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Exponential Stability of Linear Systems under a Class of Unbounded Perturbations

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Abstract—In this work, we investigate the exponential stability of a linear system described by $\dot{x}(t) = Ax(t) - \rho Bx(t)$. Here, A generates a semigroup $S(t)$ on a Hilbert space, the operator B is supposed to be of Desch-Schappacher type, which makes the investigation more interesting in many applications. The case of Miyadera-Voigt perturbations is also considered. Sufficient conditions are formulated in terms of admissibility and observability inequalities and the approach is based on some energy estimates. Finally, the obtained results are applied to prove the uniform exponential stabilization of bilinear partial differential equations.

Keywords—Exponential stabilization, unbounded operator, Desch-Schappacher, Miyadera-Voigt operator.

I. Introduction

In this work, we deal with the following perturbed linear system:

$$\dot{x}(t) = Ax(t) - \rho Bx(t), \;\; t > 0, \;\; x(0) = x_0. \qquad (1)$$

Here, X is a Hilbert space called the state space with inner product $\langle \cdot, \cdot \rangle$ and corresponding norm $\|.\|_X$. The system's operator A is the infinitesimal generator of a C_0-semigroup $S(t)$ on X, and the control operator B is an unbounded linear operator of X. In several practical situations the modeling gives rise to systems of the form (1) with a perturbation operator B which is of Miyadera-Voigt or Desch-Schappacher type. This is the case for instance when the control acts in a multiplicative way through the boundary or at a point of the system's geometrical domain and also in many situations of internal control of partial differential equations (see e.g. [4], [10], [11]). In general, due to the unbounded aspect of the operator B, the solution of (1) does not exist with values in X. Thus, in order to confront this difficulty, the concept of admissibility is needed, which requires the introduction of interpolating and extrapolating spaces of the state space X.

In this work we will investigate the problem of exponential stability of system (1). This consists of looking for a set of parameters ρ for which there exists a global $X-$valued mild solution $x(t)$ of (1) and is such that $\|x(t)\| \leq Ke^{-\sigma t}\|x_0\|, \;\; \forall t \geq 0$ for some constants $K, \sigma > 0$. As an application, one can consider the stabilization of bilinear systems by means of switching controllers. Note that such problem has been considered in several works [2], [3], [5], [12], [14]. In [12] the authors treated the case of a bounded operator B, and in [14] for a Miyadera-Voigt type operator. Moreover, in [2] the case of $1-$admissibility in Banach space has been considered either for Miyadera-Voigt or Desch-Schappacher perturbations. However, the $1-$admissibility condition excludes the case of several unbounded systems evolving in a Hilbert state space. In this paper, we will study the exponential stability of system (1) under the $p-$admissibility property of the operator B with $p \geq 1$.

In the next section, we give some preliminaries on Miyadera-Voigt and Desch-Schappacher operators. In the third section we state and prove the main results. Section IV concerns applications to feedback stabilization of the reaction-diffusion equation.

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II. Preliminaries

As pointed out in the introduction, the unbounded aspect of the operator B requires the introduction of interpolating and extrapolating spaces. Classically, the spaces X_1 and X_{-1} are defined as follows: $X_1 := (D(A), \|\cdot\|_1)$, where $\|x\|_1 := \|(\lambda I - A)x\|_X$, $x \in D(A)$ for some λ in the resolvent set $\rho(A)$ of A, and X_{-1} is the completion of X with respect to the norm $\|x\|_{-1} := \|(\lambda I - A)^{-1}x\|_X$, $x \in X$. These spaces are independent of the choice of λ and are related by the following continuous and dense embedding: $X_1 \hookrightarrow X \hookrightarrow X_{-1}$. That way the unbounded operator B becomes bounded from X_1 to X if B is a Miyadera-Voigt operator, and from X to the extrapolating space X_{-1} in the case of Desch-Schappacher operator.

A. Preliminary on Desch-Schappacher perturbations

Here, the operator B is supposed to be of Desch-Schappacher type (see below). Thus, in order to give a meaning to solutions of (1), we will use that the semigroup $S(t)$ can be extended to a C_0-semigroup $(S_{-1}(t))_{t\geq 0}$ on X_{-1} whose generator A_{-1} has $D(A_{-1}) = X$ as domain and is such that $A_{-1}x = Ax$, for any $x \in D(A)$. Note that if the semigroup $S(t)$ is a contraction, then so is $S_{-1}(t)$. The system (1) can be rewritten in the large space X_{-1} in the following abstract form:

$$\dot{x}(t) = A_{-1}x(t) - \rho Bx(t), \; x(0) = x_0,$$

which is well-posed in X whenever $A - \rho B$ is a generator of a C_0-semigroup on X (cf. [9], Section II.6).

The next result provides sufficient conditions on the Desch-Schappacher perturbation B to guarantee the existence and uniqueness of the mild solution of (1) (see [1] & ([9], p. 183)).

Theorem 1. Let A be the generator of a C_0-semigroup $S(t)$

on X and let $B \in \mathcal{L}(X, X_{-1})$ be $p-$admissible for some $1 \leq p < \infty$, i.e., there is a $T > 0$ such that

$$\int_0^T S_{-1}(T-t)Bu(s)ds \in X, \quad \forall u \in L^p(0,T;X)\cdot \quad (2)$$

Then for any ρ, the operator $(A_{-1} - \rho B)|_X$ is defined on the domain $D((A_{-1} - \rho B)|_X) := \{x \in X : (A_{-1} - \rho B)x \in X\}$ by

$$(A_{-1} - \rho B)|_X x := A_{-1}x - \rho Bx, \ \forall x \in D((A_{-1} - \rho B)|_X) \quad (3)$$

is the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on X, which verifies the following variation of parameters formula for all $x_0 \in D((A_{-1} - \rho B)|_X)$

$$T(t)x_0 = S(t)x_0 - \rho \int_0^t S_{-1}(t-s)BT(s)x_0 ds, \ \forall t \geq 0.$$

Let us recall the following lemma which can be deduced from a general one from [8]:

Lemma 2. Let A generate a C_0-semigroup $S(t)$ on the Hilbert space X. Then

- for all $f \in L^1_{loc}(0, +\infty; X)$, we have

$$\int_0^t S_{-1}(t-s)f(s)ds \in X, \ \forall t \geq 0,$$

- for any $w > 0$, there is a constant $K > 0$ (which is independent of t and f) such that

$$\| \int_0^t S_{-1}(t-s)f(s)ds\|_X \leq Ke^{wt}\|f\|_{L^1(0,t;X)} \quad (4)$$

Remarks.

• An operator $B \in \mathcal{L}(X, X_{-1})$ satisfying the condition (2) is called a Desch-Schappacher operator or perturbation.

• Notice that since $W^{1,p}(0,T;X)$ is dense in $L^p(0,T;X)$, the rank condition (2) is equivalent to the existence of some $M > 0$ such that

$$\| \int_0^T S_{-1}(T-s)Bu(s)ds\|_X \leq M\|u\|_{L^p(0,T;U)}, \quad (5)$$

for all $u \in W^{1,p}(0,T;X)$, where $\|u\|_{L^p(0,T;X)} = (\int_0^T \|u(t)\|_X^p dt)^{\frac{1}{p}}$.

Moreover, if the operator $B \in \mathcal{L}(X, X_{-1})$ is $p-$admissible in $[0,T]$, then it is so in $[0,t]$ for any $t \in [0,T]$.

• Under the assumptions of Theorem 1, the mild solution $x(t)$ of the system (1) satisfies the following estimate, for any $0 < \rho < \frac{1}{T^{\frac{1}{p}}M}$:

$$\|x(.)\|_{L^p(0,T;X)} \leq \frac{T^{\frac{1}{p}}}{1 - \rho T^{\frac{1}{p}}M}\|x_0\|_X, \ \forall x_0 \in X, \quad (6)$$

and from which, it follows that for all $t \geq 0$, we have

$$\|x(t)\|_X \leq \left(1 + \frac{\rho M T^{\frac{1}{p}}}{1 - T^{\frac{1}{p}}\rho M}\right)\|x_0\|_X. \quad (7)$$

• Under the assumptions of Theorem 1, one can show that the mild solution $x(t)$ of the system (1) satisfies the following estimate

$$\| \int_0^t S_{-1}(t-s)Bx(s)ds\|_X \leq M_\rho\|x_0\|_X, \ \ \forall t \in [T, 2T], \ (8)$$

for every $0 < \rho < \frac{1}{T^{\frac{1}{p}}M}$ with $M_\rho := \frac{MT^{\frac{1}{p}}}{1-\rho T^{\frac{1}{p}}M}\left(2 + \frac{\rho MT^{\frac{1}{p}}}{1-\rho T^{\frac{1}{p}}M}\right)$.

B. Preliminary on Miyadera-Voigt Perturbations

Let us recall the following theorem regarding the well-posedness of (1) in the case of Miyadera-Voigt perturbations (see [13]).

Theorem 2. Let $B \in \mathcal{L}(X_1, X)$ be such that

$$\int_0^T \|BS(s)x\|_X^p ds \leq \alpha\|x\|_X^p, \ \forall x \in D(A), \quad (9)$$

for some $T, \alpha > 0$ with $p > 1$ (i.e., B is $p-$admissible w.r.t $S(t)$). Then the operator A_B defined by

$$A_B x := (A+B)x, \ x \in D(A_B) := D(A),$$

is the generator of a C_0-semigroup $\mathrm{T}(t)$ on X.

Remarks.

• The semigroup $\mathrm{T}(t)$ satisfies, for any $x_0 \in D(A)$, the following integral equation:

$$\mathrm{T}(t)x_0 = S(t)x_0 + \int_0^t S(t-s)B\mathrm{T}(s)x_0 ds, \ \forall t \geq 0.$$

• Note that we can also consider the condition (24) for $p = 1$ provided that $0 < \alpha < 1$. Moreover (see [9], p. 199, [1], [14]) if the following estimate

$$\int_0^T \|BS(t)y\|dt \leq \beta\|y\|_X, \ \forall y \in D(A), \quad (10)$$

holds for some $T, \beta > 0$, then we have:

(i) for every $0 < \rho < \frac{1}{\beta}$, the operator $A - \rho B$, defined on $D(A)$, generates a C_0-semigroup $\mathrm{T}(t)$ on X given for all $x_0 \in X$ by

$$\mathrm{T}(t)x_0 = S(t)x_0 - \rho \int_0^t S(t-s)B\mathrm{T}(s)x_0 ds, \ t \geq 0.$$

(ii) Moreover, $\mathrm{T}(t)$ satisfies

$$\int_0^{\mathcal{T}} \|B\mathrm{T}(s)x\|_X ds \leq \frac{\beta}{1-\rho\beta}\|x\|_X, \ \forall x \in D(A). \quad (11)$$

III. Stabilisation Results

In the following theorem, we provide sufficient conditions for exponential stability of system (1) under Desch-Schappacher perturbations.

Theorem 3. Suppose that A is the infinitesimal generator of a linear C_0-semigroup of contractions $(S(t))_{t\geq 0}$ on X and that for some $T > 0$,

(i) there exists $1 < p < \infty$ such that for all $u \in L^p(0,T;X)$, we have
$\int_0^T S_{-1}(T-s)Bu(s)ds \in X$,

(ii) for any $t>0$, $Range(BS(t)) \subset X$,

(iii) there exists $\delta > 0$ such that

$$\int_0^T Re\,\langle BS(t)x, S(t)x\rangle_X\, dt \geq \delta\|S(T)x\|_X^2,\ \forall x \in D(A). \tag{12}$$

Then there is a $\rho_1 > 0$ such that the system (1) is exponentially stable on X for all $\rho \in (0,\rho_1)$.

Proof: for any $\rho > 0$, let us set $A_{\rho B} := (A_{-1} - \rho B)|_X$. According to Theorem II-A, the system (1) admits a unique mild solution which is given, for $x_0 \in D((A_{\rho B})|_X)$, by the variation of parameters formula (see [7]):

$$x(t) = S(t)x_0 - \rho \int_0^t S_{-1}(t-s)Bx(s)ds, \quad \forall t \geq 0. \tag{13}$$

Case 1: Let us suppose that for all $x_0 \in X$ and $t>0$, we have $Bx(t) \in X$. Let $x_0 \in D(A_{\rho B})$ be fixed. We have

$$\frac{d}{dt}\|x(t)\|_X^2 = 2Re\langle A_{\rho B}x(t), x(t)\rangle_X, \quad \forall t > 0\cdot \tag{14}$$

Moreover, by virtue of the closed graph theorem, we deduce from (i) that for some constant $M>0$ and for all $u \in L^p(0,T;X)$, we have

$$\left\|\int_0^T S_{-1}(T-s)Bu(s)ds\right\|_X \leq M\|u\|_{L^p(0,T;X)} \tag{15}$$

and

$$\left\|B\int_0^T S_{-1}(T-s)Bu(s)ds\right\|_X \leq M\|u\|_{L^p(0,T;X)}. \tag{16}$$

Then we deduce from (13)

$$\|S(t)x_0 - x(t)\|_X \leq \rho M\|x(.)\|_{L^p(0,T;X)}, \forall t \in [0,T].$$

Then, according to the estimate (7), we conclude that

$$\|S(t)x_0 - x(t)\|_X \leq \frac{\rho M T^{\frac{1}{p}}}{1-\rho T^{\frac{1}{p}}M}\|x_0\|_X \tag{17}$$

By similar arguments as above, we can see that (16) implies that for all $t\in[0,T]$ we have

$$\|B\int_0^t S_{-1}(t-s)Bx(s)ds\|_X \leq M\|x\|_{L^p(0,T;X)}, \tag{18}$$

from which it comes via (13)

$$\|B(S(t)x_0 - x(t))\|_X \leq \rho M\|x(.)\|_{L^p(0,T;X)},$$

and hence

$$\|B(S(t)x_0 - x(t))\|_X \leq \frac{\rho M T^{\frac{1}{p}}}{1-\rho T^{\frac{1}{p}}M}\|x_0\|_X. \tag{19}$$

For $x_0 \in D(A_{\rho B})$, we have the following equality

$$\begin{aligned} Re\,\langle BS(t)x_0, S(t)x_0\rangle_X &= \langle BS(t)x_0, S(t)x_0 - x(t)\rangle_X \\ &+ \langle BS(t)x_0 - Bx(t), x(t)\rangle_X \\ &+ \langle Bx(t), x(t)\rangle_X \end{aligned}$$

It follows that

$$\begin{aligned} & Re\,\langle BS(t)x_0, S(t)x_0\rangle_X \\ \leq\ & \|x_0\|_X\|(BS(t))^*(S(t)x_0 - x(t))\|_X \\ +\ & \|B(S(t)x_0 - x(t))\|_X\|x(t)\|_X \\ +\ & \langle Bx(t), x(t)\rangle_X \end{aligned}$$

Using (7), (17) and (19), comes

$$\begin{aligned} Re\,\langle BS(t)x_0, S(t)x_0\rangle_X &\leq \frac{\rho c M T^{\frac{1}{p}}}{1-\rho T^{\frac{1}{p}}M}\|x_0\|_X^2 \\ &+ \frac{\rho M T^{\frac{1}{p}}}{1-\rho T^{\frac{1}{p}}M}\left(1 + \frac{\rho M T^{\frac{1}{p}}}{1-T^{\frac{1}{p}}\rho M}\right)\|x_0\|_X^2 \\ &+ Re\,\langle Bx(t), x(t)\rangle_X, \end{aligned}$$

with $c := \|B^*\|_{\mathcal{L}(X_{-1},X)}$. Integrating the last inequality, yields

$$\begin{aligned} \int_0^T Re\,\langle BS(t)x_0, S(t)x_0\rangle_X\, dt &\leq \rho C_1\|x_0\|_X^2 \\ &+ \int_0^T Re\,\langle Bx(t), x(t)\rangle_X\, dt \end{aligned}$$

with $C_1 = \frac{MT^{\frac{1}{p}+1}}{1-\rho T^{\frac{1}{p}}M}\left(1 + c + \frac{\rho M T^{\frac{1}{p}}}{1-\rho T^{\frac{1}{p}}M}\right)$.
Applying the inequality (iii), we derive that

$$\delta\|S(T)x(t)\|_X^2 - \rho C_1\|x(t)\|_X^2 \leq \int_t^{t+T} Re\,\langle Bx(s), x(s)\rangle_X\, ds \tag{20}$$

We deduce via (24) and the variation of constants formula (13) that for all $t \in [T,2T]$, we have

$$\begin{aligned} \|x(t)\|_X &\leq \|S(T)x_0\|_X + \rho\|\textstyle\int_0^t S_{-1}(t-s)Bx(s)ds\|_X \\ &\leq \|S(T)x_0\|_X + \rho M_\rho\|x_0\|_X. \end{aligned}$$

By reiterating the processes for $t \in [kT,(k+1)T]$, $k \geq 1$, we deduce that

$$\|x(t)\|_X \leq \|S(T)x(kT)\|_X + \rho M_\rho\|x(kT)\|_X.$$

Then for all $k \geq 1$, we have

$$\|x((k+1)T)\|_X^2 \leq 2\|S(T)x(kT)\|_X^2 + 2\rho M_\rho\|x(kT)\|_X^2. \tag{21}$$

Integrating (14) and using the dissipativeness of A gives

$$2\rho\int_{kT}^{(k+1)T} Re\langle Bx(\tau), x(\tau)\rangle_X d\tau \leq \|x(kT)\|_X^2 - \|x((k+1)T)\|_X^2.$$

This together with (20) and (21) implies

$$\rho\delta\bigg(\|x((k+1)T)\|_X^2 - 2\rho M_\rho\|x(kT)\|_X^2\bigg) - 2C_1\rho^2\|x(kT)\|_X^2 \leq \|x(kT)\|_X^2 - \|x((k+1)T)\|_X^2.$$

Hence

$$(1+\rho\delta)\|x((k+1)T)\|_X^2 \leq \bigg(2\delta\rho^2 M_\rho + 2C_1\rho^2 + 1\bigg)\|x(kT)\|_X^2\cdot$$

This implies

$$\|x\left((k+1)T\right)\|_X^2 \leq C_2\|x(kT)\|_X^2$$

where $C_2 = \frac{2\rho^2(\delta M_\rho + C_1)+1}{1+\rho\delta}$, which belongs to $(0,1)$ for $\rho \to 0^+$. Using the integer part $k = E\left(\frac{t}{T}\right)$, we deduce from (8) that

$$\|x(t)\|_X^2 \leq C_3(C_2)^k\|x_0\|_X^2, \ \ (C_3 > 0)$$

which gives the following exponential decay

$$\|x(t)\|_X \leq Ke^{-\sigma t}\|x_0\|, \ \forall t \geq 0, \tag{22}$$

for some $K, \sigma > 0$. This estimate extends by density to all $x_0 \in X$. Hence the uniform exponential stability holds for any $0 < \rho < \rho_1$, where ρ_1 is such that $0 < \rho_1 < \frac{1}{T^{\frac{1}{p}}M}$ and $\frac{2\rho^2(\delta M_\rho + C_1)+1}{1+\rho\delta} \in (0,1)$.

Case 2: General case. For $\epsilon > 0$ we consider the system (1) with $B_\epsilon := S_{-1}(\epsilon)B$ instead of B. Let us fist observe that the operator B_ϵ is $p-$admissible $(p > 1)$ in the sense of (5) with the same constant M (which is independent of ϵ). Then the corresponding system admits a unique mild solution denoted by x_ϵ, which satisfies the following formula

$$x_\epsilon(t) = S(t)x_0 - \rho\int_0^t S_{-1}(t-s)B_\epsilon x_\epsilon(s)ds, \ \ \forall t \geq 0. \tag{23}$$

Using the assumption (ii) we can see that $Range(B_\epsilon S(t)) \subset X$ and then from the V.C.F (23) we have $B_\epsilon x(t) \in X, \ \forall t \geq 0$. Let us show that $\lim_{\epsilon\to 0} x_\epsilon(t) = x(t)$ on X. For all $t \geq 0$, we have

$$\begin{aligned} x_\epsilon(t) - x(t) &= \rho\int_0^t S_{-1}(t-s)B_\epsilon x_\epsilon(s)ds \\ &- \rho\int_0^t S_{-1}(t-s)Bx(s)ds \\ &= \rho\int_0^t S_{-1}(t-s)B_\epsilon(x_\epsilon(s) - x(s))ds \\ &+ \rho\int_0^t S_{-1}(t-s)(B_\epsilon x(s) - Bx(s)ds. \end{aligned}$$

Then, using the admissibility of B_ϵ, we get

$$\begin{aligned} \|x_\epsilon(t) - x(t)\|_X &\leq \rho M\|x_\epsilon(.) - x(.)\|_{L^p(0,t;X)} \\ &+ \rho\|\int_0^t S_{-1}(t-s)(B_\epsilon x(s) - Bx(s))ds\|_X \end{aligned}$$

with M is a positive constant. Integrating the last inequality over $[0,T]$, we obtain for all $t \in [0,T]$

$$\begin{aligned} &\|x_\epsilon(.) - x(.)\|_{L^p(0,T;X)} \\ \leq\ & C_\rho\int_0^T \|\int_0^t S_{-1}(t-s)(B_\epsilon x(s) - Bx(s)ds\|_X^p, \end{aligned}$$

where $C_\rho := \frac{\rho}{1-(\rho M 2T)^p}$. One can easily verify, via the admissibility assumption, that

$$\lim_{\epsilon\to 0}\|\int_0^t S_{-1}(t-s)(B_\epsilon x(s) - Bx(s)ds\|_X = 0.$$

Using again the admissibility of both operators B and B_ϵ, we get

$$\begin{aligned} &\|\int_0^t S_{-1}(t-s)(B_\epsilon x(s) - Bx(s)ds\|_X \\ \leq\ &\|\int_0^t S_{-1}(t-s)B_\epsilon x(s)\|_X \\ +\ &\|\int_0^t S_{-1}(t-s)Bx(s)\|_X \\ \leq\ &2M\|x(.)\|_{L^p(0,t;X)}. \end{aligned}$$

We conclude from the dominated convergence theorem that $\lim_{\epsilon\to 0} x_\epsilon(t) = x(t)$ in X. Then by procedure as in the first case we obtain the following estimate

$$\|x_\epsilon(t)\|_X \leq Ke^{-\sigma t}\|x_0\|, \ \forall t \geq 0,$$

where $K, \sigma > 0$ are the same as in (22). Using the convergence of $x_\epsilon(t)$ to $x(t)$ we get the estimate (22). ■

Remark. The previous idea of the proof may be applied, via the above corollary, to the case of Desch-Schapacher operator with analytic semigroup with for example, one can take $B = A^\alpha, \ 0 < \alpha < 1$.

We have the following result regarding Miyadera-Voigt perturbations.

Corollary. Let A be a generator of a C_0-semigroup of contractions $S(t)$ on X and let $B \in \mathcal{L}(X_1, X)$ satisfying (12) and such that

$$\int_0^T \|BS(s)x\|_X^p ds \leq \alpha\|x\|_X^p, \ \forall x \in D(A), \tag{24}$$

for some $T, \alpha > 0$ with $p > 1$. Then there exists $\rho_1 > 0$ such that the system (1) is exponentially stable for any $\rho \in (0, \rho_1)$.

Proof:

- According to Theorem 2, for $\rho > 0$ small enough (and for every $\rho > 0$ if $p > 1$), there exists a unique mild solution $x(t) \in D(A), \forall x_0 \in D(A)$.
- Let us prove that B admits an extension (also denoted by B). It suffices to show that B is bounded from $(D(A), \|\cdot\|_X)$ to $(X_{-1}, \|\cdot\|_{-1})$.

First let us observe that $B(D(A)) \subset X \subset X_{-1}$. Moreover, because B is bounded from X_1 to X, the operator $R(\lambda; A)B$ is a bounded operator of X for any $\lambda \in \rho(A)$ [13]. Then for any $x \in D(A)$, we have

$$\|Bx\|_{-1} = \|R(\eta; A)Bx\|_X \leq C\|x\|_X,$$

with C is a positive constant. Thus B is bounded from $(D(A), \|\cdot\|_X)$ to $(X_{-1}, \|\cdot\|_{-1})$, and by density it extends to a bounded operator from $(X, \|\cdot\|_X)$ to $(X_{-1}, \|\cdot\|_{-1})$.

- For $x_0 \in D(A)$ we have $x(t) \in D(A), t \geq 0$ and so $Bx(t) \in X, \ \forall t \geq 0$. Thus the assumption (ii) of Theorem 3 is not needed.
- From the proof of Theorem 3, we observe that it suffices that (15) holds for $u(\cdot) = x(\cdot)$. For this end, we will apply Lemma 2. Indeed, let $x_0 \in D(A)$. Thus $Bx(t) \in X$. Then applying the inequality (4), for t $\in [0,T]$ we get

$$\|\int_0^t S_{-1}(t-s)Bx(s)ds\|_X \leq Ke^{wt}\|Bx(\cdot)\|_{L^1(0,t;X)}.$$

Now we have according to the second subsection of the preliminary,

$$\|Bx(\cdot)\|_{L^1(0,t;X)} \leq \frac{T^{\frac{p-1}{p}}\alpha^{\frac{1}{p}}}{1-\rho T^{\frac{p-1}{p}}\alpha^{\frac{1}{p}}}\|x_0\|$$

where α is the $p-$admissibilty constant of (24).

The reminder of the proof is the same as in the proof of Theorem 3. ■

IV. Applications

Example 1. Let Ω be an open and bounded subset of $\mathbf{R}^d,\ d \geq 1$, and let us consider the following bilinear equation of diffusion type

$$\begin{cases} \frac{\partial}{\partial t}x = \Delta x + gx + \nu(t)(-\Delta)^{\frac{1}{2}}x & \text{on } (0,\infty) \\ x(t) = 0 & \text{on } \partial\Omega \times (0,\infty) \\ x(0) = x_0 & \text{on } \Omega \end{cases} \tag{25}$$

where $g \in L^\infty(\Omega),\ \nu$ is a real valued bilinear control and $x(t) = x(\zeta, t) \in L^2(\Omega)$ is the state. Let us observe that system (25) can be written in the form of (1) if we close it by the switching feedback control $\nu(t) = -\rho\mathbf{1}_{\{t\geq 0\ /\ x(t)\neq 0\}}$.

The strong stabilisation of (25) has been achieved in [6] by making use of a (nonlinear) monotone feedback control. Here, we aim to show the exponential stabilization of (25) using the switching control. For this end we will verify the assumptions of Theorem 4. Let us take the state space $X = L^2(\Omega)$ (endowed with its natural scalar product $\langle\cdot,\cdot\rangle_X$), and consider the control operator $B = (-\Delta)^{\frac{1}{2}}$ and the system's operator $A = \Delta + gI$ with $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$. The operator A generates an analytic semigroup $S(t)$ on X (see [9], p. 107 and p. 176) which is given by the following variation of constants formula:

$$S(t)x = S_0(t)x + \int_0^t S_0(t-s)g(\xi)S(s)xds,\ \ t \geq 0,$$

where $S_0(t)$ is the semigroup generated by A with $g = 0$. In the sequel, in order to make the computation easier, we restrict our self to the mono-dimension case, thus we consider $\Omega = (0,1)$. In this case the semigroup $(S_0(t))$ is given by

$$S_0(t)x = \sum_{j\geq 1} e^{-\alpha_j t}\langle x, \phi_j\rangle_X\ \phi_j,\ \forall x \in L^2(\Omega)$$

with $\alpha_j = j^2\pi^2, j \geq 1$ is the set of eigenvalues of $-\Delta$ with the corresponding orthonormal basis of $L^2(\Omega)$: $\phi_j(x) = \sqrt{2}\sin(j\pi x)$. Moreover, the semigroup $S(t)$ is a contraction if in addition

$$\int_\Omega g(\xi)y^2(\xi)d\xi \leq \|y\|^2_{H_0^1(\Omega)},\ \forall y \in H_0^1(\Omega).$$

Thus, in the sequel we suppose this condition satisfied. Then the operator B can be expressed as

$$Bx = \sum_{j\geq 1}\alpha_j^{\frac{1}{2}}\langle x, \phi_j\rangle_X\ \phi_j,\ x \in L^2(\Omega).$$

Here, B is unbounded on $L^2(\Omega)$ and it is bounded from $L^2(\Omega)$ into the space X_{-1}, defined as the completion of $L^2(\Omega)$ for the norm $\|y\| = \big(\sum_{j\geq 1}\frac{1}{\alpha_j}\langle y, \phi_j\rangle^2\big)^{\frac{1}{2}},\ \forall y \in L^2(\Omega)$.

Let $T > 0$ and let $u \in L^p(0,T;X)$. It follows from the analyticity of the semigroup $(S(t)$ (and so $S_{-1}(t)$) that $\int_0^T S_{-1}(T-s)(-\Delta)^{\frac{1}{2}}u(s)ds \in X$, which gives the admissibility of B (see [9], Prop. 3.3 and [15], Lemma. 4.3.9), Let us check (ii). We have (observe that $\alpha_j \geq 1$)

$$\begin{aligned} y \in D(A) &\Rightarrow \sum_{j\geq 1}\alpha_j^2\langle y, \phi_j\rangle_X^2 < \infty \\ &\Rightarrow \sum_{j\geq 1}\alpha_j\langle y, \phi_j\rangle_X^2 < \infty \\ &\Rightarrow By \in X\cdot \end{aligned}$$

In other words $D(A) \subset D(B|_X)$. Using again the analyticity of $S(t)$ we conclude that $Range(BS(t)) \subset X$.

Now, using again the series expansion of $BS(t)y$ for $y \in D(A)$, we get after integrating:

$$\begin{aligned} \int_0^T \langle BS(t)x, S(t)x\rangle_X\,dt &= \int_0^T \sum_{j\geq 1}\alpha_j^{\frac{1}{2}}\langle S(t)x, \phi_j\rangle_X^2\,dt \\ &\geq \int_0^T \|S(t)x\|_X^2 dt \\ &\geq T\|S(T)x\|_X^2, \end{aligned}$$

hence the assumption (iii) of Theorem 3 is fulfilled.

According to Theorem 3, we conclude that for $\rho > 0$ small enough, the control $\nu(t) = -\rho\mathbf{1}_{\{t\geq 0,\ x(t)\neq 0\}}$ guarantees the uniform exponential stability of the system (25).

Example 2. Let Ω be an open bounded domain in $\mathbf{R}^d,\ d \geq 1$ with sufficiently smooth boundary $\partial\Omega$. We consider on $X := L^2(\Omega)$ the following system:

$$\begin{cases} x_t(t,.) = \Delta x(t,.) + g(.)x(t,.) + v(t)\nabla(a(.)x(t,.)), & in\ \Omega \\ x = 0, & in\ \partial\Omega \end{cases} \tag{26}$$

where $a \in W^{1,\infty}(\Omega)$ s.t $\frac{\partial a}{\partial \zeta_i} > k > 0$, for $i = 1...n$ and $g \in L^\infty(\Omega)$ is such that

$$\int_\Omega g(\zeta)x^2 d\zeta \leq \|\nabla x\|^2_{X^n},\ \forall x \in H_0^1(\Omega).$$

Under this last inequality, the operator $A := \Delta + g(\zeta)I_X$ with $D(A) := H_0^1(\Omega) \cap H^2(\Omega)$ generates a C_0-semigroup $S(t)$ of contractions. Here, the operator B can be identified to the mapping: $x \to \nabla(ax)$ which is clearly $A-$bounded. For any $x \in D(A)$, we have

$$\int_0^T \|\nabla\big(a(S(t)x)\big)\|^2_{X^n}dt \leq$$

$$4\int_0^T \big(\|\nabla a(S(t)x)\|^2_{X^n} + \|a\nabla(S(t)x)\|^2_{X^n}\big)dt$$

$$\leq 4T\|\nabla a\|^2\|x\|^2_X + \|a\|\int_0^T |\langle \Delta S(t)x, S(t)x\rangle_X|dt$$

$$\leq 4T\|\nabla a\|^2\|x\|^2_X + T\|a\|_{L^\infty(\Omega)}\|g\|_{L^\infty(\Omega)}\|x\|^2_X +$$

$$\|a\|_{L^\infty(\Omega)}\int_0^T |\langle \frac{d}{dt}S(t)x, S(t)x\rangle_X|dt$$

$$\leq 4T\|\nabla a\|^2\|x\|^2_X + T\|a\|_{L^\infty(\Omega)}\|g\|_{L^\infty(\Omega)}\|x\|^2_X + \|a\|_{L^\infty(\Omega)}\|x\|^2_X.$$

Thus, the admissibility of B follows. This guarantees the well-posedness of the system (26) for ρ small enough. Now, for $x \in D(A)$ and $T > 0$ we have

$$\int_0^T \langle BS(t)x, S(t)x\rangle dt = \int_0^T \langle \nabla(a(\zeta)(S(t)x)), S(t)x\rangle_X dt$$

which gives by integrating

$$\begin{aligned}\int_0^T \langle BS(t)x, S(t)x\rangle_X \, dt &= -\int_0^T \langle aS(t)x, \nabla(S(t)x)\rangle_X \, dt \\ &= -\int_0^T \frac{1}{2}\int_0^1 a(\zeta)\nabla\left((S(t)x)^2(\zeta)\right) dt \\ &= \int_0^T \frac{1}{2}\int_0^1 \nabla(a(\zeta))(S(t)x)(\zeta))^2 dt \\ &\geq \int_0^T \frac{k}{2}\|S(t)x\|_X^2 dt.\end{aligned}$$

which gives (since $\|S(t)x\|_X$ decreases)

$$\int_0^T \langle BS(t)x, S(t)x\rangle_X \, dt \geq T\|S(T)x\|_X^2.$$

Thus, the assumption (12) is fulfilled. We conclude from corollary that the system (26) is exponentially stable on X.

V. Conclusion

This paper provides sufficient conditions for exponential stability of a linear system under a Desch-Schapacher perturbation of the dynamic. The main assumptions of sufficiency are formulated in terms of admissibility and observability inequalities of unbounded linear operators. An explicit decay rate of the stabilized state is provided. The case of a Miyadera-Voigt perturbation is discussed as well. The main stabilization result is further applied to show the uniform exponential stabilization of unbounded bilinear reaction diffusion and transport equations using a bang bang controller.

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