31)$$

$$\mathscr{b}_1 = v_1 = v(1,0.5) = 1.5191\,,$$
$$\mathscr{b}_3 = v_3 = v(1,0.4857) = 1.4835\,,$$
$$\mathscr{b} = 1 + \alpha = 1.4860\,.$$

Using the Ying Buzu Shu algorithm in (6), we have

$$\alpha_4 = \frac{\alpha_1(\mathscr{b}-\mathscr{b}_3)-\alpha_3(\mathscr{b}-\mathscr{b}_1)}{\mathscr{b}_1-\mathscr{b}_3} = \frac{0.5\,(1.4860-1.4835)-0.4857(1.4860-1.5191)}{1.5191-1.4835} = 0.4867$$

$$v_4(x) = 0.9734 + 0.4931x^2 + 0.0211x^4 - 0.0017x^6 \quad (32)$$

From (32), $v_4(1) = 1.4860$ and from (9), $v_4(1) = 1.4867$ Hence the relative error is 0.07%.

Repeating the algorithm for α_5 and v_5 with the following guesses $\alpha_1 = v_1(0) = 0.5$ and $\alpha_4 = v_4(0) = 0.4867$, $\alpha = 0.4860$, we obtain a relative error of 0.07% similar to the previous algorithm.

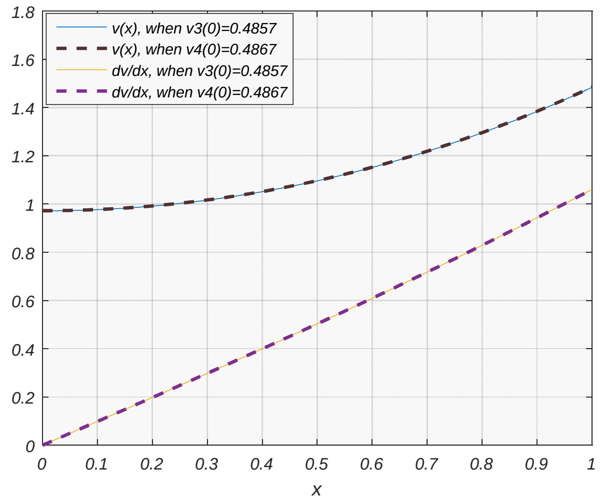

Fig. 6 The sixth order Ying Buzu Shu processes with different initial guesses

V. Discussion and Conclusion

Figs. 1-3 show plots of the Taylor series approximate solutions of the nonlinear boundary value problem in (8) and (9). The graphs show that the approximate solutions satisfy the boundary conditions criteria. Fig. 4 gives a comparison of the approximate solutions for the second, fourth and sixth order Taylor series solution. It is observed that the fourth and sixth order approximate solutions converge very closely to the boundary conditions unlike the second order approximation solution. Also, Figs. 5 and 6 show plots of the fourth and sixth order Ying Buzu Shu processes with different initial guesses. We observed that by using the initial guesses in Ying Buzu Shu algorithm for the fourth order algorithm, we obtain an approximate solution $v(1) = 1.4855$ with relative error of 0.08% different from the approximate solution $v(1) = 1.4835$ obtained for the Taylor series method with relative error of 0.13%. Also, a relative error of 0.07% was obtained for Ying Buzu Shu algorithm for the sixth order algorithm. This shows that with Ying Buzu Shu algorithm, we can obtain a better approximated solution very close to the exact solution. This can further be confirmed in [26], where the relative error for Ying Buzu Shu algorithm was 0.01% as compared to 0.07% for the solution obtained with Taylor series method. Hence, we conclude that solutions obtained with Ying Buzu Shu process are much more accurate than that of Taylor series approximate solutions. Furthermore, the method gives a simple and clear-cut means in solving two point second order nonlinear boundary value problems.

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