World Academy of Science, Engineering and Technology
International Journal of Computer and Information Engineering
Vol:15, No:11, 2021

MDS b-symbol repeated-root γ-constacylic codes over a finite chain ring of length 2

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Abstract—Let p be a prime and let $\mathbf{b}$ be an integer. MDS $\mathbf{b}$-symbol codes are a direct generalization of MDS codes. The γ-constacyclic codes of length p^s over the finite commutative chain ring $\mathbb{F}_{p^m}[u]/\langle u^2\rangle$ had been classified into four distinct types, where γ is a nonzero element of the field $\mathbb{F}_{p^m}$. Let $\mathcal{C}_3$ be a code of **Type 3**. In this paper, we obtain the $\mathbf{b}$-symbol distance $d_{\mathbf{b}}(\mathcal{C}_3)$ of the code $\mathcal{C}_3$. Using this result, necessary and sufficient conditions under which $\mathcal{C}_3$ is a MDS $\mathbf{b}$-symbol code are given.

Keywords—Constacyclic code, repeated-root code, MDS codes, b-symbol distance, finite chain rings.

I. Introduction

In information theory, at the beginning, the message communicated in a noisy channel was divided into information units, which are called symbols. The operations of reading and writing are often presumed to be performed on individual symbols, which created a lot of disruptions. However, by the recent development of emerging technologies, the symbols can only be written and read in possibly overlapping groups. In 2010, Cassuto and Blaum [1] were the first to propose this method in which the outputs of the channel is overlapping pair of symbols. They provided constructions and decoding methods of symbol-pair codes. In 2011, by using algebraic methods, Cassuto and Litsyn [2] constructed cyclic symbol-pair codes, and showed that there exist symbol-pair codes with rates strictly higher than such codes in the Hamming metric with the same relative distance. Later, Kai et al. [14] developed the theory given by Cassuto and Litsyn [2, Th. 10] for simple-root constacyclic codes. After that, the results established for symbol-pair read channels were further generalized to b-symbol read channels, where the read operation is performed as a consecutive sequence of $\mathsf{b} \geq 3$ by Yaakobi et al. [22].

One of the principal problems in error correction is to construct codes with the best possible distance. Maximum distance separable (MDS) code has the largest Hamming distance, i.e., they have the best possible error-correction capability. In recent years, many researchers want to construct MDS symbol-pair codes (see, for example [4], [5], [6], [12], [14], [15], [16], [17], [18]). As a generalization of MDS codes and MDS symbol-pair codes, MDS b-symbol codes are a kind of b-symbol codes. Hence, constructing MDS b-symbol codes are always one of the central topics in coding theory.

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In engineering, constacyclic codes are most preferred because of their rich algebraic structures and they can be methodically encoded using shift registers. This family of codes are a direct generalization of cyclic codes and perform a crucial role in the theory of error-correcting codes. For a unit element γ in a commutative ring R, γ-constacyclic codes of a given length n over R are in correspondence with ideals in the polynomial ring $R[x]/\langle x^n-\gamma\rangle$. Thus the study of γ-constacyclic codes of length n over R is equivalent to the study of ideals of the quotient ring $R[x]/\langle x^n-\gamma\rangle$. When the code length n is divisible by the characteristic p of the residue feld R, the constacyclic codes are called repeated-root constacyclic codes. Otherwise, the constacyclic codes are called single-root constacyclic codes. Repeated-root constacyclic codes were first initiated in the most generality by Castagnoli in [3] and Van Lint in [21]. Let γ be a nonzero element of the field $\mathbb{F}_{p^m}$. All γ-constacyclic code of length p^s over $\mathcal{R}=\mathbb{F}_{p^m}+u\mathbb{F}_{p^m}$ are classified into 4 distinct types and their detailed structures are also established in [8]. Let $\mathcal{C}_3=\langle(x-\gamma_0)^\delta+u(x-\gamma_0)^t h(x)\rangle$ be of Type 3, as in [8], where $h(x)$ is a unit, $0\leq t<\delta$, and $1\leq\delta\leq p^s-1$.

In [9], Dinh et al. computed the b-distances over $\mathbb{F}_{p^m}$ of repeated root constacyclic codes of prime power lengths for $1\leq \mathsf{b}\leq\lfloor\frac{p}{2}\rfloor$, where $\lfloor\frac{p}{2}\rfloor$ denotes the largest integer, which is less than or equal to $\frac{p}{2}$. As an application, they also determined all MDS b-symbol codes within this class of codes. These works motivate us to study b-symbol distance $d_{\mathsf{b}}(\mathcal{C}_3)$ of the code $\mathcal{C}_3$ for $1\leq \mathsf{b}\leq\lfloor\frac{p}{2}\rfloor$. As an application, necessary and sufficient conditions under which $\mathcal{C}_3$ is a MDS b-symbol code are given.

The remainder of this paper is organized as follows. Some preliminary results are discussed in Section 2. In Section 3, b-distance $d_{\mathsf{b}}(\mathcal{C}_3)$ of the code $\mathcal{C}_3$ is established. In Section 4, we derive necessary and sufficient conditions under which $\mathcal{C}_3$ is a MDS b-symbol code. Section 5 concludes the paper.

II. Some Preliminaries

All rings are commutative rings with identity. A ring R is called principal ideal ring if its ideals are parincipal. R is called a local ring if R has a unique maximal ideal (consisting of all the non-units of R). Finally, R is called a chain ring if all the ideals of R form a chain with respect to the inclusion relation.

The following equivalent conditions are well-known for the class of finite commutative chain rings (see [11, Proposition 2.1]).

Proposition 1: If R is a finite commutative ring, then the following conditions are equivalent:

(i) R is a local ring and the maximal ideal of R is principal,

(ii) R is a local principal ideal ring,
(iii) R is a chain ring.

If we denote by $\langle \mathfrak{a} \rangle$ the maximal ideal of the finite chain ring R, then $\mathfrak{a}$ is nilpotent with nilpotency index some integer e and the ideal of R from the following chain:

$$\langle 0 \rangle = \langle \mathfrak{a}^e \rangle \subsetneq \langle \mathfrak{a}^{e-1} \rangle \subsetneq \cdots \subsetneq \langle \mathfrak{a} \rangle \subsetneq \langle \mathfrak{a}^0 \rangle = R$$

Furthermore, we have $|\langle \mathfrak{a}^i \rangle| = |R/\langle \mathfrak{a} \rangle|^{e-i}$ for $0 \le i \le e$. (Throughout this paper, $|A|$ denotes the cardinality of the set A.)

Let R be a finite ring and let n be a positive integer. A code C of length n over R is a nonempty subset of R^n. If this subset is also an R-submodule of R^n, then C is called linear. For a unit γ of R, a code C of length n over R is said to be γ-constacyclic if $(c_0, c_1, \ldots, c_{n-1}) \in C$ implies that $(\gamma c_{n-1}, c_0, \ldots, c_{n-2}) \in C$. In the case $\gamma = 1$, those γ-constacyclic codes are called cyclic codes, and when $\gamma = -1$, such γ-constacyclic codes are called negacyclic codes. The following is a well known fact about γ-constacyclic codes.

Proposition 2 (cf. [13], [19], [20]): A linear code C of length n is γ-constacyclic over R if and only if C is an ideal of $R[x]/\langle x^n - \gamma \rangle$.

Now let us consider the code alphabet Σ with q elements, whose elements are called symbols. In b-symbol read channels, a codeword $\boldsymbol{x} = (x_0, x_1, \ldots, x_{n-1})$ in Σ^n is represented as (see [6])

$$\pi_{\mathsf{b}}(\boldsymbol{x}) = [(x_0, x_1, \ldots, x_{\mathsf{b}-1}), (x_1, x_2, \ldots, x_{\mathsf{b}}), \ldots, (x_{n-1}, x_0, \ldots, x_{\mathsf{b}-2})] \in (\Sigma^{\mathsf{b}})^n,$$

where $\mathsf{b} \ge 1$. Define the b-weight of a vector $\boldsymbol{x}$ as

$$w_{\mathsf{b}}(\boldsymbol{x}) = |\{0 \le i \le n-1 : (x_i, \ldots, x_{i+\mathsf{b}-1}) \ne (0, \ldots, 0)\}|,$$

where the subscripts are taken modulo n. For two vectors $\boldsymbol{x},\ \boldsymbol{y} \in \Sigma^n$, the b-distance between $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined as

$$d_{\mathsf{b}}(\boldsymbol{x}, \boldsymbol{y}) = |\{i : (x_i, \ldots, x_{i+\mathsf{b}-1}) \ne (y_i, \ldots, y_{i+\mathsf{b}-1})\}|,$$

where the subscripts are reduced modulo n. The minimum b-distance of a b-symbol code C is defined as

$$d_{\mathsf{b}}(C) = \min\{d_{\mathsf{b}}(\boldsymbol{x}, \boldsymbol{y}) \mid \boldsymbol{x}, \boldsymbol{y} \in C, \boldsymbol{x} \ne \boldsymbol{y}\}.$$

If C is a linear b-symbol code, its b-distance is equal to the minimum b-weight of nonzero codewords of C:

$$d_{\mathsf{b}}(C) = \min\{wt_{\mathsf{b}}(\boldsymbol{x}) \mid 0 \ne \boldsymbol{x} \in C\}.$$

When $\mathsf{b} = 1$ and $\mathsf{b} = 2$, the b-distance is the Hamming distance and the symbol-pair distance, respectively.

In this paper, let $\mathbb{F}_{p^m}$ be a finite field of p^m elements, where p is a prime number, and denote

$$\mathcal{R} = \mathbb{F}_{p^m} + u\mathbb{F}_{p^m}\ (u^2 = 0).$$

The ring $\mathcal{R}$ is a finite commutative ring with p^{2m} elements, and can be expressed as $\mathcal{R} = \mathbb{F}_{p^m}[u]/\langle u^2 \rangle = \{a + bu \mid a, b \in \mathbb{F}_{p^m}\}$. It is easy to check that $\mathcal{R}$ is a local ring with maximal ideal $\langle u \rangle = u\mathbb{F}_{p^m}$. Therefore, by propsition 1, it is a chain ring. The ring $\mathcal{R}$ has precisely $p^m(p^m - 1)$ units and every invertible element in $\mathcal{R}$ is of the form: $a + bu$ where $a, b \in \mathbb{F}_{p^m}$ and $a \ne 0$.

Throughout this paper, we always assume that b is a positive integer with $1 \le \mathsf{b} \le \lfloor \frac{p}{2} \rfloor$. For any invertible element γ of $\mathbb{F}_{p^m}$, γ-constacyclic codes of length p^s over a finite field $\mathbb{F}_{p^m}$ are precisely the ideals of the finite chain ring $\mathbb{F}_{p^m}[x]/\langle x^{p^s} - \gamma \rangle$. Since γ is a nonzero element of the field $\mathbb{F}_{p^m}$, there exists $\gamma_0 \in \mathbb{F}_{p^m}$ such that $\gamma_0^{p^s} = \gamma$.

In [7], [9] the algebraic structure and b-distance of γ-constacyclic codes of length p^s over $\mathbb{F}_{p^m}$ were established and given by the following theorem.

Theorem 1 (cf. [9]): Let $1 \le \mathsf{b} \le \lfloor \frac{p}{2} \rfloor$ and $1 \le \beta \le p-1$. Let $\mathcal{C}$ be a γ-constacyclic code of length p^s over $\mathbb{F}_{p^m}$. Then $\mathcal{C} = \langle (x - \gamma_0)^i \rangle \subseteq \mathbb{F}_{p^m}[x]/\langle x^{p^s} - \gamma \rangle$, for $i \in \{0, 1, \ldots, p^s\}$, and its b-distance $d_{\mathsf{b}}(\mathcal{C})$ is completely determined by:

$$d_{\mathsf{b}}(\mathcal{C}) = \begin{cases} \mathbf{b}, \quad if\ i = 0; \\ \boldsymbol{(\beta + \mathsf{b})(\alpha + 1)p^k}, \quad if \\ \qquad i = p^s - pr + \alpha r + \beta, \\ \qquad where\ r = p^{s-k-1},\ 0 \le k \le s-2, \\ \qquad 0 \le \alpha \le p-2,\ \beta(\alpha+1) \le \mathsf{b}, \\ \qquad and\ \beta + \mathsf{b} \le p; \\ \boldsymbol{\mathsf{b}(\alpha + 2)p^k},\ if \\ p^s - pr + \alpha r + \beta \le i \le p^s - pr + (\alpha+1)r, \\ \qquad where\ r = p^{s-k-1},\ 0 \le k \le s-2, \\ \qquad 0 \le \alpha \le p-2,\ \beta(\alpha+1) > \mathsf{b}, \\ \qquad or\ \beta + \mathsf{b} > p; \\ \boldsymbol{(\zeta + \mathsf{b})p^{s-1}}, \quad if\ i = p^s - p + \zeta, \\ \qquad where\ 0 \le \zeta \le p - \mathsf{b}; \\ \boldsymbol{p^s}, \quad if\ p^s - \mathsf{b} + 1 \le i \le p^s - 1; \\ \mathbf{0}, \quad if\ i = p^s. \end{cases}$$

III. b-Symbol distance

Let γ be a nonzero element of the field $\mathbb{F}_{p^m}$. γ-constacyclic code of length p^s over $\mathcal{R}$ are precisely the ideals of the local ring

$$\mathcal{R}_\gamma = \mathcal{R}[x]/\langle x^{p^s} - \gamma \rangle.$$

In 2010, Dinh [8] classified all γ-constacyclic code of length p^s over $\mathcal{R}$ into 4 distinct types, and their detailed structures are also obtained. Let $\mathcal{C}_3$ be of Type 3 (principal ideals with monic polynomial generators), i.e., $\mathcal{C}_3 = \langle (x-\gamma_0)^\delta + u(x-\gamma_0)^t h(x) \rangle$, where $1 \le \mathsf{T} \le \delta \le p^s - 1$, $0 \le t < \delta$, and either $h(x)$ is 0 or a unit in $\mathbb{F}_{p^m}[x]/\langle x^{p^s} - \gamma \rangle$, $\deg h(x) \le \mathsf{T} - t - 1$, and T is the smallest integer satisfying $u(x - \gamma_0)^{\mathsf{T}} \in \mathcal{C}_3$, i.e.,

$$\mathsf{T} = \begin{cases} \delta, & \text{if } h(x) = 0, \\ \min\{\delta, p^s - \delta + t\}, & \text{if } h(x) \ne 0. \end{cases}$$

Moreover, we have

$$|\mathcal{C}_3| = p^{m(2p^s - \delta - \mathsf{T})}.$$

Note that $\mathbb{F}_{p^m}$ is a subring of $\mathcal{R}$, for a code $\mathcal{C}$ over $\mathcal{R}$, we denote $d_{\mathsf{b}}(\mathcal{C}_{\mathbb{F}})$ as the b-symbol distance of $\mathcal{C}|_{\mathbb{F}_{p^m}}$.

In [10, Theorem 3.3(Type 3)], Dinh et al. stated that: $d_{\mathsf{b}}(\mathcal{C}_3) = d_{\mathsf{b}}(\langle (x - \gamma_0)^\delta \rangle_{\mathbb{F}})$. Unfortunately, this result is not true in general, which we illustrate in the following example.

Example 1: Let $\mathcal{R} = \mathbb{F}_{11^m} + u\mathbb{F}_{11^m}$, where $u^2 = 0$, and taking $\mathsf{b} = 4$. Consider the γ-constacyclic code $\mathcal{C}_3 = \langle (x - \gamma_0)^9 + u(x - \gamma_0)h(x) \rangle$ of length 11 over $\mathcal{R}$, where $h(x) \ne 0$. Here $p = 11,\ s = 1,\ \delta = 9$ and $t = 1$. Then $\mathsf{T} = 3$, which gives $\langle u(x-\gamma_0)^3\rangle \subseteq \mathcal{C}_3$. This implies that $d_4(\langle u(x-\gamma_0)^3\rangle) = d_4(\langle\langle (x-\gamma_0)^3\rangle_{\mathbb{F}}) \geq d_4(\mathcal{C}_3)$. By Theorem 1, we have $d_4(\langle\langle (x-\gamma_0)^3\rangle_{\mathbb{F}}) = 7$, which implies that

$$d_4(\mathcal{C}_3) \leq 7. \tag{1}$$

By using Theorem 1 again, we see that

$$d_4(\langle\langle (x-\gamma_0)^9\rangle_{\mathbb{F}}) = 11. \tag{2}$$

Now by (1) and (2), we see that $d_4(\mathcal{C}_3) \neq d_4(\langle\langle (x-\gamma_0)^9\rangle_{\mathbb{F}})$. This example shows that [10, Theorem 3.3(Type 3)] is incorrect.

In the following theorem, we shall rectify the error in Theorem 3.3 (Type 3) of Dinh et al. [10].

Theorem 2: Let $\mathcal{C}_3$ be a γ-constacyclic codes of length p^s over $\mathcal{R}$ of Type 3, i.e., $\mathcal{C}_3 = \langle (x-\gamma_0)^\delta + u(x-\gamma_0)^t h(x)\rangle$, where $1 \leq \mathsf{T} \leq \delta \leq p^s-1$, $0 \leq t < \delta$, either $h(x)$ is 0 or $h(x)$ is a unit, and

$$\mathsf{T} = \begin{cases} \delta, & \text{if } h(x)=0, \\ \min\{\delta, p^s-\delta+t\}, & \text{if } h(x)\neq 0. \end{cases}$$

Then the b-symbol distance $d_{\mathsf{b}}(\mathcal{C}_3)$ of the code $\mathcal{C}_3$ is given by

$$d_{\mathsf{b}}(\mathcal{C}_3) = d_{\mathsf{b}}(\langle\langle (x-\gamma_0)^{\mathsf{T}}\rangle_{\mathbb{F}})$$
$$= \begin{cases} \boldsymbol{(\beta+\mathsf{b})(\alpha+1)p^k}, & if \\ \quad \mathsf{T} = p^s - pr + \alpha r + \beta, \\ \quad where\ r = p^{s-k-1},\ 0\leq k\leq s-2, \\ \quad 0\leq\alpha\leq p-2,\ \beta(\alpha+1)\leq \mathsf{b}, \\ \quad and\ \beta+\mathsf{b}\leq p; \\ \boldsymbol{\mathsf{b}(\alpha+2)p^k}, & if \\ \quad p^s-pr+\alpha r+\beta \leq \mathsf{T} \leq p^s-pr+(\alpha+1)r, \\ \quad where\ r = p^{s-k-1},\ 0\leq k\leq s-2, \\ \quad 0\leq\alpha\leq p-2,\ \beta(\alpha+1) > \mathsf{b}, \\ \quad or\ \beta+\mathsf{b} > p; \\ \boldsymbol{(\zeta+\mathsf{b})p^{s-1}}, & if\ \mathsf{T} = p^s-p+\zeta, \\ \quad where\ 0\leq\zeta\leq p-\mathsf{b}; \\ \boldsymbol{p^s}, & if\ p^s-\mathsf{b}+1 \leq \mathsf{T} \leq p^s-1. \end{cases}$$

Proof: In order to determine $d_{\mathsf{b}}(\mathcal{C}_3)$, we first observe that

$$wt_{\mathsf{b}}(a(x)+ub(x)) \geq wt_{\mathsf{b}}(a(x)), \tag{3}$$

where $a(x),\ b(x) \in \mathbb{F}_{p^m}[x]/\langle x^{p^s}-\gamma\rangle$.

Note that T is the smallest integer such that $u(x-\gamma_0)^{\mathsf{T}} \in \mathcal{C}_3$, which implies that

$$d_{\mathsf{b}}(\langle\langle (x-\gamma_0)^{\mathsf{T}}\rangle_{\mathbb{F}}) = d_{\mathsf{b}}(\langle u(x-\gamma_0)^{\mathsf{T}}\rangle) \geq d_{\mathsf{b}}(\mathcal{C}_3) \tag{4}$$

On the other hand to prove that $d_{\mathsf{b}}(\mathcal{C}_3) \geq d_{\mathsf{b}}(\langle\langle (x-\gamma_0)^{\mathsf{T}}\rangle_{\mathbb{F}})$, we consider an arbitrary nonzero element $c(x) \in \mathcal{C}_3$, that means there exist $\varphi(x), \psi(x) \in \mathbb{F}_{p^m}[x]$ such that

$$c(x) = [\varphi(x)+u\psi(x)][(x-\gamma_0)^\delta + u(x-\gamma_0)^t h(x)].$$

Now we shall distinguish the following two cases.

Case 1: $\varphi(x)=0$. In this case, $\psi(x)\neq 0$. We have $c(x) = u\psi(x)(x-\gamma_0)^\delta$. Thus

$$\begin{aligned} wt_{\mathsf{b}}(c(x)) &= wt_{\mathsf{b}}(u\psi(x)(x-\gamma_0)^\delta) \\ &\geq d_{\mathsf{b}}(\langle u(x-\gamma_0)^\delta\rangle) = d_{\mathsf{b}}(\langle\langle (x-\gamma_0)^\delta\rangle_{\mathbb{F}}). \end{aligned}$$

Since, $\langle (x-\gamma_0)^\delta\rangle \subseteq \langle (x-\gamma_0)^{\mathsf{T}}\rangle$, we have

$$d_{\mathsf{b}}(\langle\langle (x-\gamma_0)^\delta\rangle_{\mathbb{F}}) \geq d_{\mathsf{b}}(\langle\langle (x-\gamma_0)^{\mathsf{T}}\rangle_{\mathbb{F}}).$$

Case 2: $\varphi(x)\neq 0$. Then we have

$$c(x) = \varphi(x)(x-\gamma_0)^\delta + u[\psi(x)(x-\gamma_0)^\delta + \varphi(x)(x-\gamma_0)^t h(x)].$$

By (3), we obtain that

$$\begin{aligned} wt_{\mathsf{b}}(c(x)) &\geq wt_{\mathsf{b}}(\varphi(x)(x-\gamma_0)^\delta) \\ &\geq d_{\mathsf{b}}(\langle\langle (x-\gamma_0)^\delta\rangle_{\mathbb{F}}) \\ &\geq d_{\mathsf{b}}(\langle\langle (x-\gamma_0)^{\mathsf{T}}\rangle_{\mathbb{F}}). \end{aligned}$$

From this, we get $wt_{\mathsf{b}}(c(x)) \geq d_{\mathsf{b}}(\langle\langle (x-\gamma_0)^{\mathsf{T}}\rangle_{\mathbb{F}})$ for each $c(x)$ nonzero element of $\mathcal{C}_3$. This implies that

$$d_{\mathsf{b}}(\mathcal{C}_3) \geq d_{\mathsf{b}}(\langle\langle (x-\gamma_0)^{\mathsf{T}}\rangle_{\mathbb{F}}) \tag{5}$$

Now by (4) and (5) , we get

$$d_{\mathsf{b}}(\mathcal{C}_3) = d_{\mathsf{b}}(\langle\langle (x-\gamma_0)^{\mathsf{T}}\rangle_{\mathbb{F}}).$$

The rest of the proof follows from Theorem 1 and the discussion above. ■

It is easy to observe that $\mathsf{T}=\delta$ when $h(x)=0$ or $h(x)\neq 0$ and $1\leq\delta\leq\frac{p^s+t}{2}$, and that $\mathsf{T} = p^s-\delta+t$ when $h(x)\neq 0$ and $\frac{p^s+t}{2} < \delta \leq p^s-1$.

Corollary 1: Under the same notations as in Theorem 2, we have the following results:

a) If $h(x)$ is 0 or $h(x)$ is a unit and $1\leq\delta\leq\frac{p^s+t}{2}$, then

$$d_{\mathsf{b}}(\mathcal{C}_3) = \begin{cases} \boldsymbol{(\beta+\mathsf{b})(\alpha+1)p^k}, & if \\ \quad \delta = p^s - pr + \alpha r + \beta, \\ \quad where\ r = p^{s-k-1},\ 0\leq k\leq s-2, \\ \quad 0\leq\alpha\leq p-2,\ \beta(\alpha+1)\leq \mathsf{b}, \\ \quad and\ \beta+\mathsf{b}\leq p; \\ \boldsymbol{\mathsf{b}(\alpha+2)p^k}, & if \\ \quad p^s-pr+\alpha r+\beta \leq \delta \leq p^s-pr+(\alpha+1)r, \\ \quad where\ r = p^{s-k-1},\ 0\leq k\leq s-2, \\ \quad 0\leq\alpha\leq p-2,\ \beta(\alpha+1) > \mathsf{b}, \\ \quad or\ \beta+\mathsf{b} > p; \\ \boldsymbol{(\zeta+\mathsf{b})p^{s-1}}, & if\ \delta = p^s-p+\zeta, \\ \quad where\ 0\leq\zeta\leq p-\mathsf{b}; \\ \boldsymbol{p^s}, & if\ p^s-\mathsf{b}+1 \leq \delta \leq p^s-1. \end{cases}$$

b) If $h(x)$ is a unit and $\frac{p^s+t}{2} < \delta \leq p^s-1$, then

$$d_{\mathsf{b}}(\mathcal{C}_3) = \begin{cases} \boldsymbol{(\beta+\mathsf{b})(\alpha+1)p^k}, & if \\ \quad \delta = t + p^{s-k} - \alpha p^{s-k-1} - \beta, \\ \quad where\ r = p^{s-k-1},\ 0\leq k\leq s-2, \\ \quad 0\leq\alpha\leq p-2,\ \beta(\alpha+1)\leq \mathsf{b}, \\ \quad and\ \beta+\mathsf{b}\leq p; \\ \boldsymbol{\mathsf{b}(\alpha+2)p^k}, & if \\ \quad t+pr-(\alpha+1)r \leq \delta \leq t+pr-\alpha r-\beta, \\ \quad where\ r = p^{s-k-1},\ 0\leq k\leq s-2, \\ \quad 0\leq\alpha\leq p-2,\ \beta(\alpha+1) > \mathsf{b}, \\ \quad or\ \beta+\mathsf{b} > p; \\ \boldsymbol{(\zeta+\mathsf{b})p^{s-1}}, & if\ \delta = t+p-\zeta, \\ \quad where\ 0\leq\zeta\leq p-\mathsf{b}; \\ \boldsymbol{p^s}, & if\ t+1 \leq \delta \leq t+\mathsf{b}-1. \end{cases}$$

Example 2: γ-constacyclic codes of length 5 over the chain ring $\mathcal{R} = \mathbb{F}_5 + u\mathbb{F}_5$ are precisely the ideals of $\mathcal{R}[x]/\langle x^5-\gamma\rangle$, where $\gamma \in \{1,2,3,4\}$. The following Table I gives all γ-constacyclic codes of length 5 over the chain ring $\mathbb{F}_5+u\mathbb{F}_5$ of Type 3, where $h(x)$ is a unit, $0\leq t<\delta$ and $\frac{5+t}{2} < \delta \leq 4$,

TABLE I
γ-CONSTACYCLIC CODES OF LENGTH $\mathbf{5}$ OVER THE CHAIN RING $\mathbb{F}_5 + u\mathbb{F}_5$ OF Type 3 ($h(x)$ IS A UNIT, $0 \le t < \delta$ AND $\frac{5+t}{2} < \delta \le 4$)

Ideal	N	d_H	d_{sp}
$\rightarrow \delta = 3$ and $t = 0$: $\langle (x-\gamma)^3 + h_0 u + h_1 u(x-\gamma)\rangle$	20	3	4
$\rightarrow \delta = 4$ and $t = 0$: $\langle (x-\gamma)^4 + h_0 u\rangle$	4	2	3
$\rightarrow \delta = 4$ and $t = 1$: $\langle (x-\gamma)^4 + h_0 u(x-\gamma)\rangle$	4	3	4
$\rightarrow \delta = 4$ and $t = 2$: $\langle (x-\gamma)^4 + h_0 u(x-\gamma)^2\rangle$	4	4	5

together with their number N, their Hamming distances d_H and their symbol-pair distances d_{sp}. In all codes we have $h_0 \in \{1,2,3,4\}$ and $h_1 \in \{0,1,2,3,4\}$.

Remark 1: Let $\mathcal{C}_4$ be a γ-constacyclic code of Type 4 (non principal ideals) as in [8]:

Here, we have $\mathcal{C}_4 = \langle (x-\gamma_0)^\delta + u(x-\gamma_0)^t h(x), u(x-\gamma_0)^\kappa\rangle$, with $h(x)$ as in Type 3, $\deg(h(x)) \le \kappa - t - 1$, $0 \le \kappa < \delta \le p^s - 1$ ($\kappa \ne p^s - 1$) and $0 \le t < \kappa$.

It is easy to see that $u \in \mathcal{C}_4$ when $\kappa = 0$. This means that $d_{\mathsf{b}}(\mathcal{C}_4) = \mathsf{b}$. Then the b-symbol distance $d_{\mathsf{b}}(\mathcal{C}_4)$ of the code $\mathcal{C}_4$ is given by

$$d_{\mathsf{b}}(\mathcal{C}_4) = d_{\mathsf{b}}(\langle (x-\gamma_0)^\kappa\rangle_{\mathbb{F}})$$
$$= \begin{cases} \mathbf{b}, \quad if\ \kappa = 0; \\ (\boldsymbol{\beta} + \mathbf{b})(\boldsymbol{\alpha}+\mathbf{1})\boldsymbol{p^k}, \quad if \\ \quad \kappa = p^s - pr + \alpha r + \beta, \\ \quad where\ r = p^{s-k-1},\ 0 \le k \le s-2, \\ \quad 0 \le \alpha \le p-2,\ \beta(\alpha+1) \le \mathsf{b}, \\ \quad and\ \beta + \mathsf{b} \le p; \\ \mathbf{b}(\boldsymbol{\alpha}+\mathbf{2})\boldsymbol{p^k},\ if \\ p^s - pr + \alpha r + \beta \le \kappa \le p^s - pr + (\alpha+1)r, \\ \quad where\ r = p^{s-k-1},\ 0 \le k \le s-2, \\ \quad 0 \le \alpha \le p-2,\ \beta(\alpha+1) > \mathsf{b}, \\ \quad or\ \beta + \mathsf{b} > p; \\ (\boldsymbol{\zeta} + \mathbf{b})\boldsymbol{p^{s-1}}, \quad if\ \kappa = p^s - p + \zeta, \\ \quad where\ 0 \le \zeta \le p - \mathsf{b}; \\ \boldsymbol{p^s}, \quad if\ p^s - \mathsf{b} + 1 \le \kappa \le p^s - 2. \end{cases}$$

IV. MDS b-SYMBOL CONSTACYCLIC CODES OF LENGTH p^s OVER $\mathcal{R}$

To get MDS b-symbol codes, we need to determine the singleton bound for b-symbol codes first. Singleton bound for b-symbol code C of length n over a finite commutative ring R with b-distance $d_{\mathsf{b}}(C)$ is as follows: $|C| \le |R|^{(n-d_{\mathsf{b}}(C)+\mathsf{b})}$ (see [6]). A b-symbol code C is called an MDS b-symbol codes if it attains the singleton bound for b-symbol codes, i.e.,

$$|C| = |R|^{(n-d_{\mathsf{b}}(C)+\mathsf{b})}. \tag{6}$$

Let $\mathcal{C} = \langle (x-\gamma_0)^i\rangle$ be a γ-constacyclic code of length p^s over $\mathbb{F}_{p^m}$, where $0 \le i \le p^s$. It is well known that $|\mathcal{C}| = p^{m(p^s-i)}$. So the dimension of code $\mathcal{C}$ is $p^s - i$. By (6), $\mathcal{C}$ is an MDS b-symbol code if and only if $p^s - i = p^s - d_{sp}(\mathcal{C}) + \mathsf{b}$, i.e., $i = d_{sp}(\mathcal{C}) - \mathsf{b}$. In [9], Dinh et al. identifed all the MDS b-symbol constacyclic codes of length p^s over $\mathbb{F}_{p^m}$.

Theorem 3 (cf. [9]): Let $\mathcal{C} = \langle (x-\gamma_0)^i\rangle$ be a γ-constacyclic code of length p^s over $\mathbb{F}_{p^m}$, for $i \in \{0,1,\ldots,p^s\}$. Then $\mathcal{C}$ is an MDS b-symbol constacyclic code if and only if one of the following conditions holds:

- If $s = 1$, then $\delta = \eta$ for $1 \le \eta \le p - \mathsf{b}$, in such case, $d_{\mathsf{b}}(\mathcal{C}) = \eta + \mathsf{b}$.
- If $s \ge 2$, then
  - $*$ $\delta = \sigma$ for $0 \le \sigma \le \mathsf{b}$, in such case, $d_{\mathsf{b}}(\mathcal{C}) = \sigma + \mathsf{b}$.
  - $*$ $\delta = p^s - \mathsf{b}$, $d_{\mathsf{b}}(\mathcal{C}) = p^s$.

Now, let $\mathcal{C}_3 = \langle (x-\gamma_0)^\delta + u(x-\gamma_0)^t h(x)\rangle$ be a γ-constacyclic codes of length p^s over $\mathcal{R}$ of Type 3, where $h(x) \ne 0$, $1 \le \delta \le p^s - 1$ and $0 \le t < \delta$. In this section, we shall determine necessary and sufficient conditions for $\mathcal{C}_3$ to be MDS b-symbol code. For this, the following two cases arise.

Case 1: When $\mathbf{1} \le \boldsymbol{\delta} \le \frac{\boldsymbol{p^s+t}}{\mathbf{2}}$ then, $|\mathcal{C}_3| = p^{2m(p^s-\delta)}$. By applying (6), we see that the code $\mathcal{C}_3$ is a MDS b-symbol code if and only if $p^s - \delta = p^s - d_{\mathsf{b}}(\mathcal{C}_3) + \mathsf{b}$, i.e., $\delta = d_{\mathsf{b}}(\mathcal{C}_3) - \mathsf{b}$, which is the same as the MDS γ-constacyclic b-symbol codes over $\mathbb{F}_{p^m}$ (see Theorem 3). But we have $1 \le \delta \le \frac{p^s+t}{2}$ and $0 \le t < \delta$, which gives $\max\{2\delta - p^s, 0\} \le t < \delta$. From this, we can conclude the following theorem:

Theorem 4: Let $\mathcal{C}_3 = \langle (x-\gamma_0)^\delta + u(x-\gamma_0)^t h(x)\rangle$ be of Type 3, where $h(x) \ne 0$, $0 \le t < \delta$ and $1 \le \delta \le \frac{p^s+t}{2}$, then $\mathcal{C}_3$ is a MDS b-symbol code if and only if one of the following conditions holds:

- If $s = 1$, then $\delta = \eta$, $1 \le \eta \le p - \mathsf{b}$, $\max\{2\eta - p, 0\} \le t \le \eta - 1$, then $d_{\mathsf{b}}(\mathcal{C}_3) = \eta + \mathsf{b}$.
- If $s \ge 2$, then
  - $*$ $\delta = \sigma$, $1 \le \sigma \le \mathsf{b}$, $0 \le t \le \sigma - 1$, $d_{\mathsf{b}}(\mathcal{C}_3) = \sigma + \mathsf{b}$,
  - $*$ $\delta = p^s - \mathsf{b}$, $p^s - 2\mathsf{b} \le t \le p^s - \mathsf{b} - 1$, $d_{\mathsf{b}}(\mathcal{C}_3) = p^s$.

Remark 2: Let $\mathcal{C}_3 = \langle (x-\gamma_0)^\delta + u(x-\gamma_0)^t h(x)\rangle$, where $h(x)$ is a unit, $0 \le t < \delta$ and $1 \le \delta \le p^{s-1} + \lfloor \frac{t}{2} \rfloor$. Thus by (6), $\mathcal{C}_3$ is a b-symbol MDS code if and only if $\delta = d_{\mathsf{b}}(\mathcal{C}_3) - \mathsf{b}$. In [10, Theorem 4.4], Dinh et al. stated that: When $s \ge 2$, then $\delta \ge 2$. Unfortunately, this result is not true. For example, taking $\mathsf{b} = 4$, $s = 4$, $p = 11$ and $\delta = 1$, then $t = 0$. By using Corollary 1(a), we see that $d_{\mathsf{b}}(\mathcal{C}_3) = 5$, which gives $d_{\mathsf{b}}(\mathcal{C}_3) - \mathsf{b} = 5 - 4 = 1 = \delta$. This implies that $\mathcal{C}_3$ is a MDS b-symbol code. This example shows that [10, Theorem 4.4] is incomplete.

Case 2: When $\frac{\boldsymbol{p^s+t}}{\mathbf{2}} < \boldsymbol{\delta} \le \boldsymbol{p^s} - \mathbf{1}$. In this case, $|\mathcal{C}_3| = p^{m(p^s-t)}$. Thus by (6), $\mathcal{C}_3$ is a b-symbol MDS code if and only if $p^s - t = 2(p^s - d_{\mathsf{b}}(\mathcal{C}_3) + \mathsf{b})$, i.e., $t = 2d_{\mathsf{b}}(\mathcal{C}_3) - p^s - 2\mathsf{b}$. Hence, we have the following theorem.

Theorem 5: Let $\mathcal{C}_3 = \langle (x-\gamma_0)^\delta + u(x-\gamma_0)^t h(x)\rangle$ be of Type 3, where $h(x) \ne 0$, $0 \le t < \delta$ and $\frac{p^s+t}{2} < \delta \le p^s - 1$. Then $\mathcal{C}_3$ is not a MDS b-symbol constacyclic code.

Proof: When $\frac{p^s+t}{2} < \delta \le p^s - 1$, i.e., $2\delta > p^s + t$, MDS codes can be obtained when $t = 2d_{\mathsf{b}}(\mathcal{C}_3) - p^s - 2\mathsf{b}$. The b-distance $d_{\mathsf{b}}(\mathcal{C}_3)$ is determined in Corollary 1(b). In the following, we discuss the case when $t = 2d_{\mathsf{b}}(\mathcal{C}_3) - p^s - 2\mathsf{b}$ and $2\delta > p^s + t$.

Case 1: $\boldsymbol{\delta = t + p^{s-k} - \alpha p^{s-k-1} - \beta}$, where $0 \le k \le s-2$, $0 \le \alpha \le p-2$, $\beta(\alpha+1) \le \mathsf{b}$ and $\beta + \mathsf{b} \le p$. Then $d_{\mathsf{b}}(\mathcal{C}_3) = (\beta+\mathsf{b})(\alpha+1)p^k$, and $2\delta = 2t + 2p^{s-k} - 2\alpha p^{s-k-1} -$

$2\beta > p^s + t$. Hence,

$$\begin{aligned} t &> p^s - 2p^{s-k} + 2\alpha p^{s-k-1} + 2\beta \\ &= 2p^{s-k}(p^k - 1) - p^s + 2\alpha p^{s-k-1} + 2\beta \\ &\geq 2p^2(p^k - 1) - p^s + 2\alpha p + 2\beta \\ &\geq 2p(\alpha+2)(p^k - 1) - p^s + 2\alpha p + 2\beta \\ &= 2p(\alpha+1)p^k - p^s + 2p^{k+1} - 4p + 2\beta \\ &\geq 2(\beta+\mathsf{b})(\alpha+1)p^k - p^s + 2\beta \\ &> 2(\beta+\mathsf{b})(\alpha+1)p^k - p^s - 2\mathsf{b} \\ &= 2d_{\mathsf{b}}(\mathcal{C}_3) - p^s - 2\mathsf{b}. \end{aligned}$$

Since, $t > 2d_{\mathsf{b}}(\mathcal{C}_3) - p^s - 2\mathsf{b}$, no MDS b-symbol constacyclic code can be obtained in this case.

Case 2: $\boldsymbol{t + p^{s-k} - (\alpha+1)p^{s-k-1} \leq \delta \leq t + p^{s-k} - \alpha p^{s-k-1} - \beta}$, where $0 \leq k \leq s-2$, $0 \leq \alpha \leq p-2$, $\beta(\alpha+1) > \mathsf{b}$, or $\beta + \mathsf{b} > p$. So $d_{\mathsf{b}}(\mathcal{C}_3) = \mathsf{b}(\alpha+2)p^k$. We consider $\delta = t + p^{s-k} - \alpha p^{s-k-1} - j$, where $\beta \leq j \leq p^{s-k-1}$. Then $2\delta = 2t + 2p^{s-k} - 2\alpha p^{s-k-1} - 2j > p^s + t$. Hence,

$$\begin{aligned} t &> p^s - 2p^{s-k} + 2\alpha p^{s-k-1} + 2j \\ &= 2p^{s-k}(p^k - 1) - p^s + 2\alpha p^{s-k-1} + 2j \\ &\geq 2p^2(p^k - 1) - p^s + 2\alpha p + 2\beta \\ &\geq 2p(\alpha+2)(p^k - 1) - p^s + 2\alpha p + 2\beta \\ &= p(\alpha+2)p^k + \alpha p^{k+1} + 2p^{k+1} - 2p - p^s + 2(\beta - p) \\ &> 2\mathsf{b}(\alpha+2)p^k - p^s - 2\mathsf{b} \\ &= 2d_{\mathsf{b}}(\mathcal{C}_3) - p^s - 2\mathsf{b}. \end{aligned}$$

Since, $t > 2d_{\mathsf{b}}(\mathcal{C}_3) - p^s - 2\mathsf{b}$, no MDS b-symbol constacyclic code exists in this case.

Case 3: $\boldsymbol{\delta = t + p - \zeta}$, where $0 \leq \zeta \leq p - b$. Then $d_{\mathsf{b}}(\mathcal{C}_3) = (\zeta + \mathsf{b})p^{s-1}$, and $2\delta = 2t + 2p - 2\zeta > p^s + t$. Hence,

$$\begin{aligned} t &> p^s - 2p + 2\zeta \\ &= 2p(p^{s-1} - 1) - p^s + 2\zeta \\ &\geq 2(\zeta+\mathsf{b})(p^{s-1} - 1) - p^s + 2\zeta \\ &= 2(\zeta+\mathsf{b})p^{s-1} - p^s - 2\mathsf{b} \\ &= 2d_{\mathsf{b}}(\mathcal{C}_3) - p^s - 2\mathsf{b}. \end{aligned}$$

Since, $t > 2d_{\mathsf{b}}(\mathcal{C}_3) - p^s - 2\mathsf{b}$, no MDS b-symbol constacyclic code exists in this case.

Case 4: $\boldsymbol{t + 1 \leq \delta \leq t + \mathsf{b} - 1}$. So $d_{\mathsf{b}}(\mathcal{C}_3) = p^s$. We consider $\delta = t + \mathsf{b} - j$, where $1 \leq j \leq \mathsf{b} - 1$. Then $2\delta = 2t + 2\mathsf{b} - 2j > p^s + t$. Hence,

$$\begin{aligned} t &> p^s - 2\mathsf{b} + 2j \\ &= 2p^s - p^s - 2\mathsf{b} + 2j \\ &> 2p^s - p^s - 2\mathsf{b} \\ &= 2d_{\mathsf{b}}(\mathcal{C}_3) - p^s - 2\mathsf{b}. \end{aligned}$$

Since, $t > 2d_{\mathsf{b}}(\mathcal{C}_3) - p^s - 2\mathsf{b}$, no MDS b-symbol constacyclic code exists in this case. This completes the proof. ■

Remark 3: Let $\mathcal{C}_3 = \langle (x-\gamma_0)^\delta + u(x-\gamma_0)^t h(x) \rangle$, where $h(x)$ is a unit, $0 \leq t < \delta$ and $p^{s-1} + \lfloor \frac{t}{2} \rfloor < \delta \leq p^s - 1$. Then by (6), $\mathcal{C}_3$ is a MDS b-symbol code if and only if $t = 2d_{\mathsf{b}}(\mathcal{C}_3) - p^s - 2\mathsf{b}$. In [10, Theorem 4.5], Dinh et al. stated that: When $s \geq 1$, $p^s - \mathsf{b} + 1 \leq \delta \leq p^s - 1$ and $t = p^s - 2\mathsf{b}$. Then $\mathcal{C}_3$ is a MDS b-symbol code. Unfortunately, this result is not true. For example, taking $\mathsf{b} = 5$, $s = 2$, $p = 11$, $\delta = 118$ and $t = 111$. By Corollary 1(b), we see that $d_{\mathsf{b}}(\mathcal{C}_3) = 99$, which gives $2d_{\mathsf{b}}(\mathcal{C}_3) - p^s - 2\mathsf{b} = 198 - 121 - 10 = 67 \neq t = 111$. This implies that $\mathcal{C}_3$ is not a MDS b-symbol code. This example shows that [10, Theorem 4.5] is incorrect.

We conclude this section by some examples of MDS b-symbol constacyclic codes.

Example 3: γ-constacyclic codes of length 7 over the chain ring $\mathcal{R} = \mathbb{F}_7 + u\mathbb{F}_7$ are precisely the ideals of $\mathcal{R}[x]/\langle x^7 - \gamma \rangle$, where $\gamma \in \{1,2,3,4,5,6\}$. Let $\mathcal{C}_3 = \langle (x-\gamma)^\delta + u(x-\gamma)^t h(x) \rangle$ be of Type 3, where $h(x)$ is a unit, $1 \leq \delta \leq 6$ and $0 \leq t < \delta$. We consider $\mathsf{b} = 3$. We have the following list of MDS 3-symbol γ-constacyclic codes.

- When $1 \leq \delta \leq \frac{7+t}{2}$, MDS codes are obtained by the condition $\delta = d_3(\mathcal{C}_3) - 3$ and all the distinct MDS codes are given by:
  - $\langle (x-\gamma) + h_0 u \rangle$,
  - $\langle (x-\gamma)^2 + h_0 u + h_1 u(x-\gamma) \rangle$,
  - $\langle (x-\gamma)^2 + h_0 u(x-\gamma) \rangle$,
  - $\langle (x-\gamma)^3 + h_0 u + h_1 u(x-\gamma) + h_2 u(x-\gamma)^2 \rangle$,
  - $\langle (x-\gamma)^3 + h_0 u(x-\gamma) + h_1 u(x-\gamma)^2 \rangle$,
  - $\langle (x-\gamma)^3 + h_0 u(x-\gamma)^2 \rangle$,
  - $\langle (x-\gamma)^4 + h_0 u(x-\gamma) + h_1 u(x-\gamma)^2 + h_2 u(x-\gamma)^3 \rangle$,
  - $\langle (x-\gamma)^4 + h_0 u(x-\gamma)^2 + h_1 u(x-\gamma)^2 \rangle$,
  - $\langle (x-\gamma)^4 + h_0 u(x-\gamma)^3 \rangle$,

  where $h_0 \in \{1,\ldots,6\}$, $h_1, h_2 \in \{0,1,\ldots,6\}$.
- When $\frac{7+t}{2} < \delta \leq 6$, the MDS code condition is $t = 2d_3(\mathcal{C}_3) - 13$, which is not satisfied by any value of t and $d_3(\mathcal{C}_3)$. Thus, no MDS code is obtained in this case.

Example 4: γ-constacyclic codes of length 121 over the chain ring $\mathcal{R} = \mathbb{F}_{11} + u\mathbb{F}_{11}$ are precisely the ideals of $\mathcal{R}[x]/\langle x^{121} - \gamma \rangle$, where $\gamma \in \{1,2,\ldots,10\}$. Let $\mathcal{C}_3 = \langle (x-\gamma)^\delta + u(x-\gamma)^t h(x) \rangle$ be of Type 3, where $h(x)$ is a unit, $1 \leq \delta \leq 120$ and $0 \leq t < \delta$. Let $\mathsf{b} = 4$. We have the following list of MDS 4-symbol γ-constacyclic codes.

- When $1 \leq \delta \leq \frac{121+t}{2}$, MDS codes are obtained by the condition $\delta = d_4(\mathcal{C}_3) - 4$ and all the distinct MDS codes are given by:
  - $\langle (x-\gamma) + h_0 u \rangle$,
  - $\langle (x-\gamma)^2 + h_0 u + h_1 u(x-\gamma) \rangle$,
  - $\langle (x-\gamma)^2 + h_0 u(x-\gamma) \rangle$,
  - $\langle (x-\gamma)^3 + h_0 u + h_1 u(x-\gamma) + h_2 u(x-\gamma)^2 \rangle$,
  - $\langle (x-\gamma)^3 + h_0 u(x-\gamma) + h_1 u(x-\gamma)^2 \rangle$,
  - $\langle (x-\gamma)^3 + h_0 u(x-\gamma)^2 \rangle$,
  - $\langle (x-\gamma)^4 + h_0 u + h_1 u(x-\gamma) + h_2 u(x-\gamma)^2 + h_3 u(x-\gamma)^3 \rangle$,
  - $\langle (x-\gamma)^4 + h_0 u(x-\gamma) + h_1 u(x-\gamma)^2 + h_2 u(x-\gamma)^3 \rangle$,
  - $\langle (x-\gamma)^4 + h_0 u(x-\gamma)^2 + h_1 u(x-\gamma)^3 \rangle$,
  - $\langle (x-\gamma)^4 + h_0 u(x-\gamma)^3 \rangle$,
  - $\langle (x-\gamma)^{117} + h_0 u(x-\gamma)^{113} + h_1 u(x-\gamma)^{114} + h_2 u(x-\gamma)^{115} + h_3 u(x-\gamma)^{116} \rangle$,
  - $\langle (x-\gamma)^{117} + h_0 u(x-\gamma)^{114} + h_1 u(x-\gamma)^{115} + h_2 u(x-\gamma)^{116} \rangle$,
  - $\langle (x-\gamma)^{117} + h_0 u(x-\gamma)^{115} + h_1 u(x-\gamma)^{116} \rangle$,

- $\circ$ $\langle (x-\gamma)^{117} + h_0 u(x-\gamma)^{116} \rangle$,

  where $h_0 \in \{1, \ldots, 10\}$, $h_1, h_2, h_3 \in \{0, 1, \ldots, 10\}$.
- When $\frac{121+t}{2} < \delta \leq 120$, the MDS code condition is $t = 2d_4(\mathcal{C}_3) - 129$, which is not satisfied by any value of t and $d_4(\mathcal{C}_3)$. Thus, no MDS code is obtained in this case.

V. Conclusion

Let p be a prime and γ be an any nonzero element of the finite field $\mathbb{F}_{p^m}$. Determining the b-distances of constacyclic codes and obtaining MDS b-symbol constacyclic codes are very important in coding theory. Motivated by this, in this paper, we studied the b-distances of γ-constacyclic codes of length p^s over $\mathcal{R} = \mathbb{F}_{p^m}[u]/\langle u^2 \rangle$ for $1 \leq \mathtt{b} \leq \lfloor \frac{p}{2} \rfloor$. We also completed the problem of determination MDS b-symbol γ-constacyclic codes of length p^s over $\mathcal{R}$.

Acknowledgment

The authors are very grateful the anonymous referee for his valuable comments and suggestions to improve this paper.

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