On the Algorithmic Iterative Solutions of Conjugate Gradient, Gauss-Seidel and Jacobi Methods for Solving Systems of Linear Equations

H. D. Ibrahim, H. C. Chinwenyi, H. N. Ude

Abstract**—**In this paper, efforts were made to examine and compare the algorithmic iterative solutions of conjugate gradient method as against other methods such as Gauss-Seidel and Jacobi approaches for solving systems of linear equations of the form *Ax = b,* where *A* is a real *n x n* symmetric and positive definite matrix. We performed algorithmic iterative steps and obtained analytical solutions of a typical *3 x 3* symmetric and positive definite matrix using the three methods described in this paper (Gauss-Seidel, Jacobi and Conjugate Gradient methods) respectively. From the results obtained, we discovered that the Conjugate Gradient method converges faster to exact solutions in fewer iterative steps than the two other methods which took much iteration, much time and kept tending to the exact solutions.

Keywords**—**Conjugate gradient, linear equations, symmetric and positive definite matrix, Gauss-Seidel, Jacobi, algorithm.

I. Introduction

CONJUGATE Gradient method is the most popular iterative method for solving large systems of linear equations. It is effective for systems of the form $Ax = b$ where x is an unknown vector, b is a known vector, and A is a known, square, symmetric, positive-definite (or positive-indefinite) matrix [1]. It is also an iterative method for first approximation and converges after a specified number of iterations to produce the actual solutions. In this approach, the matrix of the system is not affected in the process of computation as every iteration is only used in multiplying the resulting vector. The order of systems of equation that can actually be computed is often high and it is being ascertained by the amount of arithmetical information required to stipulate the matrix. Being a direct iterative method, its arrangement is based on the method of sequential $A-$orthogonalization of a set of vectors and is an ordinary orthogonalization process with respect to the scalar product $\langle x, y\rangle = X^T Ay$. if $\{s_i \dots s_n\}$ is an $A-$orthorgonal basis of the space, then for any initial approximation x_0, the exact solution x^* of the system can be obtained from the decomposition [2].

$$x^* - x_0 = \sum_{j=1}^{n} \alpha_j s_j, \quad \alpha_j = \frac{\langle r_0, s_j\rangle}{\langle s_j, As_j\rangle}$$

where $r_0 = b - Ax_0$ is the discrepancy of x_0. In the conjugate-gradient method, the A-orthogonal vectors $s_i \dots s_n$ are constructed by A-orthogonalizing the discrepancies $r_0 \dots r_{n-1}$ of the sequence of approximations $x_i \dots x_{n-1}$, given by:

$$x_k = x_0 + \sum_{j=1}^{k} \alpha_j s_j, \quad \alpha_j = \frac{\langle r_0, s_j\rangle}{\langle s_j, As_j\rangle}.$$

The vectors $r_0 \dots r_{n-1}$ and $s_i \dots s_n$ constructed in this way have the following properties:

$$\langle r_i, r_j\rangle = 0, i \neq j; \ \langle r_i, s_j\rangle = 0, \ j = 1 \dots i. \tag{1}$$

The conjugate-gradient method is now defined by the following recurrence relations:

$$\begin{cases} s_1 = r_0; \ x_i = x_{i-1} + \alpha_i s_i, \ \alpha_i = -\frac{\langle s_i, r_{i-1}\rangle}{\langle s_i, As_i\rangle}, \\ r_i = r_{i-1} + \alpha_i As_i, \ s_{i+1} = r_i + \beta_i s_i, \\ \beta_i = -\frac{\langle r_i, As_i\rangle}{\langle s_i, As_i\rangle}. \end{cases} \tag{2}$$

The process ends at some $k \leq n$ for which $r_k = 0$. Then, $x^* = x_k$. The convergence point is known by the first approximation x_0. It follows from (2) that recurring vectors $r_0 \dots r_i$ are direct combinations of the vectors $r_0, Ar_0 \dots . A^i r_0$. However, since the vectors $r_0 \dots r_i$ are orthogonal, r_i can only disappear when the vectors $r_0, Ar_0 \dots . A^i r_0$ are linearly dependent. For instance, when there are only i non-zero components in the decomposition of r_0 with respect to a basis of eigenvectors of A. It suffices to show that this can actually impact on the choice of initial approximation [2].

A. System of Linear Equation

A linear equation in the xy plane can be represented algebraically by an equation of the form:

$$a_1 + a_2 y = b$$

An equation in this kind is called a linear equation in the variables x and y. In general terms, we can say a linear equation in n variables $x_1, x_2, \dots x_n$, is one that can be written in the form:

$$a_1 x_1 + a_2 x_2 + \cdots a_n x_n = b$$

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where, $a_1, a_2 \ldots a_n$, and are real constants. In most times, the variables in a typical linear equation are called the roots of the linear equations or solution. For instance, a typical system of linear equations in four unknows is given as:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4 = b_3$$

It is pertinent to state that the double subscript on the coefficients of the unknown parameters is often used specify location of the coefficients in the given systems of equations in that order.

II. Methods

A. Gauss-Seidel Method

This is a minor modification of the Jacobi method that often reduces the number of iterations needed to obtain a given degree of accuracy. The technique is called Gauss-Seidel iteration or the method of successive displacements. Gauss-Seidel method is more appropriate for programming as the successive approximations do not need to be stored in two separate arrays; the new values can be overwritten immediately on the old values [3]; which in turn requires less storage on a computer and that will be significantly effective in case of huge systems, for instance when solving for millions of unknowns [4].

$$x_1^{(n+1)} = \frac{1}{a_{11}}\left[b_1 - a_{12}x_2^{(n)} - a_{13}x_3^{(n)}\right]$$
$$x_2^{(n+1)} = \frac{1}{a_{22}}\left[b_2 - a_{21}x_1^{(n+1)} - a_{23}x_3^{(n)}\right]$$
$$x_3^{(n+1)} = \frac{1}{a_{33}}\left[b_3 - a_{31}x_1^{(n+1)} - a_{32}x_2^{(n+1)}\right]$$

$$x_1^{(n+1)} = \frac{1}{a_{ii}}\left[b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(n+1)} - \sum_{j=1+1}^{n} a_{ij}x_j^{(n)}\right], i = 1, 2, \ldots, n \quad (3)$$

In a solved example applying Gauss-Seidel method, given the system:

$$\begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$$

Comparing with $Ax = b$,

$$A = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix} x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, b = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

A is symmetric since $A^T A$. It is also positive definite since the determinant of every principal sub-matrix is positive i.e.,

$$\begin{vmatrix} 3 & -1 \\ -1 & 2 \end{vmatrix} = (3x_2) - (-1x - 1) = 6 -= 5$$

Also,

$$3\begin{vmatrix} 3 & -1 \\ -1 & 2 \end{vmatrix} + 1\begin{vmatrix} -1 & -1 \\ 0 & 3 \end{vmatrix} + 0$$

$$3(6 - 1) + 1(-3 - 0) = 15 - 3 = 12$$

which are all positive. Therefore, the matrix A is symmetric and positive definite. Now applying the Gauss-Seidel algorithm for a three-by-three matrix we have,

$$x^1{}_1 = \frac{1}{3}(2 - (-1)(-0) - (0)(0)) = \frac{2}{3} = 0.66666666$$
$$x^1{}_2 = \frac{1}{2}(3 - (-1)(0.66666666) - (-1)(0)) = 1.83333333$$
$$x^1{}_3 = \frac{1}{3}(5 - (0)(0.66666666) - (-1)(1.83333333)) = 2.277777778$$
$$x^2{}_1 = \frac{1}{3}(2 - (-1)(1.83333333) - (0)(2.277777778)) = 1.277777778$$
$$x^2{}_2 = \frac{1}{2}\big(3 - (-1)(1.277777778) - (-1)(2.277777778)\big) = 3.277777778$$
$$x^2{}_3 = \frac{1}{3}\big(5 - (0)(1.277777778) - (-1)(2.27777778)\big) = 2.7592595259$$
$$x^3{}_1 = \frac{1}{2}\big(3 - (-1)\big)(1.759259259 + 2.759259259) = 3.759259259$$
$$x^3{}_2 = \frac{1}{3}(5 - (0)(1.759259259) - (-1)(3.759259259)) = 6.2530864197)$$
$$x^3{}_3 = \frac{1}{3}(8.759259259) = 2.91753086$$
$$x^4{}_1 = \frac{1}{2}(3 - (-1)(1.919864197) - (-1)(2.919753086)) = 1.919864197$$
$$x^4{}_2 = \frac{1}{2}(7.839617283) = 3.919808642 = \frac{1}{2}(7.839617283) = 3.919808642$$
$$x^4{}_3 = \frac{1}{3}(5 - (0)(1.919864197) - (-1)(3.919808642)) = 2.973269547$$
$$x^5{}_1 = \frac{1}{3}\big(2 - (1 - 1)(3.919808642) - (0)(2.973269547)\big) = 1.973269547$$
$$x^5{}_2 = \frac{1}{2}\big(3 - (-1)(1.973269547) - (-1)(3.919808642)\big) = 3.973269547$$
$$x^5{}_3 = \frac{1}{3}\big(5 - (0)(1.973269547) - (-1)(3.973269547)\big) = 2.4991089849$$
$$x^6{}_1 = \frac{1}{3}(5.973269547) = 1.991089849$$
$$x^6{}_2 = \frac{1}{2}\big(3 - (-1)(1.991089849) - (-1)(2.991089849)\big) = 3.991089849$$
$$x^6{}_3 = \frac{1}{3}\big(5 - (0)(1.991089849) - (-1)(3.991089849)\big) = 2.99702995$$

$$x_1{}^7 = \frac{1}{3}\big(2-(-1)(3.991089849)-(0)(2.99702995)\big) = 1.99702995$$
$$x_2{}^7 = \frac{1}{2}\big(3-(-1)(1.99702995)-(-1)(2.99702995)\big) = 3.99702995$$
$$x_3{}^7 = \frac{1}{3}\big(5-(0)(1.99702995)-(-1)(3.99702995)\big) = 2.999009983$$
$$x_1 = 1.99702995,$$
$$x_2 = 3.99702995,$$
$$x_3 = 2.999009983,$$

B. *Jacobi Method*

The simplest iterative method, which is called Jacobi iteration or the method of simultaneous displacements, applies to linear systems of n equations in n unknowns. We suppose that the system

$$\left.\begin{array}{llll} a_1x_1 & +a_{12}x_2+\cdots & +a_{1n}x_n = & b_1 \\ a_{21}x_1 & +a_{22}x_2+\cdots & +a_{2n}x_n = & b_2 \\ \vdots & & \vdots & \vdots \\ a_{1n}x_1 & +a_{n2}x_2+\cdots & +a_{nn}x_n = & b_n \end{array}\right\} \quad (4)$$

has exactly one solution and that the diagonal entries $a_{11}, a_{22}, \ldots a_{nn}$ are nonzens. To start, we rewrite (4) by solving the first equation for x_1 in terms of the remaining unknowns and solving the second equation for x_2 and x_3 in terms of the existing unknowns in the nth equation produces:

$$\left.\begin{array}{l} x_1 = \frac{1}{a_{11}}(b_1 - a_{12}x_2 - a_{13}x_3 - \cdots - a_{1n}x_n) \\ x_2 = \frac{1}{a_{22}}(b_2 - a_{21}x_1 - a_{23}x_3 - \cdots - a_{2n}x_n) \\ \vdots \\ x_1 = \frac{1}{a_{nn}}\big(b_n - a_{n1}x_1 - a_{n2}x_2 - \cdots a_{n,n-1}x_{n-1}\big) \end{array}\right\} \quad (5)$$

examining a solved example applying Jacobi method, given the system

$$\begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$$

comparing with $AX = b$

$$A = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \qquad b = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

A is a symmetric since $A^T = A$. It is also positive definitely since the determinant of every principal sub-matrix is positive i.e.

$$\begin{vmatrix} 3 & -1 \\ -1 & 2 \end{vmatrix} = (3 \times 2) - (-1x - 1) = -6 -= 5$$

Also,

$$3\begin{vmatrix} 3 & -1 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} -1 & -1 \\ 0 & 3 \end{vmatrix} + 0 = 3(6-1) + 1(-3-0)$$
$$= 15 - 3 = 12$$

which are all positive. Therefore, the matrix A is symmetric and positive definite.

Let $x = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$x_1{}^1 = \frac{1}{3}\big(2-(-1)(0)-(0)(0)\big) = \frac{2}{3} = 0.6666666$$
$$x_2{}^1 = \frac{1}{2}\big(3-(-1)(0)-(-1)(0)\big) = \frac{3}{2} = 1.5$$
$$x_3{}^1 = \frac{1}{3}\big(5-(0)(0)-(-1)(0)\big) = \frac{5}{3} = 0.66666667$$
$$x_1{}^2 = \frac{1}{3}\big(2-(-1)(1.5)-0(1.66666667)\big) = 1.66666667$$
$$x_2{}^2 = \frac{1}{2}\big(3-(-1)(1.6666666)-(-1)(1.66666667)\big) = 2.6666666$$
$$x_3{}^2 = \frac{1}{3}(5-(0)(0)(0.6666666)-(-1)(1.5)) = 2.16666666$$
$$x_1{}^3 = \frac{1}{3}(2-(-1)(2.6666666)-(0)(2.166666667)) = \frac{4.66666667}{3} = 1.5555556$$
$$x_2{}^3 = \frac{1}{2}\big(3-(-1)(1.66666667)-(-1)(2.16666667)\big) = 3.16666685$$
$$x_3{}^3 = \frac{1}{3}(3-(0)(1.166666667)-(-1)(2.66666666)) = 2.55555555$$
$$x_1{}^4 = \frac{1}{3}\big(2-(-1)(3.66666685)-(-0)(2.55555555)\big) = 1.7222222228$$
$$x_2{}^4 = \frac{1}{2} = (3-(-1)(1.55555556) - (-1)(2.55555555)) = 3.555555553$$
$$x_3{}^4 = \frac{1}{3}\big(5-(0)(1.55555556)-(-1)(3.1666666685)\big) = 2.7222228$$
$$x_1{}^5 = \frac{1}{3}\big(2-(-1)(3.5555555)-(0)(2.7222222228)\big) = 1.851851852$$
$$x_2{}^5 = \frac{1}{2}\big(3-(-1)(1.722222228)-(-1)(2.7222228)\big) = 3.7222228$$
$$x_3{}^5 = \frac{1}{3}\big(5-(0)(1.722222228)-(-1)(3.555555555)\big) = 2.851851252$$
$$x_1{}^6 = \frac{1}{3}\big(2-(-1)(3.72222228)-(0)(2.851851252)\big) = 1.907407427$$
$$x_2{}^6 = \frac{1}{2}\big(3-(-1)(1.8511851852) - (-1)(2.851851882)\big) = 3.851851852$$
$$x_3{}^6 = \frac{1}{3}\big(5-(0)(1.851851852)-(-1)(3.72222228)\big) = 2.907407427$$
$$x_1{}^7 = \frac{1}{3}\big(2-(-1)(3.851851852)-(0)(2.907407427)\big) = 1.950617284$$
$$x_2{}^7 = \frac{1}{3}\big(3-(-1)(1.907407427) - (-1)(1.907407427)\big) = 3.907407427$$
$$x_3{}^7 = \frac{1}{3}\big(5-(0)(1.907407427)-(-1)(3.85185152)\big) =$$

$$2.950617284$$
$$x_1{}^8 = \frac{1}{3}\big(2 - (-1)(9.907407427) - (0)(2.950617284)\big) = 1.969135809$$
$$x_2{}^8 = \frac{1}{3}\big(3 - (-1)(1.950617284) - (-1)(2.950617284)\big) = 3.950617284$$
$$x_3{}^8 = \frac{1}{3}\big(5 - (0)(1.950617284) - (-1)(3.907407427)\big) = 2.969135809$$
$$x_1{}^9 = \frac{1}{3}\big(2 - (-1)(3.950617284)\big) = \frac{1}{3}(3.950617284) = 1.983539095$$
$$x_2{}^9 = \frac{1}{2}\big(3 - (-1)(1.969135809) - (-1)(2.969135809)\big) = 3.969135809$$
$$x_3{}^9 = \frac{1}{2}\big(5 - (0)(1.969135809) - (-1)(3.950617284)\big) = 2.93539095$$
$$x_1{}^{10} = \frac{1}{3}\big(2 - (-1)(3.969135809) - (0)(2.93539095)\big) = 1.989711936$$
$$x_2{}^{10} = \frac{2}{3}\big(3 - (-1)(1.983539095) - (-1)(2.983539095)\big) = 3.983539095$$
$$x_3{}^{10} = \frac{1}{3}\big(5 - (0)(1.983539095) - (-1)(3.96399177)\big) = 2.987997257$$
$$x_1{}^{11} = \frac{1}{3}\big(2 - (-1)(3.983539095) - (0)(2.98799257)\big) = 1.994513032$$
$$x_2{}^{11} = \frac{1}{3}\big(3 - (-1)(1.98799257) - (-1)(2.98799257)\big) = 3.98799257$$
$$x_3{}^{11} = \frac{1}{3}\big(5 - (0)(1.98799257) - (-1)(3.983539095)\big) = 2.994513032$$
$$x_1{}^{12} = \frac{1}{3}\big(2 - (-1)(3.983539095) - (0)(2.994513032)\big) = 1.99599999919$$
$$x_2{}^{12} = \frac{1}{2}\big(3 - (-1)(1.994513032)\big) = \frac{1}{2}(7.989026064) = 3.994513032$$
$$x_3{}^{12} = \frac{1}{3}\big(5 - (0)(1.994513032) - (-1)(3.987997257)\big) = 2.995999919$$
$$x_1{}^{13} = \frac{1}{3}(2 - (-1)(3.994513032) - (0)(2.99599919)) = 1.998171011$$
$$x_2{}^{13} = \frac{1}{2}(3 - (-1)(1.995999919) - (-1)(2.995999919) = 3.995999919$$
$$x_3{}^{13} = \frac{1}{3}\big(5 - (0)(1.995999919) - (-1)(3.994513032)\big) = 2.998171011$$

Therefore,

$$x_1 = 1.998171011$$
$$x_2 = 3.995999919$$
$$x_3 = 2.99871011$$

C. Conjugate Gradient

One of the major problems in machine computations is to find an effective method of solving a system of n simultaneous equations in *n* unknowns, particularly if *n* is large [5]. The conjugate gradient method solves systems of linear equations of the form $Ax = b$, where A is a real $n \times n$ symmetric and positive definite matrix. This implies that,

$$A^T = A \text{ and } x^T Ax > 0 \text{ for } x \neq 0.$$

Theorem 1. A symmetric matrix A is a positive definite if and only if all the eigenvalue of A are positive.

Theorem 2. A symmetric matrix A is positive definite if and only if the determinant of every principal sub-matrix is positive. i.e., $A = \begin{bmatrix} 2 & -1 & -3 \\ -1 & 2 & 4 \\ -3 & 4 & 9 \end{bmatrix}$ is positive definite since, $|2| = 2$, $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = 3$, $\begin{bmatrix} 2 & -1 & -3 \\ -1 & 2 & 4 \\ -3 & 4 & 9 \end{bmatrix} = 1$ all of which are positive. Thus, we are guaranteed that all eigenvalues of A are positive and $x^1 Ax > 0$ for all $x \neq 0$. The inner-product notation for real vectors x and y are used in the conjugate gradient method.

$$\langle x_1 y \rangle = x^T y = \sum_{i+1}^{n} x_i y_i \tag{6}$$

1. Algorithm Conjugate Gradient

In this section, we are going to apply the algorithm to solve the system of linear equation of the form $Ax = b$ is a positive definite and symmetric matrix; using the conjugate gradient method ("Algorithm Conjugate Gradient").

Step 1: Input x^0, m, A, b, e
Step 2: set $r^0 = b - Ax^0$
Step 3: $V^0 = r^0$
Step 4: For $K = 0,1,2, \dots, m-1$ do, Begin
Step 4a: if $V^k = 0$ then stop
Else
Step 4b: set $C =< r^k, r^k, >= (r^k)^T r^k$
Step 4c: set
$t_k = \frac{c}{<V^{(k)}, z>}$
Step 4e: set $x^{(k+1)} = x^{(k)} + t_k V^{(k)}$
Step 4f: set $r^{(k+1)} = r^{(k)} - t_k z$
Step 4g: if $(\|k + k\|)_2{}^2 < \Sigma$ then stop
End do
Step 5: set
$S_K = \frac{<r^{(k+k)}, r^{(k+k)}>}{C}$
Step 6: set $V^{(k+k)}, r^{(k+k)} + S_K V^{(k)}$
Step 7: output $k+1, x^{(k+k)}, r^{(k+k)}$
Step 8: stop

A solved example applying the conjugate gradient method by tracing the algorithm with the given matrix system in (7) yields:

$$\begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$$

Step 1: $V^0 = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}, x^0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Step 2: $V^0 = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$

Step 3: $C = V^T V = (2 \quad 3 \quad 5)\begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} = 38$

Step 4: $z = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 12 \end{bmatrix}$

Step 5: $t = \frac{C}{<V,Z>} = \frac{C}{V^T Z} = \frac{38}{(2 \quad 3 \quad 5)\begin{bmatrix} 3 \\ -1 \\ 12 \end{bmatrix}}$

$$t = t = \frac{38}{63} = 0.603174603$$

$$x^{(1)} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + 0.603174603\begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 1.206349206 \\ 1.809523809 \\ 3.015873015 \end{pmatrix}$$

$$V = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} - 0.603174603\begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} - \begin{pmatrix} 1.809523809 \\ -0.603174603 \\ 3.015873015 \end{pmatrix}$$

$$V = \begin{pmatrix} 0.194761913 \\ 3.603174603 \\ -2.238095236 \end{pmatrix}$$

$$C = (0.194761913\ 3.603174603\ -2.238095236)\begin{pmatrix} 0.194761913 \\ 3.603174603 \\ -2.238095236 \end{pmatrix}$$

$$C = (0.036281179 + 12.9828672197 + 5.009070285) = (18.0287492947)$$

$$x^{(2)} = \begin{pmatrix} 1.206349206 \\ 1.809523809 \\ 3.015873015 \end{pmatrix} + 0.43256906\begin{pmatrix} 1.139318967 \\ 5.026438767 \\ 0.134011694 \end{pmatrix}$$

$$x^{(2)} = \begin{pmatrix} 0.206349206 \\ 1.809523809 \\ 3.015873015 \end{pmatrix} + \begin{pmatrix} 0.492828262 \\ 2.16281953 \\ 0.057968621 \end{pmatrix}$$

$$x^{(2)} = \begin{pmatrix} 1.699177468 \\ 3.97243339 \\ 3.073841637 \end{pmatrix}$$

$$V = \begin{pmatrix} 0.190476191 \\ 3.603174603 \\ -2.238095236 \end{pmatrix} - 0.432563906 = \begin{pmatrix} -1.608481866 \\ 8.779546923 \\ -4.624403685 \end{pmatrix}$$

$$V = \begin{pmatrix} 0.190476191 \\ 3.603174603 \\ -2.238095236 \end{pmatrix} - \begin{pmatrix} -0.69571198 \\ 3.69771511 \\ -2.000350121 \end{pmatrix}$$

$$V = \begin{pmatrix} 0.886247389 \\ -0.194540507 \\ -0.237745115 \end{pmatrix}$$

$$C = (0.886247389\ -0.194540507\ -0.237745115)\begin{pmatrix} 0.886247389 \\ -0.194540507 \\ -0.237745115 \end{pmatrix}$$

$$C = 0.785434434 + 0.037846008 + 0.056522739 = 0.879803181$$

$$V = \begin{pmatrix} 0.886247389 \\ -0.194540507 \\ -0.237745115 \end{pmatrix} + \left(\frac{0.879803181}{18.02801277}\right)\begin{pmatrix} -1.139318967 \\ 5.026438767 \\ 0.134011694 \end{pmatrix}$$

$$V = \begin{pmatrix} 0.886247389 \\ -0.194540507 \\ -0.237745115 \end{pmatrix} + 0.048802005\begin{pmatrix} -1.139318967 \\ 5.026438767 \\ 0.134011694 \end{pmatrix}$$

$$V = \begin{pmatrix} 0.886247389 \\ -0.194540507 \\ -0.237745115 \end{pmatrix} + \begin{pmatrix} 0.067990449 \\ 0.245300293 \\ 0.006540039 \end{pmatrix} = \begin{pmatrix} 0.954237838 \\ -0.050759786 \\ -0.231205075 \end{pmatrix}$$

$$C = 0.879803181$$

$$Z = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{pmatrix}\begin{pmatrix} 0.954237838 \\ -0.050759786 \\ -0.231205075 \end{pmatrix}$$

$$Z = \begin{pmatrix} 2.811953728 \\ -0.621513191 \\ -0.744375011 \end{pmatrix}$$

$$t = \frac{C}{< V,Z >} = \frac{0.879803181}{< V,Z >}$$

$$< V,Z > = (0.954237838\ 0.050759786\ -0.231205075)\begin{pmatrix} 2.811953728 \\ -0.621513191 \\ -0.744375011 \end{pmatrix}$$

$$< V,Z > = 2.655153109 - 0.031547876 + 0.17210328$$

$$< V,Z > = 2.795708512$$

$$t = \frac{0.879083181}{2.795708512} = 0.314285459$$

$$x^{(3)} = \begin{pmatrix} 1.699177468 \\ 3.97243339 \\ 3.073841637 \end{pmatrix} + 0.314285459\begin{pmatrix} 0.954237838 \\ 0.050759786 \\ -0.231205075 \end{pmatrix}$$

$$x^{(3)} = \begin{pmatrix} 1.699177468 \\ 3.97243339 \\ 3.073841637 \end{pmatrix} + \begin{pmatrix} 0.299903076 \\ 0.015953062 \\ -0.072664393 \end{pmatrix} = \begin{pmatrix} 1.999080545 \\ 3.988386452 \\ 3.001177244 \end{pmatrix}$$

Hence,

$$x_1 = 1.999080545,$$
$$x_2 = 3.988386452,$$
$$x_3 = 3.001177244$$

III. Convergence

The Conjugate Gradient method converges faster to the exact solutions, the Gauss-Seidel and Jacobi method do not always work. In few cases, the two methods may fail to produce a good approximation to the solution, regardless of the number of iterations performed. In such cases, the approximations are said to diverge.

$$A = \begin{bmatrix} a_{11} & a_{12} \dots. & a_{1n} \\ a_{21} & a_{22} \dots. & a_{2n} \\ \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} \dots. & a_{nn} \end{bmatrix}$$

However, if by performing sufficiently much iteration, the solution can be obtained to any desired degree of accuracy, the approximations are said to converge. We shall now discuss some conditions that ensure convergence. A square matrix

$$|a_{11}| > |a_{12}| + |a_{13}| + \cdots + |a_{1n}|$$
$$|a_{22}| > |a_{21}| + |a_{23}| + \cdots + |a_{2n}|$$
$$|a_{nn}| > |a_{ns}| + |a_{n2}| + \cdots + |a_n(n-1)|$$

is called strictly diagonally dominant, if the absolute value of each diagonal entry is greater than of the absolute value of the remaining entries in the same row. That is, if A is strictly diagonally dominant, then the Gauss-Seidel and Jacobi approximations to the solution of $Ax = b$ both converge to exact solutions to the system for all choices of the initial approximation. In practical problems we are concerned not only with the convergence of iterative method but also with how fast they converge; for example, there are linear system in which the Gauss-Seidel approximation converge, but the convergence is such that millions of iterations would be required to obtain any reasonable accuracy. With clever programming, less computer memory is needed for iterative methods than direct methods. Thus, if memory space is a problem, iterative methods may be essential. If the iterative methods diverge or the rate of convergence is too slow, direct methods may be essential.

A. *Comparisons with Other Methods*

TABLE I
DIFFERENCE BETWEEN CONJUGATE GRADIENT, JACOBI AND GAUSS SEIDEL METHODS

Conjugate Gradient	Jacobi and Gauss-Seidel
For any initial approximation, the conjugate gradient method converges after a finite number of iterations	It is only an iterative method for any initial approximation, it converges after many iterations given the convergent condition
It gives the exact solution	It keeps tending to the solution
As a direct method, its structure is based in the process of sequential A −Orthogonalization process with respect to the scalar product $< x, y >$	It is more iterative and is based on the process of continuous and recessive replacement of proves results of the iteration and it is less matrix and dimensional oriented
It is best in strategic layout, i.e., it gives the maximal minimization after n step	It is more sensitive to rounding-off error. Therefore, the method is continued beyond n- iterations.

IV. CONCLUSION

In this research, we examined the solutions of three different iterative methods for solving systems of linear equations of the form $Ax = b$, where A is a real $n \times n$ symmetric and positive definite matrix. From the results obtained from the preceding sections, it is clear to state that the conjugate gradient method produces more exact and efficient solutions in small iterative steps than the other two methods (Gauss-Seidel and Jacobi). The essential advantage of the conjugate gradient method over the other two methods is that it does not require knowledge of the boundaries of the spectrum. It also places small demands on computer memory when compared to other methods. Since it gives the maximal minimization after n step, the conjugate Gradient method generally should be adopted for solving systems of linear equations of the form $Ax = b$, where the matrix A must be positive definite and symmetric.

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