Experimental Investigation of Natural Frequency and Forced Vibration of Euler-Bernoulli Beam under Displacement of Concentrated Mass and Load

Aref Aasi, Sadegh Mehdi Aghaei, Balaji Panchapakesan

***Abstract*—This work aims to evaluate the free and forced vibration of a beam with two end joints subjected to a concentrated moving mass and a load using the Euler-Bernoulli method. The natural frequency is calculated for different locations of the concentrated mass and load on the beam. The analytical results are verified by the experimental data. The variations of natural frequency as a function of the location of the mass, the effect of the forced frequency on the vibrational amplitude, and the displacement amplitude versus time are investigated. It is discovered that as the concentrated mass moves toward the center of the beam, the natural frequency of the beam and the relative error between experimental and analytical data decreases. There is a close resemblance between analytical data and experimental observations.

***Keywords*—Euler-Bernoulli beam, natural frequency, forced vibration, experimental setup.

I. Introduction

THE vibration analysis of beams carrying concentrated masses at arbitrary locations has been the subject of extensive research for many years. To find a solution for a beam carrying a mass two models have been used: Euler-Bernoulli beam theory and Timoshenko beam theory [1]. To analyze the transverse vibrations of beams, Euler-Bernoulli is an effective and simple method to predict the natural frequencies of the beams carrying concentrated masses at arbitrary positions.

The vibrations of uniform and non-uniform Euler Bernoulli beam with a concentrated mass under distributed load have been addressed by many researchers in which different boundary conditions have been considered. Chun [2] analyzed the free vibrations of a Euler-Bernoulli beam, in which one end is connected to torsional spring with a constant spring coefficient, and the other end is free. Laura et al. [3] investigated a Euler–Bernoulli beam subjected to a point mass and disregarded the effect of shear deformation. Goel [4] inspected beam vibrations with a concentrated mass at the desired location with rotation-resistant support using the Euler-Bernoulli model and the Laplace transform. He focused on the effects of concentrated mass to beam mass ratio, stiffness of springs at two joints and location of concentrated mass on natural frequencies. Parnell and Cobble [5] studied the vibrations of a Euler-Bernoulli beam of one fixed end and with a mass at the other end by the Laplace transform. Their analysis was based on a beam with a fixed cross-sectional area, but different load distribution, boundary conditions, and initial conditions. An analysis on the problem of the free vibration of a uniform beam with one end attached to a torsion spring and its other end restricted by a linear spring motion using trigonometric expressions and hyperbolic functions was carried out by Maurizi et al. [6]. To [7] studied the vibration of a cantilever beam with a concentrated mass and base excitation. He also explored the impact of the distance between the tip mass center of gravity and to the point where it is connected to the end of the beam. Laura and Gutierrez [8] explored the vibrations of a beam with an elastic end and the variable cross-sectional area, on which a concentrated mass is placed, they used the Rayleigh-Schmidt method to solve it. Liu and Huang [9] examined the free vibrations of a Euler-Bernoulli beam with two concentrated masses, one in the middle of the beam and the other at the free end, under boundary conditions of one end involved with an elastic base. Abramovich and Hamburger [1] calculated the natural frequencies of a cantilever beam carrying a tip mass at its free end with having translational and rotational springs.

Wang and Lin [10] calculated the dynamic analysis of beams that have arbitrary boundary conditions by using Fourier series. Yeih et al. [11] used the dual multiple reciprocity method (MRM) to determine the natural frequencies and natural modes of a Bernoulli beam. Kim and Kim [12] used Fourier series to obtain vibration frequencies of uniform Euler-Bernoulli beams with finite boundary conditions. Low [13], by using the Euler-Bernoulli beam with a concentrated mass and the Rayleigh approximation method, studied the transverse vibrations and natural frequencies of a beam. The case of transverse vibration of uniform Euler-Bernoulli beams under variable axial load has been inspected by Noguleswaran [14]. He acquired the mode shapes using the Frobenius method. Yaman [15] used the finite element method to analyze a beam with one fixed end and a mass on the other free end. The use of Euler-Bernoulli beam theory and accurate beam solving with finite concentrated masses on it and considering the mass inertia is a research conducted by Maiz et al. [16]. In work by Lai et al. [17], the Adomian decomposition method (ADM) was used to solve the free vibration of Euler-Bernoulli beams with different elastic support conditions. Liu et al. [18] computed the natural frequencies and mode shapes of a Euler-Bernoulli beam under

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different end support conditions using the iterative method. Hozhabrossadati et al. [19] studied the free vibration of a beam with intermediate sliding connection joined by a mass-spring system. Chen et al. [20] employed the dynamic stiffness method to study the vibrations of combined beam and 2DOF spring-mass system. Ganguli and Gouravaraju [21] investigated the detection of damages in a cantilever beam by using spatial Fourier coefficients and mode shape method. De Rosa et al. [22] investigated the free vibrations of a tapered beam with one elastic end and with a concentrated mass and a viscous damper at the other end. Liu and Barkey [23] analyzed the nonlinear vibration behavior of a cantilever beam with a breathing crack and simplified the beam into a 1 DOF model by the Galerkin method. Rezaiee-pajand et al. [24] inspected the vibration suppression of a double-beam system by a 2D mass-spring system. Korayem et al. [25] explored the vibration control of an atomic force microscope (AFM) Euler-Bernoulli micro-beam by a piezoelectric layer on top of it. They displayed that vibrations of the beam can be reduced by viscoelasticity. Ahmadi et al. [26] performed Finite element method (FEM) for studying the free and forced vibration of rectangular and V-shaped AFM piezoelectric microcantilevers. Jafarzadeh et al. [27] analyzed the effects of concentrated masses on the transverse vibration of nanobeams. They imposed the mathematical model of the concentrated masses into the equations of motion by using Dirac's delta function. Pouretemad et al. [28] examined the free vibration of a non-uniform nano-beam that carrying arbitrary concentrated masses.

However, to the best of our knowledge, no experimental study to this point has considered a beam with a concentrated mass under a concentrated load caused by an unbalanced mass at different locations. Due to this reason, it encouraged us to investigate and provide an experimental study on the free and forced vibration behavior with the concentrated mass. Here, we use the Euler-Bernoulli method to analyze the free and forced vibration of a beam with two-joint boundary conditions subjected to a concentrated moving mass and a load. By applying concentrated mass and load on the beam, its natural frequency is calculated by using an analytical method, then, the results are compared with the experimental results. To this end, for various states of concentrated load and mass at different distances on the beam, the governing Euler-Bernoulli beam equations are extracted. Accuracy is determined by comparing the obtained theoretical and experimental data.

This paper is organized as follows. Section II introduces the theories and governing equations for the free and forced vibration equations. Section III describes the experimental setup used in this work. The results and discussion are given in Section IV. Finally, last section presents our conclusions.

II. Theoretical Section

A. Notation and Preliminaries

The equation for transverse vibrations of a beam is as follows:

$$\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 w(x,t)}{\partial x^2}\right] + \rho A(x)\frac{\partial^2 w(x,t)}{\partial t^2} = f(x,t) \qquad (1)$$

where ρ is the density of the beam (kg /m^3), E is Young's modulus (kg /m^2), I is the moment of inertia (m^4), A is the cross-sectional area (m^2), w is the transverse displacement (function of displacement and time), x is the beam location, t is time (seconds), and f is the external force (function of displacement and time).

In this study, the beam is simply supported, as shown in Fig. 1.

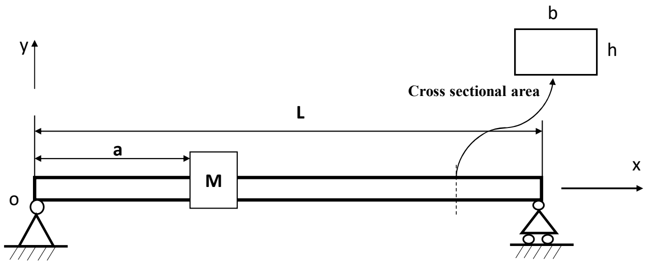

Fig. 1 Simple beam with concentrated mass

B. Free Vibration

To calculate the natural frequency of the beam, the external force f was assumed to be 0 in (1). In the two-jointed beam system, the boundary conditions of the system are in the form of (2). The effect of the concentrated mass should also be applied to the boundary conditions of the beam.

$$w(0,t) = 0, \frac{\partial^2 w}{\partial x^2}(0,t) = 0, w(L,t) = 0, \frac{\partial^2 w}{\partial x^2}(L,t) = 0 \; (2)$$

First, using the characteristics and parameters of the beam listed in Table I, for a given mass, at different distances from one end of the beam, the natural frequency of the beam is calculated.

TABLE I
BEAM CHARACTERISTICS AND PARAMETERS

Parameters of the beam		Value
Length	L (cm)	82
Young's modulus	E (Kg/cm^2)	2.109×10^6
Density	ρ (Kg/m^3)	7832
Cross-sectional area	b (cm)	2.54
	h (cm)	1.27
Mass of motor and belongings	M_m (gr)	4450
Beam mass	m (gr)	2000

An approximate solution to obtain natural frequency is the Dunkerley equation, which is expressed as follows:

$$\frac{1}{f^2} = \frac{1}{{f_b}^2} + \frac{1}{{f_1}^2} \qquad (3)$$

where f_b is the natural frequency of the beam without the external load, f_1 is the natural frequency of the beam with the external load, and f is the natural frequency of the system. For different distances from the left of the beam, the natural frequency is calculated.

C. Forced Vibration

The solution to (1) is assumed by using the method of separation of variables as follows:

$$w(x,t) = \sum_{i=1}^{\infty} W_i(x)\eta_i(t) \qquad (4)$$

where $W_i(x)$ is the normal modes of the beam and $\eta_i(t)$ is a function of time.

The system response is as follows:

$$w(x,t) = W_i(x) \times \sum_{i=1}^{\infty}[A_i cos(\omega_i t) + B_i sin(\omega_i t) + \frac{1}{\rho A b \omega_i}\int_0^t Q_i(\tau) sin\big(\omega_i(t-\pi)\big)d\tau] \qquad (5)$$

where b is:

$$b = \int_0^L {W_i}^2(x)dx \qquad (6)$$

The effect of concentrated force is considered as:

$$f(x,t) = F(t).\delta(x-a) \qquad (7)$$

where δ is the Kronecker delta, which is defined as follows:

$$\delta(x-a) = \begin{cases} 1 & x = a \\ 0 & x \neq a \end{cases} \qquad (8)$$

where the excitation force with **Ω** (excitation frequency or forced frequency) is applying to the beam, the force is:

$$F(t) = f(\Omega)\sin(\Omega t) \qquad (9)$$

where $f(\Omega)$ is as follows:

$$f(\Omega) = m_e r \Omega^2 \qquad (10)$$

The specifications of the unbalanced mass are shown in Table II.

TABLE II
SPECIFICATIONS OF THE UNBALANCED MASS

Parameters		Value
Unbalanced mass	m_e(Kg)	0.0768
Distance between unbalanced mass and center of the disc	R (mm)	57.4

III. EXPERIMENTAL SETUP

In this test rig, a beam of rectangular cross-section with two hinged ends is considered. The beam carries a synchronous motor with an unbalanced disc which applies a harmonic force to the beam [29]. An adjustable concentrated mass is attached to the bottom of the motor. The natural frequency of the beam relates to the disc rotation, which can be measured using a motor speed controller or a stroboscope. Fig. 2 shows our experimental setup.

Experiments are performed by increasing the rotational speed of the synchronous motor driver, as illustrated in Fig. 3.

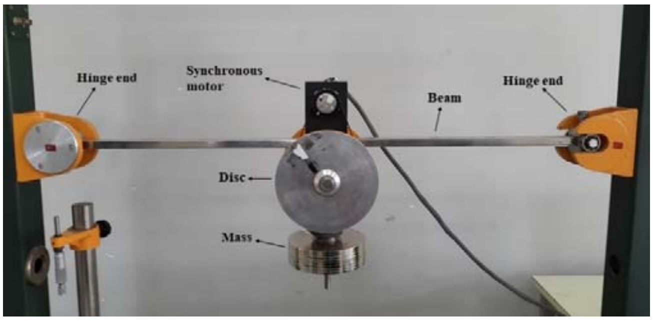

Fig. 2 The test rig

Fig. 3 The driver of synchronous motor to increase rotation speed

IV. RESULTS

The natural frequency of the beam is calculated by considering the speed ratio between the motor and the main system that comprises the beam. Table III shows the experimental results at different locations of the mass.

TABLE III
THE NATURAL FREQUENCY OF THE BEAM FOR DIFFERENT DISTANCES

Distances of the concentrated mass (M_m) from the left end (mm)	The first natural frequency of the beam (ω_{n1}) by analytical solution (rad/s)	The first natural frequency of the beam (ω_{n1}) by experimental solution (rad/s)
a_1= 150	107.36	101.34
a_2= 230	99.43	94.58
a_3= 300	96.14	92.89
a_4= 380	91.78	89.52
a_5= 410	88.95	87.83

As can be seen from Table III, the natural frequency of the beam and the relative error value between the results of analytical and experiment solutions reduce by moving the concentrated mass toward the middle of the beam. The first natural frequencies of the system extracted from analytical and experiment results as a function of the locations of the mass are presented in Fig. 4. The relative error for different locations is also calculated using (11) and plotted in Fig. 4.

$$\text{Relative error (\%)} = \frac{(Theoretical\ value - Experimental\ value)}{(Theoretical\ value)} \times 100\% \qquad (11)$$

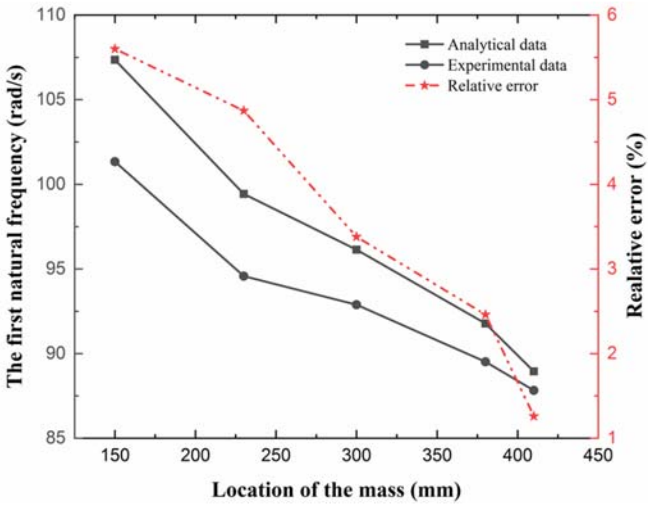

Fig. 4 The results of the first natural frequency due to analytical and experimental solution

As one can see from Fig. 4, the relative error percentage decreases as the mass approaches the center of the beam.

In Fig. 5, the vibrational amplitude at x = 410 mm (center of the beam) as a function of the frequency of the excitation force is obtained for different distances.

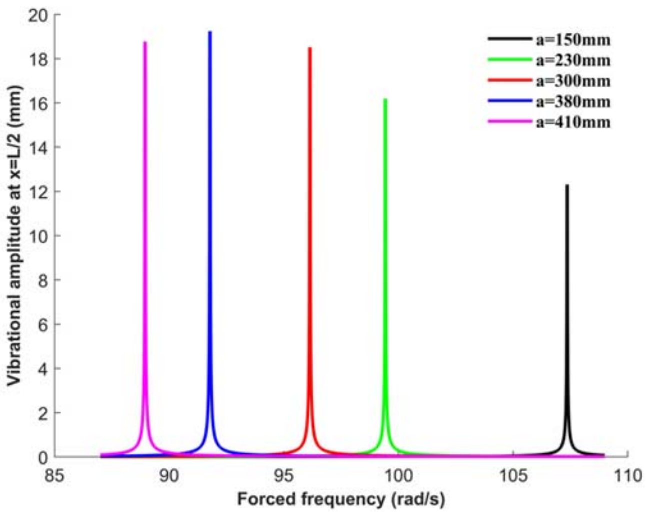

Fig. 5 The vibrational amplitude in the middle of the beam in terms of the frequency excitation

One can conclude from Fig. 5 and Table III that the minimum natural frequency of the beam is obtained when the concentrated mass is located in the middle of the beam.

The range of transverse vibration of the beam is calculated at different locations and for different excitation frequencies. For Ω = 100 rad/s and x = 410 mm, in Fig. 6, as the excitation frequency, approaches the natural frequency, the range of displacement increases. Here, the amplitude at a = 230 mm is greater than that of at a = 300 mm and a = 380 mm.

It can be seen from Fig. 6 that, the maximum vibrational amplitude occurs at the center of the beam and as the mass and load moves toward the edges of the beam, the vibrational amplitude decreases.

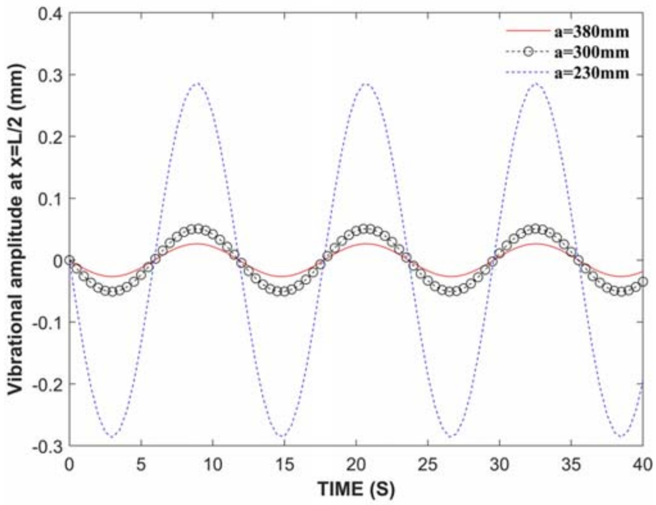

Fig. 6 The range of displacement of the beam with a known x and Ω

V. Discussion

In various engineering systems, designing a beam with carrying masses is a very common model [30]. In order to prevent the system from failure, it is important to know the natural frequencies of the beam-mass systems, improve the design, and to obtain a proper design. In addition, encountering resonance could cause failure in the machine, so it is vital to study forced vibration.

It can be deduced that when the concentrated mass is placed in the middle of the beam, the natural frequency is minimum in comparison with other locations of the mass [31]. Besides, when the forced frequency due to the mass and load is in the proximity of the natural frequency of the beam, resonance phenomena would happen and the vibrational amplitude will be maximum [32], [33]. The aim of this study was to provide experimental verification of the analytical hypotheses which are defined before and give sight for further studies and the acquired results prove the accuracy of the experiment and compatibility with analytical results.

VI. Conclusion

Several works have been done and published on the vibration analysis of beams carrying concentrated masses, but a few of them considered the experimental study. So, it motivated the author to prepare a test setup and an experimental investigation was done to study the natural frequency and forced vibration of the beam with carrying concentrated masses.

The Euler-Bernoulli method was employed to analyze the free and forced vibration of a two end joints beam with a concentrated moving mass and a load. The experimental results were verified analytically. It was found that as the frequency of excitation increases, the resonance phenomena occur at the desired working locations. Moreover, as the concentrated mass moves toward the center of the beam, the natural frequency of the beam decreases, and the minimum the natural frequency achieved when the mass is located in the middle of the beam. The resonance happened when the forced frequency is equal to the natural frequency. It was found that there is a good correlation between predictive and experimental results. This paper took into consideration the effect of the concentrated masses on the vibration behavior of the beam. Furthermore, it provides an experimental reference for future studies. The author is planning to consider other beams with different geometries and boundary conditions for future work.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

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