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Bound State Solutions of the Schrödinger Equation for Hulthen-Yukawa Potential in D-Dimensions

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***Abstract*—**In this work, we used the Hulthen-Yukawa potential to obtain the bound state energy eigenvalues of the Schrödinger equation in D-dimensions within the frame work of the Nikiforov-Uvarov (NU) method. We demonstrated the graphical behaviour of the Hulthen and the Yukawa potential and investigated how the screening parameter and the potential depth affected the structure and the nature of the bound state eigenvalues. The results we obtained showed that increasing the screening parameter lowers the energy eigenvalues. Also, the eigenvalues acted as an inverse function of the potential depth. That is, increasing the potential depth reduces the energy eigenvalues.

***Keywords*—**Schrödinger equation, bound state, Hulthen-Yukawa potential, Nikiforov-Uvarov, D-dimensions.

I. Introduction

SCHRÖDINGER equation (SE) is known to be one of the vital and fundamental wave equations in quantum physics. Obtaining the solution of SE for a physical system in quantum mechanics is of great importance because when the information about the eigenvalue E, and the eigenvector or wave function $\varphi_{(r,t)}$ are known, the physical properties of the system like the quantum state amongst others can be evaluated [1], [2]. But exact solution of the SE is only possible for harmonic oscillator potential, Coulomb potential, Kratzer potential etc [3]. However, hypergeometric-type SE that contains both exponential and radial terms cannot be solved analytically without an approximation [1]. So, an appropriate approximation scheme is always introduced to handle the centrifugal term. Several authors using the exponential-type potentials have employed different analytical methods to obtain bound state solutions of the SE. Notable amongst them are the NU method, asymptotic iteration method (AIM) factorization method supersymmetric shape invariance approach (SUSYQM), Path integral solution, variational method, the exact quantization rule, the shifted 1/N expansion the hypervirial perturbation the algebraic approach, etc [4]-[6].However, in this study, we intend to use Hulthen-Yukawa potential as our confining potential model to obtain an approximate bound state solution of the radial part of the SE within the frame work of the NU method. The essence of combining these potentials is to allow for wider applications. A bound state depicts quantum state of a particle confined in a potential in a way the particle shows the tendency of being localized in one or more region of space. The solutions to the energy eigenvalues are discrete. The energy becomes quantized. There is the existence of bound states when the energy eigenvalues (E<0) in a field vanish at infinity [7].

We will investigate how the screening parameters and potential depth affect the structure of the eigenvalues and wave function of the system. Reference [8] in their study of the bound state solution with Hulthen plus exponential Coulombic potential showed that the bound state energy decreased as the screening parameter was increased. In terms of the potential depth, [9] used variational method to investigate two electrons in a three-dimensional quantum dot with Gaussian confinement. They calculated the energy values for two-particle system and plotted the values as a function of the depth of the potential. Their results showed that the energy values decreased as the potential depth was increased. Meanwhile to obtain the solutions for $l \neq 0$, a suitable approximation scheme is always introduced to deal with the centrifugal term.

The Huthen potential had always played a pivotal role in condensed matter physics, atomic physics, nuclear physics and solid state physics. It has been used to describe the molecular structure of an atom and nuclear interaction. It is a short-range potential that exhibits a Coulomb-like behaviour for small values of the screening parameter,α with an exponential decrease for large values of r [10]-[12]. Yukawa potential is a non-relativistic potential used to describe the interactions between nucleons. Also, the stability of the local fluid structure which led to the provision of a good expression for the free energy and the pair correlation function for a system was a result of the introduction of Yukawa potential into the system [11].The value of the screening parameter of this potential reveals the physics behind it [13].

The organization of this paper is as follows; in Section II, a brief concept of the NU and its parametric form will be given. We will show in Section III the graphical behaviour of the Hulthen and Yukawa potentials. In Section IV, we will solve analytically the radial part of the Schrödinger for the Hulthen-Yukawa potentials in D-dimensions to obtain the bound state energy eigenvalues and the wave function. Section V will show the numerical values and the plots for the screening parameter,α and potential depth V_1. The discussion of the results and conclusion will be done in Sections VI and VII. All the graphs plotted and numerical computations carried out were done with Maple software.

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II. Brief Concept of NU Method and Its Parametric Form

In order to solve hypergeometric-type second–order differential equations with special orthogonal functions, A. F Nikiforov and V. B Uvarov formulated the NU method. Based on the particular potential, the Schrödinger or Schrödinger-like equations are reduced using appropriate coordinate transformation $r \to s$ to hypergeometric-type. Such hypergeometric-type second-order differential equation is given as [1]:

$$\psi''_{(s)}+\frac{\tilde{\tau}_{(s)}}{\sigma_{(s)}}\psi'_{(s)}+\frac{\tilde{\sigma}_{(s)}}{\sigma^2_{(s)}}\psi_{(s}=0 \tag{1}$$

In (1)$\sigma_{(s)}$ and $\breve{\sigma}_{(s)}$are polynomials at most second degree, $\breve{\tau}_{(s)}$ is a first-degree polynomial and$\psi_{(s)}$ is a function of the hypergeometric-type. In order to obtain the particular solution of (1), one can carry out the following transformation

$$\psi_{(s)}=\phi_{(s)}y_{(s)} \tag{2}$$

This transformation yields an equation of the hypergeometric–type given as:

$$\sigma_{(s)}y''\psi_{(s)} + \tau_{(s)}y'_{(s)} + \lambda y_{(s)} = 0 \tag{3}$$

where $y_n(s)$ satisfies the Rodrigues relation [14] and can be written as

$$y_n(s)=\frac{Bn}{\rho(s)}\frac{d^n}{ds^n}(\sigma^n(s)\rho(s)) \tag{4}$$

Here, B_n is the normalization constant and $\rho(s)$ is the weight function which must satisfy the condition

$$\frac{d}{d(s)}[\sigma(s)\rho(s) = \tau(s)\rho(s)] \tag{5}$$

Also,

$$\frac{\phi'(s)}{\phi(s)} = \frac{\pi(s)}{\sigma(s)} \tag{6}$$

$$\tau(s) = \tilde{\tau}(s)+ 2\pi(s), \quad \tau(s)<0 \tag{7}$$

Bear in mind that the derivative of $\tau(s)$ with respect to s, is less than zero i.e. negative.

$$\Lambda_n = -n\tau''(s) - \frac{n(n-1)}{2}\sigma'(s), n = 0, 1, 2 \tag{8}$$

where

$$\pi(s) = \frac{\sigma'(s)-\tilde{\tau}(s)}{2} \pm \sqrt{\frac{\sigma'(s)-\tilde{\tau}(s)-\tilde{\sigma}(s)+k\sigma(s)}{2}} \tag{9}$$

where

$$k = \Lambda - \pi'(s) \tag{10}$$

To determine the constant k, the discriminant of the quadratic of (9) must be set to zero. It should be noted that since $\pi\ (s)$ is a polynomial of degree at most one, the expression under the square root sign must be the square of that polynomial. With this requirement met, and having obtained the value of k, the polynomial $\pi\ (s)$ is obtained from (8) [1], [5], [15].However, Tezcan and Sever went ahead to formulate the parametric form of the NU method that can be employed for central, non-central and hypergeometric-type potential. This can be obtained when we compare the hypergeometric-type Schrödinger equation written as [16]

$$\left[\frac{d^2}{ds^2}+\frac{\alpha_1-\alpha_2}{s(1-\alpha_3 s)}\frac{d}{ds}+\frac{-\xi_1 s^2+\xi_2 s-\xi_3}{s^2(1-\alpha_3 s)^2}\right]\psi = 0 \tag{11}$$

with (1), the following parametric polynomials of (12)-(14) will be obtained [17]

$$\tilde{\tau} = \alpha_1 - \alpha_2 s \tag{12}$$

$$\sigma = s(1 - \alpha_3) \tag{13}$$

$$\tilde{\sigma} = -\xi_1 + \xi_2 - \xi_3 \tag{14}$$

Equation (15) is obtained when (12)-(14) are substituted into (9)

$$\pi = \alpha_4 + \alpha_5 \pm \sqrt{(\alpha_6 - \alpha_5)s^2(\alpha_7 + k)s + \alpha_8} \tag{15}$$

where the following parameters are written as

$$\alpha_4 = \frac{1}{2}(1 - \alpha_1) \tag{16}$$

$$\alpha_5 = \frac{1}{2}(\alpha_2 - 2\alpha_3) \tag{17}$$

$$\alpha_6 = \alpha_5 + \xi_1 \tag{18}$$

$$\alpha_7 = 2\alpha_4\alpha_5 - \xi_2 \tag{19}$$

$$\alpha_8 = \alpha^2{}_4 + \xi_3 \tag{20}$$

The parameter k is obtained from the condition that the expression under the square root sign should be the square of a polynomial in accordance to the NU method. Therefore,

$$K\pm= -(\alpha_7 + 2\alpha_3\alpha_8) \pm \sqrt{\alpha_8\alpha_9} \tag{21}$$

where we define

$$\alpha_9 = \alpha_3\alpha_7 + \alpha^2{}_3\alpha_8 + \alpha_6 \tag{22}$$

The function πrepresented in (15) becomes

$$\pi = \alpha_4 + \alpha_5 s - [(\sqrt{\alpha_9} + \alpha_3\sqrt{\alpha_8})s-\sqrt{\alpha_8}] \tag{23}$$

For the k-value that is negative, it is obtained as

$$k = -(\alpha_7 + 2\alpha_3\alpha_8) - 2\sqrt{\alpha_8\alpha_9} \tag{24}$$

and τ from (7) we have

$$\tau = \alpha_1 + 2\alpha_4 - (\alpha_2 - \alpha_5)s - 2[(\sqrt{\alpha_9} + \alpha_3\sqrt{\alpha_8})s - \sqrt{\alpha_8}] \quad (25)$$

whose derivative is less than zero, that is negative. So,

$$\tau = -2\alpha_3 - 2[\sqrt{\alpha_9} + \alpha_3\sqrt{\alpha_8} < 0 \quad (26)$$

From (10), (23) and (25) the parametric energy eigenvalue equation is obtained as

$$\alpha_2 n - (2n+1)\alpha_{5\alpha} + (2n+1)[\sqrt{\alpha_9} + \alpha_3\sqrt{\alpha_8}\,] + n(n-1)\alpha_3 + \alpha_7\alpha + 2\alpha_3\alpha_8 + 2\sqrt{\alpha_8\alpha_9} = 0 \quad (27)$$

With (4), the weight function is obtained as

$$\rho_{(s)} = s^{\alpha_{10}-1}(1-\alpha_3 s)^{\frac{\alpha_{11}}{\alpha_3}-\alpha_{10}-1} \quad (28)$$

and when (28) is used in (5), we obtained (29) as

$$\gamma_n = P_n^{(\alpha_{10}-1,\ \frac{\alpha_{11}}{\alpha_3}-\alpha_{10}-1)}(1-2\alpha_3 s) \quad (29)$$

where

$$\alpha_{10} = \alpha_1 + 2\alpha_4 + 2\sqrt{\alpha_8} \quad (30)$$

$$\alpha_{11} = \alpha_2 - 2\alpha_5 + 2(\sqrt{\alpha_9} + \alpha_3\sqrt{\alpha_8}) \quad (31)$$

and $P_n^{(\alpha_{10},\alpha_{11})}$ are Jacobi polynomial [18]. By using (6) we obtain

$$\emptyset = S^{\alpha_{12}}(1-\alpha_3 S)^{-\alpha_{12}-\frac{\alpha_{13}}{\alpha_3}} \quad (32)$$

where

$$\alpha_{12} = \alpha_4 + \sqrt{\alpha_8} \quad (33)$$

$$\alpha_{13} = \alpha_5 - (\sqrt{\alpha_9} + \alpha_3\sqrt{\alpha_8}) \quad (34)$$

So, the general solution or wave function is

$$\psi = S^{\alpha_{12}}(1-\alpha_3 S)^{-\alpha_{12}-\frac{\alpha_{13}}{\alpha_3}} P_n^{(\alpha_{10}-1,\ \frac{\alpha_{11}}{\alpha_3}-\alpha_{10}-1)}(1-2\alpha_3 s) \quad (35)$$

When the value of $\alpha_3 = 0$, (35) becomes

$$\psi = S^{\alpha_{12}} e^{\alpha_{13}s}\, l_n^{\alpha_{10}-1}(\alpha_{11}S) \quad (36)$$

III. Graphical Behaviour of Hulthen-Yukawa Potential

We choose our confining potential $V_{(r)}$ to be that of [19], [20]. This we shall term Hulthen-Yukawa potential written as:

$$V_{(r)} = -\left(\frac{V_o e^{-\alpha r}}{(1-e^{-\alpha r})} + V_1\left(\frac{e^{-\alpha r}}{r}\right)\right) \quad (37)$$

This is a combination of two different potentials. Equation (37) becomes our confining potential. Here, V_o represents the strength, V_1 and α are the potential depth and the screening parameter.

Let us have a look at the graphical behaviour of the Hulthen-Yukawa potential. If we assume $V_1 = 0$ in (37), it reduces to the Hulthen potential which is of the form:

$$V_{(r)} = -\frac{V_o e^{-\alpha r}}{(1-e^{-\alpha r})} \quad (38)$$

The plot is shown in Fig. 1.

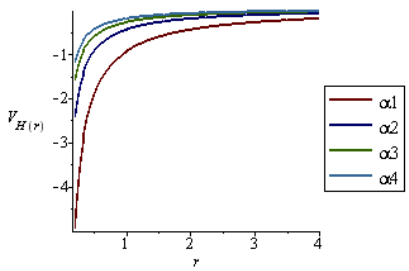

Fig. 1 Hulthen Potential versus r with $V_o = 0.1meV$ for $\alpha = 0.1, 0.2, 0.3$ and 0.4

When $V_o = 0$, (37) reduces to the Yukawa potential written as

$$V_{(r)} = -V_1\left(\frac{e^{-\alpha r}}{r}\right) \quad (39)$$

The plot is shown in Fig. 2.

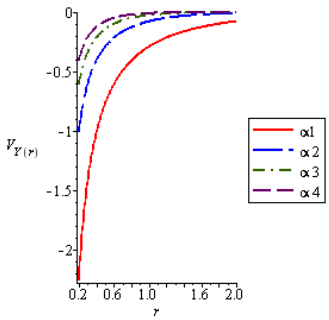

Fig. 2 Yukawa Potential versus r with $V_o = 0.1meV$ and $V_1 = 0.5meV$ for $\alpha = 1, 2, 3$ and 4

The plot of the combined potential is also shown in Fig. 3.

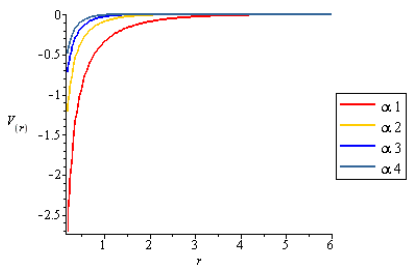

Fig. 3 Hulthen Potential versus r with $V_o = 0.1meV$ and $V_1 = 0.5meV$ for $\alpha = 1, 2, 3$ and 4

IV. Bound State Solution of the SE

The SE in D-dimensions is written as [6]:

$$\frac{d^2U_{n,l}(r)}{dr^2}+\frac{2\mu}{\hbar^2}\left[E-V_{(r)}\right]U_{n,l}-\frac{1}{r^2}\left[\frac{(D-1)(D-3)}{4}+l(l+D-2)U_{n,l}(r)=0\right. \quad (40)$$

Now, we want to solve the radial part of the SE in D-dimensions given in (40) for the Hulthen-Yukawa potential given in (37) as our confining potential.

Substituting (37) into (40) we will obtain the following equation given as

$$\left(\begin{array}{c}\frac{d^2U_{n,l}(r)}{dr^2}+\frac{2\mu}{\hbar^2}\left(E+\frac{Voe^{-\alpha r}}{(1-e^{-\alpha r})^2}+V_1\left(\frac{e^{-\alpha r}}{r}\right)\right)\\ -\frac{1}{r^2}\left(\frac{(D-1)(D-3)}{4}+l(l+D-2)\right)U_{n,l}\end{array}\right)=0 \quad (41)$$

In order to solve (41), we employ the approximation scheme given in (42) [10]. This approximation scheme is for short range potential to enable us handle the centrifugal term of (40)

$$\frac{1}{r^2}\approx\frac{\alpha^2}{(1-e^{-\alpha r})^2},\frac{1}{r}\approx\frac{\alpha}{(1-e^{-\alpha r})} \quad (42)$$

With this approximation scheme of (42), (41) can be re-written as

$$\left(\begin{array}{c}\frac{d^2U_{n,l}(r)}{dr^2}+\frac{2\mu}{\hbar^2}\left(E-\frac{Voe^{-\alpha r}}{(1-e^{-\alpha r})}+\frac{V_1\alpha}{(1-e^{-\alpha r})}\right)\\ -\frac{\alpha^2}{(1-e^{-\alpha r})^2}\left(\frac{(D-1)(D-3)}{4}+l(l+D-2)\right)U_{n,l}\end{array}\right)=0 \quad (43)$$

By using the coordinate transformation, $s=e^{-\alpha r}$(43) yields the following hypergeometric equation given as

$$\left(\begin{array}{c}\frac{d^2U_{n,l}(s)}{ds^2}+\frac{(1-s)}{s(1-s)}\frac{dU_{n,l}}{ds}\\ +\frac{1}{s^2(1-s)^2}[-\varepsilon(1-s)^2+\beta(1-s)s\\ +\rho(1-s)s-\lambda]U_{n,l}(s)\end{array}\right)=0 \quad (44)$$

Subsequently (44) can be evaluated further to obtain an equation of the form

$$\left(\begin{array}{c}\frac{d^2U_{n,l}(s)}{ds^2}+\frac{(1-s)}{s(1-s)}\frac{dU_{n,l}}{ds}\\ +\frac{1}{s^2(1-s)^2}\left[\begin{array}{c}-(\varepsilon+\beta+\rho)s^2\\ +(2\varepsilon+\beta+\rho)s-(\varepsilon+\lambda)\end{array}\right]U_{n,l}(s)\end{array}\right)=0 \quad (45)$$

where

$$-\varepsilon=\frac{2\mu E}{\hbar^2\alpha^2} \quad (46)$$

$$\beta=\frac{2\mu V_o}{\hbar^2\alpha^2} \quad (47)$$

$$\rho=\frac{2\mu V_1}{\hbar^2\alpha} \quad (48)$$

$$\lambda=\frac{1}{r^2}\left(\frac{(D-1)(D-3)}{4}+l(l+D-2)\right) \quad (49)$$

Comparing (45) with the parametric form of the NU equation written as:

$$\psi''+\frac{\alpha_1-\alpha_2 s}{s(1-\alpha_3)s}\psi'+\left[\frac{-\xi_1 s^2+\xi_2-\xi_3}{s^2(1-\alpha_3 s)^2}\right]\psi_{(s)}=0 \quad (50)$$

the following parameters can be found:

$$\alpha_1=\alpha_2=\alpha_3=1,\alpha_4=\frac{1}{2}(1-\alpha_1)=0,$$

$$\alpha_5=\frac{1}{2}(\alpha_2-2\alpha_3)=-\frac{1}{2}, \quad (51)$$

$$\xi_1=\varepsilon+\beta+\rho\ ,\xi_2=2\varepsilon+\beta+\rho\ ,\xi_3=\varepsilon+\lambda. \quad (52)$$

$$\alpha_6=\alpha^2{}_5+\xi_1=\frac{1}{4}+\varepsilon+\beta+\lambda, \quad (53)$$

$$\alpha_7=2\alpha_4\alpha_5-\xi_2=-(2\varepsilon+\beta+\rho), \quad (54)$$

$$\alpha_8=\alpha^2{}_4+\xi_3=\varepsilon+\lambda \quad (55)$$

$$\alpha_9=\alpha_3\alpha_7+\alpha^2{}_3\alpha_8+\alpha_6=\frac{1}{4}+\lambda \quad (56)$$

The NU energy eigenvalues equation is written as:

$$\alpha_2 n-(2n+1)\alpha_5+(2n+1)[\sqrt{\alpha_9}+\alpha_3\sqrt{\alpha_8}]+n(n-1)\alpha_3+\alpha_7+2\alpha_3\alpha_8+2\sqrt{\alpha_8\alpha_9}=0 \quad (57)$$

With (51)-(57), the energy eigenvalues of the Hulthen-Yukawa potentials are obtained as:

$$E_{n,l}=-\frac{\hbar^2\alpha^2}{2\mu}\left(\begin{array}{c}\left[\frac{\sigma+\eta}{2(n+\sqrt{\sigma})}+\frac{(n+\sqrt{\sigma})}{2}\right]^2\\ -\frac{1}{r^2}\left(\frac{(D-1)(D-3)}{4}+l(l+D-2)\right)\end{array}\right) \quad (58)$$

where

$$\sigma=\frac{1}{4}+\lambda\ , \quad (59)$$

$$\eta=\beta+\rho+2\lambda \quad (60)$$

The corresponding wave function is given as:

$$\psi_{(s)}=\left(\begin{array}{c}\times\ s^{\alpha_{12}}(1-\alpha_3 s)^{-\alpha_{12}-\frac{\alpha_{13}}{\alpha_3}}\\ p_n{}^{\alpha_{10-1},\left(\frac{\alpha_{11}}{\alpha_3}\right)-(\alpha_{10}-1)}(1-2\alpha_3 s)\end{array}\right) \quad (61)$$

$$\psi_{(s)}=s^{\sqrt{\varepsilon+\lambda}}(1-s)^{\frac{1}{2}+\sqrt{\frac{1}{4}+\lambda}}p_n{}^{2\sqrt{\varepsilon+\lambda},\ 2\sqrt{\frac{1}{4}+\lambda}}(1-2s) \quad (62)$$

V. Numerical Results

In this section, we present tables I and II for the bound state energy levels (in units of me$V$) for various values of n,l with $\hbar=\mu=0.1,\alpha=0.2,Vo=1.0$for D =3, 4, 5 and 6 when $V_1=0.1$ and 0.3 respectively.

TABLE I
BOUND STATE ENERGY VALUES WHEN $V_1 = 0.1$

n	l	D = 3	D = 4	D = 5	D = 6
0	0	-520.2000000	-129.2846250	-56.89288890	-31.55578124
1	0	-57.34755555	-31.93978124	-20.15632000	-13.74284722
2	0	-20.32128000	-13.91518056	-10.03746939	-7.510945310
	1	-10.03746939	-7.510945310	-5.772098765	-4.523545000
3	0	-10.12261224	-7.609195310	-5.876098765	-4.629345000
	1	-5.876098765	-4.629345000	-3.701652892	-2.992086806
	2	-3.701652892	-2.992086806	-2.436781066	-1.993706632
4	0	-5.928395060	-4.693305000	-3.772595040	-3.067086806
	1	-3.772595040	-3.067086806	-2.513964498	-2.071849490
	2	-2.513964498	-2.071849490	-1.712568889	-1.416392578
	3	-1.712568889	-1.416392578	-1.169162630	-0.9605015430
5	0	-3.808264464	-3.112420138	-2.565893492	-2.128278062
	1	-2.565893492	-2.128278062	-1.772035556	-1.477861328
	2	-1.772035556	-1.477861328	-1.231889274	-1.023945988
	3	-1.231889274	-1.023945988	-0.8464265930	-0.6935512500
	4	-0.84642665930	-0.6935512500	-0.5608616780	-0.4448729340

TABLE II
BOUND STATE ENERGY LEVELS WHEN $V_1 = 0.3$

n	l	D = 3	D = 4	D = 5	D = 6
0	0	-3.591239670	-2.848253472	-2.270035504	-1.811237244
1	0	-2.422414202	-1.953706632	-1.574880000	-1.264267578
2	0	-1.692320000	-1.376392578	-1.113619378	-0.892625000
	1	-10.86359184	-8.136570310	-6.260000000	-4.912745000
3	0	-10.95200000	-8.238570310	-6.367950615	-5.022545000
	1	-6.367950615	-5.022545000	-4.021652892	-3.256253472
	2	-4.021652892	-3.256253472	-2.657372782	-2.179625000
4	0	-6.422222220	-5.088905000	-4.095239668	-3.334031250
	1	-4.095239668	-3.334031250	-2.737396450	-2.260625000
	2	-2.737396450	-2.260625000	-1.873280000	-1.554048828
	3	-1.873280000	-1.554048828	-1.287640138	-1.062847222
5	0	-4.132231404	-3.381031250	-2.791218934	-2.319094388
	1	-2.791218934	-2.319094388	-1.934880000	-1.617705078
	2	-1.934880000	-1.617705078	-1.352581315	-1.128513888
	3	-1.352581315	-1.128513888	-0.9372853190	-0.7726512500
	4	-0.9372853190	-0.7726512500	-0.629759180	-0.5049555785

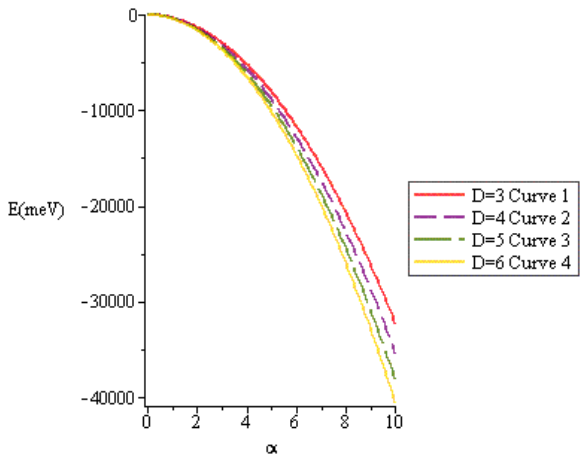

Fig. 4 Plot of Energy E vs Screening parameter α

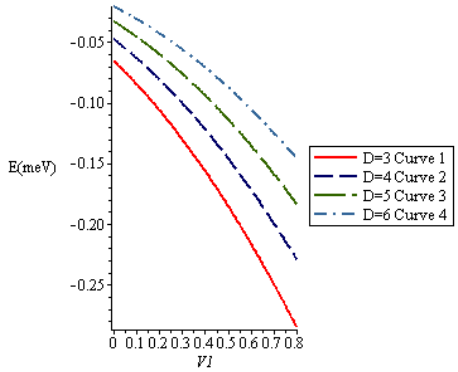

Fig. 5 Plot of Energy E vs Potential depthVI

VI. DISCUSSION

The bound state energy for the different quantum numbers can be seen in Tables I and II. There is an increase in the bound state energy as the quantum state increases. This trend is replicated for the various dimensions, that is for D=3, 4, 5 and 6. So we can see a correlation between the quantum numbers and the bound state energy. We observed that the energy eigenvalues become more bounded with increasing quantum states. Fig. 1 shows us the variation of the bound state energy against the screening parameter, α. It is seen that as the screening parameter increases, the bound state energy decreases. This result is consistent with the work of [8].

The behaviour of the bound state energy with the potential depth is shown in Fig. 2. From the plot, we discovered that the bound state energy decreases as the potential depth increases. From the classical point of view, the farther away an electron is from the proton, the screening effect reduces and the tendency of unbound state is sure to occur. So, from inference, the energy eigenvalues reduce as the potential depth increases. This result is in agreement with [9] and [10].

VII. CONCLUSION

We have used the Hulthen-Yukawa potential to study analytically and numerically the bound state solutions of the SE in D-dimensions so as to have a spectroscopic view of how the screening parameter and the potential depth affect the structure and the nature of the energy eigenvalues and its wave function. And the results obtained were in good agreement with [8].

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