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Pension Plan Member's Investment Strategies with Transaction Cost and Couple Risky Assets Modelled by the O-U Process

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Abstract**—**This paper studies the optimal investment strategies for a plan member (PM) in a defined contribution (DC) pension scheme with transaction cost, taxes on invested funds and couple risky assets (stocks) under the Ornstein-Uhlenbeck (O-U) process. The PM's portfolio is assumed to consist of a risk-free asset and two risky assets where the two risky assets are driven by the O-U process. The Legendre transformation and dual theory is use to transform the resultant optimal control problem which is a nonlinear partial differential equation (PDE) into linear PDE and the resultant linear PDE is then solved for the explicit solutions of the optimal investment strategies for PM exhibiting constant absolute risk aversion (CARA) using change of variable technique. Furthermore, theoretical analysis is used to study the influences of some sensitive parameters on the optimal investment strategies with observations that the optimal investment strategies for the two risky assets increase with increase in the dividend and decreases with increase in tax on the invested funds, risk averse coefficient, initial fund size and the transaction cost.

Keywords**—**Ornstein-Uhlenbeck process, portfolio management, Legendre transforms, CARA utility.

I. Introduction

THE DC pension scheme is a common practice among the working class whose aim is to prepare for a future after working age. This scheme has been on in many countries across the world due to its ability to provide a comfortable platform for its members to be involved in the day-to-day activities of the pension fund. The most interesting part of the DC plan is that members are involved in planning for their retirement benefits. Since member's benefit depend on investments in the financial market, there is need to develop and understand how best these accumulated funds can be invested for optimal profit; this has given birth to a new research area known as optimal investment strategies for a DC pension plan.

A lot of researches have been carried out on portfolio optimization, some of which include [1]-[3]. Reference [4] studied the optimal investment problem with taxes, dividend and transaction cost using different utility functions under the CEV process. Reference [5] studied the reinsurance problem and optimal portfolio strategies under the CEV model. In [6], [7], the optimal portfolio strategies for a DC pension plan with return of premiums clauses under CEV process was studied. Also, the optimal investment strategies with stochastic interest rate under GBM have been studied by some authors; these include [8], who studied the investment portfolio under stochastic interest rate for a case of protected DC fund. Also, [9] and [10] used stochastic interest rate to obtain optimal investment plan for a DC plan. In [11] and [12], the optimal investment plan with the Vasicek interest rate was studied and in [13] and [14], the optimal investment strategies in a DC plan under affine interest rate were studied. In [15], the optimal investment strategy for an insurer with stochastic interest rate under the CEV model was studied; in their work, they studied the effect of some sensitive parameters on the investment strategy when the interest rate is driven by the Cox-Ingersoll-Ross (CIR) model. Reference [16] studied the optimal portfolio strategies for investors with exponential utility under the modified CEV process. Here, the interest rate follows the O-U process.

Most of the literatures mentioned above used the CIR model, Vasicek model, affine model etc. to model their interest rate, however very few used O-U process. According to [17], the O-U process models both interest rate and stock market price since it reflects changes in the interest rates and asset prices. In [17], the optimal investment strategies for a DC plan under the O-U process were studied. In their work, a single risky asset modelled by the O-U process was combined with a risk-free asset and also investment in loan.

From this important assertion on the O-U process and considering the unstable nature of the stock market prices, we are inspired to build a strategic investment plan for a PM with exponential utility which considers changes in stock market prices. This is done by studying the optimal investment strategies for a PM exhibiting CARA and whose risky assets follow O-U process. The main difference between our work and that of [17] is that we will be considering a PM in a DC plan with exponential utility instead of logarithm utility and investment in two risky assets instead of one risky asset modelled by the O-U process.

II. The Financial Market Model

We consider a portfolio comprising of one risk free asset and two risky assets in a financial market which is open continuously over an interval $t \in [0, T]$ where T is the expiration date of the investment. Let $\{\mathcal{Z}_1(t), \mathcal{Z}_2(t): t \geq 0\}$ be standard Brownian motion defined on a complete probability

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space (Ω, F, P) where Ω is a real space and P is a probability measure and F is the filtration which represents the information generated by the Brownian motions.

Let $r > 0$ be the risk-free interest rate and $\mathcal{S}_0(t)$ the price of the risk-free asset at time t. Then the stochastic differential equation associated with the risk-free asset price is given as:

$$d\mathcal{S}_0(t) = r\mathcal{S}_0(t)dt \qquad \mathcal{S}_0(0) = s_0 > 0 \tag{1}$$

where $r > 0$ is the risk free interest rate.

Let $\mathcal{S}_1(t)$ and $\mathcal{S}_2(t)$ denote the prices of the two risky assets (stocks) which are described by the O-U process which describe the fluctuation in the stock market prices and the dynamics of the stock market prices are described similar to [17] by the stochastic differential equations at $t \geq 0$ as follows

$$d\mathcal{S}_1(t) = k_1(n_1 - \mathcal{S}_1)dt + v_{11}d\mathcal{Z}_1(t) + v_{12}d\mathcal{Z}_2(t) \tag{2}$$

$$d\mathcal{S}_2(t) = k_2(n_2 - \mathcal{S}_2)dt + v_{21}d\mathcal{Z}_1(t) + v_{22}d\mathcal{Z}_2(t) \tag{3}$$

where $k_1 > 0$ and $k_2 > 0$ are the recovery rates of the risky assets, n_1 and n_2 are the response centers of the two risky assets, v_{11}, v_{12}, $v_{21}$and v_{22} are instantaneous volatilities and form a 2×2 matrix $v = \{v_{pq}\}_{2\times 2}$ such that $vv^T$is positive definite. see [19] for details.

III. Optimization Problem

A. Wealth Formulations and Hamilton Jacobi Bellman Equation

Let $\varphi = \{\varphi_1, \varphi_2\}$ be the optimal investment strategy and we define the utility attained by the investor from a given state y at time t as

$$\mathcal{G}_\varphi(t, s_1, s_2, x) = E_\varphi\left[U(\mathcal{Y}(T)) \mid \mathcal{S}_1(t) = s_1, \mathcal{S}_2(t) = s_2, \mathcal{Y}(t) = y\right], \tag{4}$$

where t is the time, r is the risk free interest rate and y is the wealth.

The objective here is to determine the optimal investment strategy and the optimal value function of the investor given as

$$\varphi^* \text{ and } \mathcal{G}(t, s_1, s_2, y) = \sup_{\varphi} \mathcal{G}_\varphi(t, s_1, s_2, y) \tag{5}$$

Respectively such that

$$\mathcal{G}_{\varphi^*}(t, s_1, s_2, y) = \mathcal{G}(t, s_1, s_2, y) \tag{6}$$

Let $\mathcal{Y}(t)$ be the member's surplus wealth at time t and suppose the tax rate in the financial market is τ, the member's contribution rate at any given time is c,d_1 and d_2 represent the dividend incomes of the two risky assets and the rate of transaction cost which covers the administrative fee and stamp duties is β. Also we assume $\frac{k_1(n_1-s_1)+\left(d_1-\frac{\beta}{2}-r\right)s_1}{s_1} > 0$ and $\frac{k_2(n_2-s_2)+\left(d_2-\frac{\beta}{2}-r\right)s_2}{s_2} > 0$ then the differential form associated with the member's surplus fund is given as:

$$\mathrm{d}\mathcal{Y}(t) = \begin{pmatrix} \mathcal{Y}(t)\left(\varphi_0 \frac{d\mathcal{S}_0(t)}{\mathcal{S}_0(t)} + \varphi_1 \frac{d\mathcal{S}_1(t)}{\mathcal{S}_1(t)} + \varphi_2 \frac{d\mathcal{S}_2(t)}{\mathcal{S}_2(t)}\right) \\ -\tau\mathcal{Y}(t)dt + cdt \end{pmatrix} \tag{7}$$

substituting (1)-(3) into (7), we have

$$\mathrm{d}\mathcal{Y}(t) = \begin{pmatrix} \begin{pmatrix} \mathcal{Y}(t)\begin{pmatrix} \varphi_1\left(\frac{k_1(n_1-s_1)+\left(d_1-\frac{\beta}{2}-r\right)s_1}{s_1}\right) \\ +\varphi_2\left(\frac{k_2(n_2-s_2)+\left(d_2-\frac{\beta}{2}-r\right)s_2}{s_2}\right)\end{pmatrix} \\ +\left(\mathcal{Y}(t)(r-\tau)\right) + c \end{pmatrix} dt \\ \\ +\mathcal{Y}(t)\left(\left(\frac{\varphi_1 v_{11}}{s_1} + \frac{\varphi_2 v_{21}}{s_2}\right)d\mathcal{B}_1 + \left(\frac{\varphi_1 v_{12}}{s_1} + \frac{\varphi_2 v_{22}}{s_2}\right)d\mathcal{B}_2\right) \\ \mathcal{Y}(0) = y_0 \end{pmatrix} \tag{8}$$

where φ_0, φ_1 and φ_2 are the optimal investment plans for the risk-free asset and the two risky assets respectively, such that $\varphi_0 = 1 - \varphi_1 - \varphi_2$.

From [19], applying the Ito's lemma and maximum principle, the Hamilton Jacobi Bellman (HJB) equation which is a nonlinear PDE associated with (8) is obtained by maximizing $\mathcal{G}_{\varphi^*}(t, r, s_1, s_2, y)$ subject to the PPM's wealth as follows

$$\left.\begin{matrix} \mathcal{G}_t + k_1(n_1 - s_1)\mathcal{G}_{s_1} + k_2(n_2 - s_2)\mathcal{G}_{s_2} \\ +((r-\tau)x + c)\mathcal{G}_y + \frac{1}{2}(v_{11}^2 + v_{12}^2)\mathcal{G}_{s_1 s_1} \\ +\frac{1}{2}(v_{21}^2 + v_{22}^2)\mathcal{G}_{s_2 s_2} + (v_{11}v_{21} + v_{12}v_{22})\mathcal{G}_{s_1 s_2} \\ +\sup\limits_{\varphi_1,\varphi_2} \\ \left\{\begin{matrix} \frac{y^2}{2}\begin{pmatrix} \left(\frac{\varphi_1 v_{11}}{s_1} + \frac{\varphi_2 v_{21}}{s_2}\right)^2 \\ +\left(\frac{\varphi_1 v_{12}}{s_1} + \frac{\varphi_2 v_{22}}{s_2}\right)^2 \end{pmatrix}\mathcal{G}_{yy} \\ +y\begin{pmatrix} \left(\frac{k_1(n_1-s_1)+\left(d_1-\frac{\beta}{2}-r\right)s_1}{s_1}\right)\varphi_1 \\ +\left(\frac{k_2(n_2-s_2)+\left(d_2-\frac{\beta}{2}-r\right)s_2}{s_2}\right)\varphi_2 \end{pmatrix}\mathcal{G}_y \\ y\begin{pmatrix} \left(\frac{\varphi_1 v_{11}}{s_1} + \frac{\varphi_2 v_{21}}{s_2}\right)v_{11} \\ +\left(\frac{\varphi_1 v_{12}}{s_1} + \frac{\varphi_2 v_{22}}{s_2}\right)v_{12} \end{pmatrix}\mathcal{G}_{y s_1} \\ y\begin{pmatrix} \left(\frac{\varphi_1 v_{12}}{s_1} + \frac{\varphi_2 v_{22}}{s_2}\right)v_{22} \\ +\left(\frac{\varphi_1 v_{11}}{s_1} + \frac{\varphi_2 v_{21}}{s_2}\right)v_{21} \end{pmatrix}\mathcal{G}_{y s_2} \end{matrix}\right\} \end{matrix}\right\} = 0 \tag{9}$$

Differentiating (9) with respect to $\varphi_1 and \varphi_2$, we obtain the first order maximizing condition for (9) as

$$\varphi_1^* = \frac{\begin{bmatrix} (v_{11}v_{21}+v_{12}v_{22})\begin{pmatrix} k_2(n_2-s_2) \\ +\left(d_2-\frac{\beta}{2}-r\right)s_2 \end{pmatrix} \\ -(v_{21}^2+v_{22}^2)\begin{pmatrix} k_1(n_1-s_1) \\ +\left(d_1-\frac{\beta}{2}-r\right)s_1 \end{pmatrix} \end{bmatrix}}{y\begin{pmatrix} (v_{11}^2+v_{12}^2)(v_{21}^2+v_{22}^2) \\ -(v_{11}v_{21}+v_{12}v_{22})^2 \end{pmatrix}} \frac{s_1\mathcal{G}_y}{\mathcal{G}_{yy}} - \frac{s_1\mathcal{G}_{y s_1}}{y\mathcal{G}_{yy}} \tag{10}$$

$$\varphi_2^* = \frac{\begin{bmatrix} (v_{11}v_{21}+v_{12}v_{22})\begin{pmatrix} k_2(n_2-s_2) \\ +\left(d_2-\frac{\beta}{2}-r\right)s_2 \end{pmatrix} \\ -(v_{11}^2+v_{12}^2)\begin{pmatrix} k_2(n_2-s_2) \\ +\left(d_2-\frac{\beta}{2}-r\right)s_1 \end{pmatrix} \end{bmatrix}}{y\begin{pmatrix} (v_{11}^2+v_{12}^2)(v_{21}^2+v_{22}^2) \\ -(v_{11}v_{21}+v_{12}v_{22})^2 \end{pmatrix}} \frac{s_2 \mathcal{G}_y}{\mathcal{G}_{yy}} - \frac{s_2 \mathcal{G}_{ys_2}}{y\mathcal{G}_{yy}} \quad (11)$$

Substituting (10) and (11) into (9), we have

$$\left\{\begin{array}{c} \mathcal{G}_t + k_1(n_1-s_1)\mathcal{G}_{s_1} + k_2(n_2-s_2)\mathcal{G}_{s_2} \\ +\left((r-\tau)y+c\right)\mathcal{G}_y + \frac{1}{2}\mathcal{F}_1\mathcal{G}_{s_1s_1} \\ +\frac{1}{2}\mathcal{F}_2\mathcal{G}_{s_2s_2} + \mathcal{F}_3\mathcal{G}_{s_1s_2} + \frac{1}{2}(\mathcal{F}_4-\mathcal{F}_5-\mathcal{F}_6)\frac{\mathcal{G}_y^2}{\mathcal{G}_{yy}} \\ -\left(k_1(n_1-s_1)+\left(d_1-\frac{\beta}{2}-r\right)s_1\right)\frac{\mathcal{G}_y\mathcal{G}_{ys_1}}{\mathcal{G}_{yy}} \\ -\left(k_2(n_2-s_2)+\left(d_2-\frac{\beta}{2}-r\right)s_2\right)\frac{\mathcal{G}_y\mathcal{G}_{ys_2}}{\mathcal{G}_{yy}} \\ -\frac{1}{2}\mathcal{F}_1\frac{\mathcal{G}_{ys_1}^2}{\mathcal{G}_{yy}} - \frac{1}{2}\mathcal{F}_2\frac{\mathcal{G}_{ys_2}^2}{\mathcal{G}_{yy}} - \mathcal{F}_3\frac{\mathcal{G}_{ys_1}\mathcal{G}_{ys_2}}{\mathcal{G}_{yy}} \end{array}\right\} = 0 \quad (12)$$

where

$$\left\{\begin{array}{l} \mathcal{F}_1 = (v_{11}^2+v_{12}^2),\ \mathcal{F}_2 = (v_{21}^2+v_{22}^2), \\ \mathcal{F}_3 = (v_{11}v_{21}+v_{12}v_{22}), \\ \mathcal{F}_4 = \frac{2\mathcal{F}_3\begin{pmatrix} k_1(n_1-s_1) \\ +\left(d_1-\frac{\beta}{2}-r\right)s_1 \end{pmatrix}\begin{pmatrix} k_2(n_2-s_2) \\ +\left(d_2-\frac{\beta}{2}-r\right)s_2 \end{pmatrix}}{(\mathcal{F}_1\mathcal{F}_2-\mathcal{F}_3^2)}, \\ \mathcal{F}_5 = \frac{\mathcal{F}_2\left(k_1(n_1-s_1)+\left(d_1-\frac{\beta}{2}-r\right)s_1\right)^2}{(\mathcal{F}_1\mathcal{F}_2-\mathcal{F}_3^2)}, \\ \mathcal{F}_6 = \frac{\mathcal{F}_1\left(k_2(n_2-s_2)+\left(d_2-\frac{\beta}{2}-r\right)s_2\right)^2}{(\mathcal{F}_1\mathcal{F}_2-\mathcal{F}_3^2)}, \end{array}\right. \quad (13)$$

Equations (10) and (11) become

$$\varphi_1^* = \frac{\begin{bmatrix} \left(k_2(n_2-s_2)+\left(d_2-\frac{\beta}{2}-r\right)s_2\right)\mathcal{F}_3 \\ -\left(k_1(n_1-s_1)+\left(d_1-\frac{\beta}{2}-r\right)s_1\right)\mathcal{F}_2 \end{bmatrix}}{y(\mathcal{F}_1\mathcal{F}_2-\mathcal{F}_3^2)} \frac{s_1\mathcal{G}_y}{\mathcal{G}_{yy}} - \frac{s_1\mathcal{G}_{ys_1}}{y\mathcal{G}_{yy}} \quad (14)$$

$$\varphi_2^* = \frac{\begin{bmatrix} \left(k_2(n_2-s_2)+\left(d_2-\frac{\beta}{2}-r\right)s_2\right)\mathcal{F}_3 \\ -\left(k_2(n_2-s_2)+\left(d_2-\frac{\beta}{2}-r\right)s_1\right)\mathcal{F}_1 \end{bmatrix}}{y(\mathcal{F}_1\mathcal{F}_2-\mathcal{F}_3^2)} \frac{s_2\mathcal{G}_y}{\mathcal{G}_{yy}} - \frac{s_2\mathcal{G}_{ys_2}}{y\mathcal{G}_{yy}} \quad (15)$$

B. Legendre Transformation and Dual Theory

The differential equation obtained in (12) is a nonlinear PDE and is somehow complex to solve. In this section, we will introduce the Legendre transformation and dual theory and use it to transform the nonlinear PDE to a linear PDE.

Theorem 1. Let $f: R^n \to R$ be a convex function for $z > 0$, define the Legendre transform

$$N(z) = \max_{y}\{f(x) - zy\}, \quad (16)$$

The function $N(z)$ is the Legendre dual of the function $f(y)$, see [20].

Since $f(y)$ is convex, from Theorem 1 and [17], [18], the Legendre transform for the value function $\mathcal{L}(t, s_1, s_2, y)$ can be defined as follows

$$\hat{\mathcal{G}}(t, s_1, s_2, z) = \sup\{\mathcal{G}(t, s_1, s_2, y) - zy \mid 0 < x < \infty\}\ 0 < t < T (17)$$

where $\hat{\mathcal{G}}$ is the dual of $\mathcal{G}$ and $z > 0$ is the dual variable of y.

The value of y where this optimum is achieved is represented by $g(t, s_1, s_2, z)$, such that

$$g(t, s_1, s_2, z) = \inf\left\{ y \;\middle|\; \begin{array}{c} \mathcal{G}(t, s_1, s_2, y) \geq \\ zy + \hat{\mathcal{G}}(t, s_1, s_2, z) \end{array} \right\}\ 0 < t < T \ (18)$$

From (18), the function g and $\hat{\mathcal{G}}$ are very much related where g refers to as the dual of $\mathcal{G}$. The two functions are related thus

$$\hat{\mathcal{G}}(t, s_1, s_2, z) = \mathcal{G}(t, s_1, s_2, h) - zh. \quad (19)$$

where

$$h(t, s_1, s_2, z) = y, \mathcal{G}_y = z,\ h = -\hat{\mathcal{G}}_z. \quad (20)$$

differentiating (19) with respect to $t, , s_1, s_2\ and y$

$$\left\{\begin{array}{c} \mathcal{G}_t = \hat{\mathcal{G}}_t,\ \mathcal{G}_{s_1} = \hat{\mathcal{G}}_{s_1}, \mathcal{G}_{s_2} = \hat{\mathcal{G}}_{s_2},\ \mathcal{G}_y = z, \\ \mathcal{G}_{s_1y} = \frac{-\hat{\mathcal{G}}_{s_1z}}{\hat{\mathcal{G}}_{zz}}, \mathcal{G}_{s_2y} = \frac{-\hat{\mathcal{G}}_{s_2z}}{\hat{\mathcal{G}}_{zz}}, \\ \mathcal{G}_{yy} = \frac{-1}{\hat{\mathcal{G}}_{zz}},\ \mathcal{G}_{s_1s_1} = \hat{\mathcal{G}}_{s_1s_1} - \frac{\hat{\mathcal{G}}_{s_1z}^{\ 2}}{\hat{\mathcal{G}}_{zz}}, \\ \mathcal{G}_{s_2s_2} = \hat{\mathcal{G}}_{s_2s_2} - \frac{\hat{\mathcal{G}}_{s_2z}^{\ 2}}{\hat{\mathcal{G}}_{zz}} \end{array}\right. \quad (21)$$

At terminal time T, we define the dual utility in terms of the original utility function $U(y)$ as

$$\widehat{U}(z) = sup\{U(y) - zy \mid 0 < x < \infty\},$$

and

$$G(z) = sup\ \{y \mid U(y) \geq zy + \widehat{U}(z).$$

As a result$\hat{\mathcal{G}}(t, r, s_1, s_2, z) = \mathcal{G}(t, r, s_1, s_2, h) - zh.$

$$G(z) = (U')^{-1}(z), \quad (22)$$

where G is the inverse of the marginal utility U and note that $\mathcal{G}(T, r, s_1, s_2, y) = U(y)$.

At terminal time T, we can define

$$\begin{array}{l} h(T, r, s_1, s_2, z) \\ = \inf\limits_{y>0}\{y \mid U(x) \geq zy + \hat{\mathcal{G}}(t, r, s_1, s_2, z)\}\, and\, \mathcal{G}(t, r, s_1, s_2, z) \\ = \sup\limits_{y>0}\{U(x) - zy\} \end{array}$$

so that

$$h(T, r, s_1, s_2, z) = (U')^{-1}(z). \quad (23)$$

Substituting (21) into (12), (14) and (15), we have

$$\left\{\begin{array}{c} \hat{\mathcal{G}}_t + k_1(n_1-s_1)\hat{\mathcal{G}}_{s_1} + k_2(n_2-s_2)\hat{\mathcal{G}}_{s_2} \\ +\left((r-\tau)y+c\right)z + \frac{1}{2}\mathcal{F}_1\hat{\mathcal{G}}_{s_1s_1} \\ +\frac{1}{2}\mathcal{F}_2\hat{\mathcal{G}}_{s_2s_2} + \mathcal{F}_3\hat{\mathcal{G}}_{s_1s_2} - \frac{1}{2}(\mathcal{F}_4-\mathcal{F}_5-\mathcal{F}_6)z^2\hat{\mathcal{G}}_{zz} \\ -\left(k_1(n_1-s_1)+\left(d_1-\frac{\beta}{2}-r\right)s_1\right)z\hat{\mathcal{G}}_{s_1z} \\ -\left(k_2(n_2-s_2)+\left(d_2-\frac{\beta}{2}-r\right)s_2\right)z\hat{\mathcal{G}}_{s_2z} \end{array}\right\} = 0 \quad (24)$$

$$\varphi_1^* = \frac{\begin{bmatrix} \left(k_2(n_2-s_2)+\left(d_2-\frac{\beta}{2}-r\right)s_2\right)\mathcal{F}_3 \\ -\left(k_1(n_1-s_1)+\left(d_1-\frac{\beta}{2}-r\right)s_1\right)\mathcal{F}_2 \end{bmatrix}}{y(\mathcal{F}_1\mathcal{F}_2-\mathcal{F}_3^2)} s_1 z\hat{\mathcal{G}}_{zz} - \frac{s_1\hat{\mathcal{G}}_{s_1 z}}{y} \tag{25}$$

$$\varphi_2^* = \frac{\begin{bmatrix} \left(k_2(n_2-s_2)+\left(d_2-\frac{\beta}{2}-r\right)s_2\right)\mathcal{F}_3 \\ -\left(k_2(n_2-s_2)+\left(d_2-\frac{\beta}{2}-r\right)s_1\right)\mathcal{F}_1 \end{bmatrix}}{y(\mathcal{F}_1\mathcal{F}_2-\mathcal{F}_3^2)} s_2 z\hat{\mathcal{G}}_{zz} - \frac{s_2\hat{\mathcal{G}}_{s_1 z}}{y} \tag{26}$$

From (20), differentiating (24)-(26) with respect to z, we have

$$\left\{\begin{matrix} \hbar_t + rs_1\hbar_{s_1} + rs_2\hbar_{s_2} - \left((r-\tau)\hbar + c\right) \\ -(r-\tau)z\,\hbar_z + \frac{1}{2}\mathcal{F}_1\hbar_{s_1s_1} + \frac{1}{2}\mathcal{F}_2\hbar_{s_2s_2} \\ +\mathcal{F}_3\hbar_{s_1s_2} + \frac{1}{2}(\mathcal{F}_4-\mathcal{F}_5-\mathcal{F}_6)z^2\hbar_{zz} \\ +\frac{1}{2}(\mathcal{F}_4-\mathcal{F}_5-\mathcal{F}_6)z\hbar_z \\ -\begin{pmatrix} k_1(n_1-s_1) \\ +\left(d_1-\frac{\beta}{2}-r\right)s_1 \end{pmatrix} z\hbar_{s_1z} \\ -\left(k_2(n_2-s_2)+\left(d_2-\frac{\beta}{2}-r\right)s_2\right)z\hbar_{s_2z} \end{matrix}\right\} = 0 \tag{27}$$

$$\varphi_1^* = \frac{\begin{bmatrix} \left(k_2(n_2-s_2)+\left(d_2-\frac{\beta}{2}-r\right)s_2\right)\mathcal{F}_3 \\ -\left(k_1(n_1-s_1)+\left(d_1-\frac{\beta}{2}-r\right)s_1\right)\mathcal{F}_2 \end{bmatrix}}{y(\mathcal{F}_1\mathcal{F}_2-\mathcal{F}_3^2)} s_1 z\,\hbar_z - \frac{s_1\hbar_{s_1}}{y} \tag{28}$$

$$\varphi_2^* = \frac{\begin{bmatrix} \left(k_2(n_2-s_2)+\left(d_2-\frac{\beta}{2}-r\right)s_2\right)\mathcal{F}_3 \\ -\left(k_2(n_2-s_2)+\left(d_2-\frac{\beta}{2}-r\right)s_1\right)\mathcal{F}_1 \end{bmatrix}}{y(\mathcal{F}_1\mathcal{F}_2-\mathcal{F}_3^2)} s_2 z\,\hbar_z - \frac{s_2\hbar_{s_2}}{y} \tag{29}$$

where, $\mathcal{G}(T, s_1, s_2, z) = U(z)$ and $U(z)$ is the marginal utility of the PM. Next, we proceed to solve (27) for $\hbar$ considering a PM with exponential utility, then substitute the solution into (28) and (29) for the optimal investment plan.

C. Optimal Investment Plan for a Member with Exponential Utility

We consider a member with a utility function displaying CARA. Since we are interested in finding the optimal portfolio strategies a DC member with CARA utility, we choose the exponential utility function.

The exponential utility function is given as

$$U(y) = -\frac{1}{m}e^{-my}, \qquad m > 0 \tag{30}$$

From (23),

$$\hbar(T, s_1, s_2, z) = (U')^{-1}(z) = -\frac{1}{m}lnz \tag{31}$$

Next, we conjecture a solution to (27) as:

$$\begin{cases} \hbar(t, s_1, s_2, z) = -\frac{1}{m}\left[y(t)\begin{pmatrix} lnz + a(t, s_1) \\ +b(t, s_2) \end{pmatrix} + w(t)\right] \\ y(T) = 1,\ a(T, s_1) = b(T, s_2) = w(T) = 0, \end{cases} \tag{32}$$

$$\left.\begin{matrix} \hbar_t = -\frac{1}{m}[y_t(t)\begin{pmatrix} lnz + a_t(t, s_1) \\ +b_t(t, s_2) \end{pmatrix}] + w_t, \\ \hbar_{s_1} = -\frac{y}{m}a_{s_1}, \hbar_{s_2} = -\frac{y}{m}b_{s_2}, \\ \hbar_{s_1s_1} = -\frac{y}{m}a_{s_1s_1}, \hbar_{s_2s_2} = -\frac{y}{m}b_{s_2s_2}, \hbar_{zz} = \frac{y}{mz^2}, \\ \hbar_z = -\frac{y}{mz}, \hbar_{s_1z} = \hbar_{s_2z} = \hbar_{s_1s_2} = 0 \end{matrix}\right\} \tag{33}$$

Substituting (33) into (27), we have

$$\left\{\begin{matrix} lnz[y_t - (r-\tau)y] \\ +m[w_t - (r-\tau)w - c] \\ +y\begin{bmatrix} a_t + b_t + rs_1a_{s_1} + rs_2g_{s_2} - \frac{1}{2}\mathcal{F}_2b_{s_2s_2} \\ +r - \tau - \frac{1}{2}(\mathcal{F}_4-\mathcal{F}_5-\mathcal{F}_6) - \frac{1}{2}\mathcal{F}_1a_{s_1s_1} \\ +(r-\tau)(a+b) - \frac{y_t}{y}(a+b) \end{bmatrix} \end{matrix}\right\} = 0 \tag{34}$$

Splitting (34) we have

$$\begin{cases} y_t - (r-\tau)y = 0 \\ y(T) = 1 \end{cases} \tag{35}$$

$$\begin{cases} w_t - (r-\tau)w - c = 0 \\ w(T) = 0 \end{cases} \tag{36}$$

$$\begin{cases} \begin{bmatrix} a_t + b_t + rs_1a_{s_1} + rs_2g_{s_2} + r - \tau \\ -\frac{1}{2}(\mathcal{F}_4-\mathcal{F}_5-\mathcal{F}_6) - \frac{1}{2}\mathcal{F}_1a_{s_1s_1} - \frac{1}{2}\mathcal{F}_2b_{s_2s_2} \end{bmatrix} = 0 \\ a(T, s_1) = b(T, s_2) = 0 \end{cases} \tag{37}$$

Solving (35) and (36), we have

$$y(t) = e^{(r-\tau)(t-T)} \tag{38}$$

$$w(t) = \frac{c}{\tau - r}\left[1 - e^{(r-\tau)(t-T)}\right] \tag{39}$$

Next, we propose a solution for (37):

$$\begin{cases} a(t, s_1) + b(t, s_2) = A(t) + s_1B(t) + s_2C(t) \\ A(T) = 1, B(T) = C(T) = 0 \end{cases} \tag{40}$$

$$\begin{cases} a_t + b_t = A_t + s_1B_t + s_2C_t\,, a_{s_1} = B, \\ b_{s_1} = C, a_{s_1s_1} = b_{s_2s_2} = 0 \end{cases} \tag{41}$$

Substituting (41) into (37), we have

$$\begin{cases} A_t + (r-\tau) - \frac{1}{2}(\mathcal{F}_4-\mathcal{F}_5-\mathcal{F}_6) \\ +s_1(B_t - rB) + s_2(C_t - rC) = 0 \end{cases} \tag{42}$$

Splitting (42), we have

$$B_t - rB = 0 \tag{43}$$

$$C_t - rC = 0 \tag{44}$$

$$A_t + (r-\tau) - \frac{1}{2}(\mathcal{F}_4-\mathcal{F}_5-\mathcal{F}_6) = 0 \tag{45}$$

Solving (43)-(45), we have

$$B(t) = 0 \tag{46}$$

$$C(t) = 0 \tag{47}$$

$$A(t) = \left(\frac{\mathcal{F}_4-\mathcal{F}_5-\mathcal{F}_6}{2} + \tau - r\right)(t-T) \tag{48}$$

Therefore, the solution of (37) is given as

$$a(t,s_1) + b(t,s_2) = \left(\frac{\mathcal{F}_4-\mathcal{F}_5-\mathcal{F}_6}{2} + \tau - r\right)(t-T) \tag{49}$$

Substituting (38), (39) and (49) into (32), we have

$$\begin{cases} h(t,s_1,s_2,z) \\ = -\frac{1}{m}\left[e^{(r-\tau)(t-T)}\left(lnz + \left(\frac{\mathcal{F}_4-\mathcal{F}_5-\mathcal{F}_6+\tau-r}{2}\right)(t-T)\right)\right] \\ +\frac{c}{\tau-r}\left[1-e^{(r-\tau)(t-T)}\right] \end{cases} \tag{50}$$

Result 1. The optimal portfolio strategies for a member with exponential utility function are given as

$$\varphi_1^* = \frac{s_1 e^{(r-\tau)(t-T)}\left(1+2k_1+r-\frac{\beta}{2}+d_1\right)(T-t)\begin{bmatrix}\begin{pmatrix}k_2(n_2-s_2)\\-\left(d_2-\frac{\beta}{2}+r\right)s_2\end{pmatrix}\mathcal{F}_3\\-\begin{pmatrix}k_1(n_1-s_1)\\-\left(d_1-\frac{\beta}{2}+r\right)s_1\end{pmatrix}\mathcal{F}_2\end{bmatrix}}{my(\mathcal{F}_1\mathcal{F}_2-\mathcal{F}_3^2)} \tag{51}$$

$$\varphi_2^* = \frac{s_2 e^{(r-\tau)(t-T)}\left(1+2k_2+r-\frac{\beta}{2}+d_2\right)(T-t)\begin{bmatrix}\begin{pmatrix}k_1(n_1-s_1)\\-\left(d_1-\frac{\beta}{2}-r\right)s_1\end{pmatrix}\mathcal{F}_3\\-\begin{pmatrix}k_2(n_2-s_2)\\-\left(d_2-\frac{\beta}{2}-r\right)s_2\end{pmatrix}\mathcal{F}_1\end{bmatrix}}{my(\mathcal{F}_1\mathcal{F}_2-\mathcal{F}_3^2)} \tag{52}$$

Proof. From (50), we have

$$h_{s_1} = \frac{-e^{(r-\tau)(t-T)}\begin{bmatrix}2\mathcal{F}_3(-r-k_1)\begin{pmatrix}k_2(n_2-s_2)\\+\left(d_2-\frac{\beta}{2}-r\right)s_2\end{pmatrix}\\-2\mathcal{F}_2(-r-k_1)\begin{pmatrix}k_1(n_1-s_1)\\+\left(d_1-\frac{\beta}{2}-r\right)s_1\end{pmatrix}\end{bmatrix}(t-T)}{m(\mathcal{F}_1\mathcal{F}_2-\mathcal{F}_3^2)}$$

$$h_{s_2} = \frac{-e^{(r-\tau)(t-T)}\begin{bmatrix}2\mathcal{F}_3(-r-k_2)\begin{pmatrix}k_1(n_1-s_1)\\+\left(d_1-\frac{\beta}{2}-r\right)s_1\end{pmatrix}-\\2\mathcal{F}_1(-r-k_1)\begin{pmatrix}k_2(n_2-s_2)\\+\left(d_2-\frac{\beta}{2}-r\right)s_2\end{pmatrix}\end{bmatrix}(T-t)}{m(\mathcal{F}_1\mathcal{F}_2-\mathcal{F}_3^2)} \tag{53}$$

$$h_z = -\frac{e^{(r-\tau)(t-T)}}{mz}$$

Substituting (53) into (28) and (29), result1 is proven.

IV. Theoretical Analysis

In this section, we present some results to demonstrate the impact of some parameters on the optimal investment plan.

Result 2. Suppose $m > 0, \beta > 0, r > 0, \tau > 0, t \in [0,T]$, $k_1 > 0, k_2 > 0, d_1 > 0, d_2 > 0, s_1 > 0, s_2 > 0, y > 0$, $(\mathcal{F}_1\mathcal{F}_2 - \mathcal{F}_3^2) > 0$ and $T - t > 0$ then

(i) $\frac{\partial\varphi_1^*}{\partial y} < 0$, (ii) $\frac{\partial\varphi_1^*}{\partial m} < 0$, (iii) $\frac{\partial\varphi_1^*}{\partial d_1} > 0$, (iv) $\frac{\partial\varphi_1^*}{\partial \tau} < 0$ (v) $\frac{\partial\varphi_1^*}{\partial \beta} < 0$

Proof.

(i) Recall that

$$\varphi_1^* = \frac{s_1 e^{(\tau-r)(t-T)}\left(1+2k_1+r-\frac{\beta}{2}+d_1\right)(T-t)\begin{bmatrix}\begin{pmatrix}k_2(n_2-s_2)\\-\left(d_2-\frac{\beta}{2}+r\right)s_2\end{pmatrix}\mathcal{F}_3\\-\begin{pmatrix}k_1(n_1-s_1)\\-\left(d_1-\frac{\beta}{2}+r\right)s_1\end{pmatrix}\mathcal{F}_2\end{bmatrix}}{my(\mathcal{F}_1\mathcal{F}_2-\mathcal{F}_3^2)}$$

$$\frac{\partial\varphi_1^*}{\partial y} = -\frac{s_1 e^{(\tau-r)(t-T)}\left(1+2k_1+r-\frac{\beta}{2}+d_1\right)(T-t)\begin{bmatrix}\begin{pmatrix}k_2(n_2-s_2)\\-\left(d_2-\frac{\beta}{2}+r\right)s_2\end{pmatrix}\mathcal{F}_3\\-\begin{pmatrix}k_1(n_1-s_1)\\-\left(d_1-\frac{\beta}{2}+r\right)s_1\end{pmatrix}\mathcal{F}_2\end{bmatrix}}{my^2(\mathcal{F}_1\mathcal{F}_2-\mathcal{F}_3^2)}$$

Since $e^{(\tau-r)(t-T)}\left(1+2k_1+r-\frac{\beta}{2}+d_1\right)(T-t) > 0$ and $\left[\left(k_2(n_2-s_2)-\left(d_2-\frac{\beta}{2}+r\right)s_2\right)\mathcal{F}_3 - \left(k_1(n_1-s_1) - \left(d_1-\frac{\beta}{2}+r\right)s_1\right)\mathcal{F}_2\right] > 0$ and $(\mathcal{F}_1\mathcal{F}_2-\mathcal{F}_3^2) > 0$, then

$$\frac{\partial\varphi_1^*}{\partial y} = -\frac{s_1 e^{(\tau-r)(t-T)}\begin{pmatrix}1+2k_1\\+r-\frac{\beta}{2}+d_1\end{pmatrix}(T-t)\begin{bmatrix}\begin{pmatrix}k_2(n_2-s_2)\\-\left(d_2-\frac{\beta}{2}+r\right)s_2\end{pmatrix}\mathcal{F}_3\\-\begin{pmatrix}k_1(n_1-s_1)\\-\left(d_1-\frac{\beta}{2}+r\right)s_1\end{pmatrix}\mathcal{F}_2\end{bmatrix}}{my^2(\mathcal{F}_1\mathcal{F}_2-\mathcal{F}_3^2)} < 0$$

Therefore $\frac{\partial\varphi_1^*}{\partial y} < 0$

(ii) $$\frac{\partial\varphi_1^*}{\partial m} = -\frac{s_1 e^{(\tau-r)(t-T)}\left(1+2k_1+r-\frac{\beta}{2}+d_1\right)(T-t)\begin{bmatrix}\begin{pmatrix}k_2(n_2-s_2)\\-\left(d_2-\frac{\beta}{2}+r\right)s_2\end{pmatrix}\mathcal{F}_3\\-\begin{pmatrix}k_1(n_1-s_1)\\-\left(d_1-\frac{\beta}{2}+r\right)s_1\end{pmatrix}\mathcal{F}_2\end{bmatrix}}{ym^2(\mathcal{F}_1\mathcal{F}_2-\mathcal{F}_3^2)}$$

Since $e^{(\tau-r)(t-T)}\left(1+2k_1+r-\frac{\beta}{2}+d_1\right)(T-t) > 0$ and $\begin{bmatrix}\left(k_2(n_2-s_2)-\left(d_2-\frac{\beta}{2}+r\right)s_2\right)\mathcal{F}_3\\-\left(k_1(n_1-s_1)-\left(d_1-\frac{\beta}{2}+r\right)s_1\right)\mathcal{F}_2\end{bmatrix} > 0$ and from $(\mathcal{F}_1\mathcal{F}_2 - \mathcal{F}_3^2) > 0$, then

$$\frac{\partial\varphi_1^*}{\partial m} = -\frac{s_1 e^{(\tau-r)(t-T)}\left(1+2k_1+r-\frac{\beta}{2}+d_1\right)(T-t)\begin{bmatrix}\begin{pmatrix}k_2(n_2-s_2)\\-\left(d_2-\frac{\beta}{2}+r\right)s_2\end{pmatrix}\mathcal{F}_3\\-\begin{pmatrix}k_1(n_1-s_1)\\-\left(d_1-\frac{\beta}{2}+r\right)s_1\end{pmatrix}\mathcal{F}_2\end{bmatrix}}{ym^2(\mathcal{F}_1\mathcal{F}_2-\mathcal{F}_3^2)} < 0$$

Therefore $\frac{\partial\varphi_1^*}{\partial m} < 0$

$$\varphi_1^* = \frac{s_1 e^{(\tau-r)(t-T)}\left(1+2k_1+r-\frac{\beta}{2}+d_1\right)(T-t)\begin{bmatrix}\begin{pmatrix}k_2(n_2-s_2)\\-\left(d_2-\frac{\beta}{2}+r\right)s_2\end{pmatrix}\mathcal{F}_3\\-\begin{pmatrix}k_1(n_1-s_1)\\-\left(d_1-\frac{\beta}{2}+r\right)s_1\end{pmatrix}\mathcal{F}_2\end{bmatrix}}{my(\mathcal{F}_1\mathcal{F}_2-\mathcal{F}_3^2)}$$

(iii) $$\frac{\partial\varphi_1^*}{\partial d_1} = \left(\frac{s_1 e^{(\tau-r)(t-T)}(T-t)\begin{bmatrix}\left(1+2k_1+r-\frac{\beta}{2}+d_1\right)s_1\mathcal{F}_2\\+\begin{bmatrix}\left(k_2(n_2-s_2)-\left(d_2-\frac{\beta}{2}+r\right)s_2\right)\mathcal{F}_3\\-\left(k_1(n_1-s_1)-\left(d_1-\frac{\beta}{2}+r\right)s_1\right)\mathcal{F}_2\end{bmatrix}\end{bmatrix}}{my(\mathcal{F}_1\mathcal{F}_2-\mathcal{F}_3^2)}\right)$$

Since $\left(1+2k_1+r-\frac{\beta}{2}+d_1\right) > 0$

, $\begin{bmatrix}\left(k_2(n_2-s_2)-\left(d_2-\frac{\beta}{2}+r\right)s_2\right)\mathcal{F}_3\\-\left(k_1(n_1-s_1)-\left(d_1-\frac{\beta}{2}+r\right)s_1\right)\mathcal{F}_2\end{bmatrix}>0$ and $(\mathcal{F}_1\mathcal{F}_2-\mathcal{F}_3^2)>0$; therefore $\frac{\partial\varphi_1^*}{\partial d_1}>0$.

(iv) $$\frac{\partial\varphi_1^*}{\partial\tau}=\frac{(t-T)s_1e^{(\tau-r)(t-T)}\left(1+2k_1+r-\frac{\beta}{2}+d_1\right)(T-t)\begin{bmatrix}\begin{pmatrix}k_2(n_2-s_2)\\-\left(d_2-\frac{\beta}{2}+r\right)s_2\end{pmatrix}\mathcal{F}_3\\-\begin{pmatrix}k_1(n_1-s_1)\\-\left(d_1-\frac{\beta}{2}+r\right)s_1\end{pmatrix}\mathcal{F}_2\end{bmatrix}}{ym(\mathcal{F}_1\mathcal{F}_2-\mathcal{F}_3^2)}$$

$$=-\frac{(T-t)s_1e^{(\tau-r)(t-T)}\left(1+2k_1+r-\frac{\beta}{2}+d_1\right)(T-t)\begin{bmatrix}\begin{pmatrix}k_2(n_2-s_2)\\-\left(d_2-\frac{\beta}{2}+r\right)s_2\end{pmatrix}\mathcal{F}_3\\-\begin{pmatrix}k_1(n_1-s_1)\\-\left(d_1-\frac{\beta}{2}+r\right)s_1\end{pmatrix}\mathcal{F}_2\end{bmatrix}}{ym(\mathcal{F}_1\mathcal{F}_2-\mathcal{F}_3^2)}$$

Since $e^{(\tau-r)(t-T)}\left(1+2k_1+r-\frac{\beta}{2}+d_1\right)(T-t)>0$ and $\begin{bmatrix}\left(k_2(n_2-s_2)-\left(d_2-\frac{\beta}{2}+r\right)s_2\right)\mathcal{F}_3-\\\left(k_1(n_1-s_1)-\left(d_1-\frac{\beta}{2}+r\right)s_1\right)\mathcal{F}_2\end{bmatrix}>0$ and from $(\mathcal{F}_1\mathcal{F}_2-\mathcal{F}_3^2)>0$, then

$$\frac{\partial\varphi_1^*}{\partial\tau}=-\frac{(T-t)s_1e^{(\tau-r)(t-T)}\left(1+2k_1+r-\frac{\beta}{2}+d_1\right)(T-t)\begin{bmatrix}\begin{pmatrix}k_2(n_2-s_2)\\-\left(d_2-\frac{\beta}{2}+r\right)s_2\end{pmatrix}\mathcal{F}_3\\-\begin{pmatrix}k_1(n_1-s_1)\\-\left(d_1-\frac{\beta}{2}+r\right)s_1\end{pmatrix}\mathcal{F}_2\end{bmatrix}}{ym(\mathcal{F}_1\mathcal{F}_2-\mathcal{F}_3^2)}<0$$

Therefore $\frac{\partial\varphi_1^*}{\partial\tau}<0$

(v) $$\frac{\partial\varphi_1^*}{\partial\beta}=\left(\begin{matrix}s_1e^{(\tau-r)(t-T)}\left(1+2k_1+r-\frac{\beta}{2}+d_1\right)(T-t)\left[\frac{\mathcal{F}_3-\mathcal{F}_2}{2}\right]\\\dfrac{-(T-t)s_1e^{(\tau-r)(t-T)}\begin{bmatrix}\left(k_2(n_2-s_2)-\left(d_2-\frac{\beta}{2}+r\right)s_2\right)\mathcal{F}_3\\-\left(k_1(n_1-s_1)-\left(d_1-\frac{\beta}{2}+r\right)s_1\right)\mathcal{F}_2\end{bmatrix}}{ym(\mathcal{F}_1\mathcal{F}_2-\mathcal{F}_3^2)}\end{matrix}\right)$$

$$\frac{\partial\varphi_1^*}{\partial\beta}=\left(\begin{matrix}-s_1e^{(\tau-r)(t-T)}\left(1+2k_1+r-\frac{\beta}{2}+d_1\right)(T-t)\left[\frac{\mathcal{F}_2-\mathcal{F}_3}{2}\right]\\\dfrac{-(T-t)s_1e^{(\tau-r)(t-T)}\begin{bmatrix}\left(k_2(n_2-s_2)-\left(d_2-\frac{\beta}{2}+r\right)s_2\right)\mathcal{F}_3\\-\left(k_1(n_1-s_1)-\left(d_1-\frac{\beta}{2}+r\right)s_1\right)\mathcal{F}_2\end{bmatrix}}{ym(\mathcal{F}_1\mathcal{F}_2-\mathcal{F}_3^2)}\end{matrix}\right)$$

$$\frac{\partial\varphi_1^*}{\partial\beta}=-\left(\begin{matrix}s_1e^{(\tau-r)(t-T)}\left(1+2k_1+r-\frac{\beta}{2}+d_1\right)(T-t)\left[\frac{\mathcal{F}_2-\mathcal{F}_3}{2}\right]\\+\dfrac{(T-t)s_1e^{(\tau-r)(t-T)}\begin{bmatrix}\left(k_2(n_2-s_2)-\left(d_2-\frac{\beta}{2}+r\right)s_2\right)\mathcal{F}_3\\-\left(k_1(n_1-s_1)-\left(d_1-\frac{\beta}{2}+r\right)s_1\right)\mathcal{F}_2\end{bmatrix}}{ym(\mathcal{F}_1\mathcal{F}_2-\mathcal{F}_3^2)}\end{matrix}\right)$$

Since $e^{(\tau-r)(t-T)}\left(1+2k_1+r-\frac{\beta}{2}+d_1\right)(T-t)>0$, $\mathcal{F}_2-\mathcal{F}_3>0$ and $\left[\left(k_2(n_2-s_2)-\left(d_2-\frac{\beta}{2}+r\right)s_2\right)\mathcal{F}_3-\left(k_1(n_1-s_1)-\left(d_1-\frac{\beta}{2}+r\right)s_1\right)\mathcal{F}_2\right]>0$ and from the lemma $(\mathcal{F}_1\mathcal{F}_2-\mathcal{F}_3^2)>0$; therefore $\frac{\partial\varphi_1^*}{\partial\beta}<0$.

Result 3. Suppose $m>0,\beta>0,r>0,\tau>0,t\in[0,T],k_1>0,k_2>0,d_1>0,d_2>0,s_1>0,s_2>0,y>0,(\mathcal{F}_1\mathcal{F}_2-\mathcal{F}_3^2)>0$ and $T-t>0$ then

(i) $\frac{\partial\varphi_2^*}{\partial y}<0$, (ii) $\frac{\partial\varphi_2^*}{\partial m}<0$, (iii) $\frac{\partial\varphi_2^*}{\partial d_2}>0$, (iv) $\frac{\partial\varphi_2^*}{\partial\tau}<0$, (v) $\frac{\partial\varphi_2^*}{\partial\beta}<0$

Proof.

(i) Recall that

$$\varphi_2^*=\frac{s_2e^{(\tau-r)(t-T)}\left(1+2k_2+r-\frac{\beta}{2}+d_2\right)(T-t)\begin{bmatrix}\begin{pmatrix}k_1(n_1-s_1)\\-\left(d_1-\frac{\beta}{2}-r\right)s_1\end{pmatrix}\mathcal{F}_3\\-\begin{pmatrix}k_2(n_2-s_2)\\-\left(d_2-\frac{\beta}{2}-r\right)s_2\end{pmatrix}\mathcal{F}_1\end{bmatrix}}{my(\mathcal{F}_1\mathcal{F}_2-\mathcal{F}_3^2)}$$

$$\frac{\partial\varphi_2^*}{\partial y}=-\frac{s_2e^{(\tau-r)(t-T)}\left(1+2k_2+r-\frac{\beta}{2}+d_2\right)(T-t)\begin{bmatrix}\begin{pmatrix}k_1(n_1-s_1)\\-\left(d_1-\frac{\beta}{2}-r\right)s_1\end{pmatrix}\mathcal{F}_3\\-\begin{pmatrix}k_2(n_2-s_2)\\-\left(d_2-\frac{\beta}{2}-r\right)s_2\end{pmatrix}\mathcal{F}_1\end{bmatrix}}{my^2(\mathcal{F}_1\mathcal{F}_2-\mathcal{F}_3^2)}$$

Since $e^{(\tau-r)(t-T)}\left(1+2k_2+r-\frac{\beta}{2}+d_2\right)(T-t)>0$ and $\begin{bmatrix}\left(k_1(n_1-s_1)-\left(d_1-\frac{\beta}{2}-r\right)s_1\right)\mathcal{F}_3\\-\left(k_2(n_2-s_2)-\left(d_2-\frac{\beta}{2}-r\right)s_2\right)\mathcal{F}_1\end{bmatrix}>0$ and from the lemma $(\mathcal{F}_1\mathcal{F}_2-\mathcal{F}_3^2)>0$, then

$$\frac{\partial\varphi_2^*}{\partial y}=-\frac{s_2e^{(\tau-r)(t-T)}\begin{pmatrix}1+2k_2+r\\-\frac{\beta}{2}+d_2\end{pmatrix}(T-t)\begin{bmatrix}\begin{pmatrix}k_1(n_1-s_1)\\-\left(d_1-\frac{\beta}{2}-r\right)s_1\end{pmatrix}\mathcal{F}_3\\-\begin{pmatrix}k_2(n_2-s_2)\\-\left(d_2-\frac{\beta}{2}-r\right)s_2\end{pmatrix}\mathcal{F}_1\end{bmatrix}}{my^2(\mathcal{F}_1\mathcal{F}_2-\mathcal{F}_3^2)}<0$$

Therefore $\frac{\partial\varphi_2^*}{\partial y}<0$

(ii) $$\frac{\partial\varphi_2^*}{\partial m}=-\frac{s_2e^{(\tau-r)(t-T)}\left(1+2k_2+r-\frac{\beta}{2}+d_2\right)(T-t)\begin{bmatrix}\begin{pmatrix}k_1(n_1-s_1)\\-\left(d_1-\frac{\beta}{2}-r\right)s_1\end{pmatrix}\mathcal{F}_3\\-\begin{pmatrix}k_2(n_2-s_2)\\-\left(d_2-\frac{\beta}{2}-r\right)s_2\end{pmatrix}\mathcal{F}_1\end{bmatrix}}{ym^2(\mathcal{F}_1\mathcal{F}_2-\mathcal{F}_3^2)}$$

Since $e^{(\tau-r)(t-T)}\left(1+2k_2+r-\frac{\beta}{2}+d_2\right)(T-t)>0$ and $\begin{bmatrix}\left(k_1(n_1-s_1)-\left(d_1-\frac{\beta}{2}-r\right)s_1\right)\mathcal{F}_3\\-\left(k_2(n_2-s_2)-\left(d_2-\frac{\beta}{2}-r\right)s_2\right)\mathcal{F}_1\end{bmatrix}>0$ and from the lemma $(\mathcal{F}_1\mathcal{F}_2-\mathcal{F}_3^2)>0$, then

$$\frac{\partial\varphi_2^*}{\partial m}=-\frac{s_2e^{(\tau-r)(t-T)}\begin{pmatrix}1+2k_2+r\\-\frac{\beta}{2}+d_2\end{pmatrix}(T-t)\begin{bmatrix}\begin{pmatrix}k_1(n_1-s_1)\\-\left(d_1-\frac{\beta}{2}-r\right)s_1\end{pmatrix}\mathcal{F}_3\\-\begin{pmatrix}k_2(n_2-s_2)\\-\left(d_2-\frac{\beta}{2}-r\right)s_2\end{pmatrix}\mathcal{F}_1\end{bmatrix}}{ym^2(\mathcal{F}_1\mathcal{F}_2-\mathcal{F}_3^2)}<0$$

Therefore $\frac{\partial\varphi_2^*}{\partial m}<0$

(iii) $$\frac{\partial\varphi_2^*}{\partial d_2}=\left(\dfrac{s_2e^{(\tau-r)(t-T)}(T-t)\begin{bmatrix}\left(1+2k_2+r-\frac{\beta}{2}+d_2\right)s_2\mathcal{F}_1\\+\begin{bmatrix}\left(k_1(n_1-s_1)-\left(d_1-\frac{\beta}{2}-r\right)s_1\right)\mathcal{F}_3\\-\left(k_2(n_2-s_2)-\left(d_2-\frac{\beta}{2}-r\right)s_2\right)\mathcal{F}_1\end{bmatrix}\end{bmatrix}}{my(\mathcal{F}_1\mathcal{F}_2-\mathcal{F}_3^2)}\right)$$

Since $\left(1+2k_2+r-\frac{\beta}{2}+d_2\right)>0$, $\begin{bmatrix}\left(k_1(n_1-s_1)-\left(d_1-\frac{\beta}{2}-r\right)s_1\right)\mathcal{F}_3\\-\left(k_2(n_2-s_2)-\left(d_2-\frac{\beta}{2}-r\right)s_2\right)\mathcal{F}_1\end{bmatrix}>0$ and $(\mathcal{F}_1\mathcal{F}_2-\mathcal{F}_3^2)>$

0; therefore $\frac{\partial \varphi_2^*}{\partial d_2} > 0$

(iv) $$\frac{\partial \varphi_2^*}{\partial \tau} = \frac{(t-T)s_2 e^{(\tau-r)(t-T)}\left(1+2k_2+r-\frac{\beta}{2}+d_2\right)(T-t)\begin{bmatrix} \begin{pmatrix} k_1(n_1-s_1) \\ -\left(d_1-\frac{\beta}{2}-r\right)s_1 \end{pmatrix}\mathcal{F}_3 \\ -\begin{pmatrix} k_2(n_2-s_2) \\ -\left(d_2-\frac{\beta}{2}-r\right)s_2 \end{pmatrix}\mathcal{F}_1 \end{bmatrix}}{ym(\mathcal{F}_1\mathcal{F}_2-\mathcal{F}_3^2)}$$

$$= -\frac{(T-t)s_2 e^{(\tau-r)(t-T)}\left(1+2k_2+r-\frac{\beta}{2}+d_2\right)(T-t)\begin{bmatrix} \begin{pmatrix} k_1(n_1-s_1) \\ -\left(d_1-\frac{\beta}{2}-r\right)s_1 \end{pmatrix}\mathcal{F}_3 \\ -\begin{pmatrix} k_2(n_2-s_2) \\ -\left(d_2-\frac{\beta}{2}-r\right)s_2 \end{pmatrix}\mathcal{F}_1 \end{bmatrix}}{ym\left(\mathcal{F}_1\mathcal{F}_2-\mathcal{F}_3^2\right)}$$

Since$e^{(\tau-r)(t-T)}\left(1+2k_2+r-\frac{\beta}{2}+d_2\right)(T-t) > 0$ and $\begin{bmatrix} \left(k_1(n_1-s_1)-\left(d_1-\frac{\beta}{2}-r\right)s_1\right)\mathcal{F}_3 \\ -\left(k_2(n_2-s_2)-\left(d_2-\frac{\beta}{2}-r\right)s_2\right)\mathcal{F}_1 \end{bmatrix} > 0$ and from the lemma $(\mathcal{F}_1\mathcal{F}_2-\mathcal{F}_3^2) > 0$, then

$$\frac{\partial \varphi_1^*}{\partial \tau} = -\left(\frac{(T-t)s_2 e^{(\tau-r)(t-T)}\begin{pmatrix} 1+2k_2+r \\ -\frac{\beta}{2}+d_2 \end{pmatrix}(T-t)\begin{bmatrix} \begin{pmatrix} k_1(n_1-s_1) \\ -\left(d_1-\frac{\beta}{2}-r\right)s_1 \end{pmatrix}\mathcal{F}_3 \\ -\begin{pmatrix} k_2(n_2-s_2) \\ -\left(d_2-\frac{\beta}{2}-r\right)s_2 \end{pmatrix}\mathcal{F}_1 \end{bmatrix}}{ym(\mathcal{F}_1\mathcal{F}_2-\mathcal{F}_3^2)}\right) < 0$$

Therefore $\frac{\partial \varphi_1^*}{\partial \tau} < 0$

(v) $$\frac{\partial \varphi_2^*}{\partial \beta} = \left(\frac{\begin{matrix} s_2 e^{(\tau-r)(t-T)}\left(1+2k_2+r-\frac{\beta}{2}+d_2\right)(T-t)\left[\frac{\mathcal{F}_3-\mathcal{F}_1}{2}\right] \\ -(T-t)s_2 e^{(\tau-r)(t-T)}\begin{bmatrix} \left(k_1(n_1-s_1)-\left(d_1-\frac{\beta}{2}-r\right)s_1\right)\mathcal{F}_3 \\ -\left(k_2(n_2-s_2)-\left(d_2-\frac{\beta}{2}-r\right)s_2\right)\mathcal{F}_1 \end{bmatrix} \end{matrix}}{ym(\mathcal{F}_1\mathcal{F}_2-\mathcal{F}_3^2)}\right)$$

$$\frac{\partial \varphi_2^*}{\partial \beta} = \left(\frac{\begin{matrix} -s_2 e^{(\tau-r)(t-T)}\left(1+2k_2+r-\frac{\beta}{2}+d_2\right)(T-t)\left[\frac{\mathcal{F}_1-\mathcal{F}_3}{2}\right] \\ -(T-t)s_1 e^{(\tau-r)(t-T)}\begin{bmatrix} \left(k_1(n_1-s_1)-\left(d_1-\frac{\beta}{2}-r\right)s_1\right)\mathcal{F}_3 \\ -\left(k_2(n_2-s_2)-\left(d_2-\frac{\beta}{2}-r\right)s_2\right)\mathcal{F}_1 \end{bmatrix} \end{matrix}}{ym\left(\mathcal{F}_1\mathcal{F}_2-\mathcal{F}_3^2\right)}\right)$$

$$\frac{\partial \varphi_2^*}{\partial \beta} = -\left(\begin{matrix} s_1 e^{(\tau-r)(t-T)}\left(1+2k_1+r-\frac{\beta}{2}+d_1\right)(T-t)\left[\frac{\mathcal{F}_2-\mathcal{F}_3}{2}\right] \\ +\frac{(T-t)s_1 e^{(\tau-r)(t-T)}\begin{bmatrix} \left(k_1(n_1-s_1)-\left(d_1-\frac{\beta}{2}-r\right)s_1\right)\mathcal{F}_3 \\ -\left(k_2(n_2-s_2)-\left(d_2-\frac{\beta}{2}-r\right)s_2\right)\mathcal{F}_1 \end{bmatrix}}{ym\left(\mathcal{F}_1\mathcal{F}_2-\mathcal{F}_3^2\right)} \end{matrix}\right)$$

Since$e^{(\tau-r)(t-T)}\left(1+2k_1+r-\frac{\beta}{2}+d_1\right)(T-t) > 0$, $\mathcal{F}_1-\mathcal{F}_3 > 0$ and $\left[\left(k_1(n_1-s_1)-\left(d_1-\frac{\beta}{2}-r\right)s_1\right)\mathcal{F}_3-\left(k_2(n_2-s_2)-\left(d_2-\frac{\beta}{2}-r\right)s_2\right)\mathcal{F}_1\right] > 0$ and from the lemma $(\mathcal{F}_1\mathcal{F}_2-\mathcal{F}_3^2) > 0$; therefore $\frac{\partial \varphi_2^*}{\partial \beta} < 0$.

V. Discussion

In this section, the influences of some parameters on the optimal investment strategies are investigated. In Results 2 and 3, we observed that the optimal investment strategies for the two risky assets decrease as the initial fund size $y$of the PM increases; this is because the PM with less fund at the early stage of the investment may wish to increase his or her accumulated funds before retirement by investing more in the risky assets which have the tendency of yielding high dividend in shorter time compared to the interest made from investment in the risk-free asset within such time frame. On the contrary, a PM with large fund at the beginning of the investment may not be too desperate but may decide to invest more in the risk-free asset and balance it with some fraction of the risky assets to avoid risk that is generated by the O-U process as a result of the changes in the stock market price at different time intervals. Also, we observed that optimal investment strategies of the two risky assets decrease with an increase in the risk aversion coefficient m of the PM; the implication here is that a PM with high risk aversion coefficient is scared of taking risk, hence will prefer to invest less in risky assets and possibly more in the risk-free asset and vice versa. We also observed that the fraction invested in the risky assets increases with an increase in dividend; this is reliably so since more dividend makes such investment attractive and lucrative. Most interestingly, the optimal investment strategies are decreasing functions of taxes imposed on the investment in the risky assets; this is because high tax rate discourages investment hence the PM may be discouraged in investing in any asset attracting high rate of taxation and may switch to less or non-taxable assets.

Finally, we observed that the transaction cost also decreases the fraction of the PM fund invested in the risky assets. This so because most pension administrators charge for portfolio management and since investment in stock is highly volatile, the transaction cost may be high and, in some cases, discouraging for the PM member to accept hence push the PM to invest more in the risk-free asset.

VI. Conclusion

In conclusion, the paper investigated the strategic portfolio management for a PM in a DC pension plan with couple risky assets (stocks), transaction cost and taxes on the invested fund under the O-U process. A portfolio consisting of a risk-free asset and two risky assets was considered where the two risky assets were modelled by the O-U process. We used the Legendre transform and dual theory technique to transform the HJB equation into a linear PDE which is then solved using change of variable technique under exponential utility function for the optimal investment strategies. In addition, we used a theoretical analysis to examine the effects of some sensitive parameters on the optimal investment strategies where we observed that as the dividend from the risky assets increase, the optimal investment strategies increase and vice versa. Also, as the tax on the invested funds, risk averse coefficient, initial fund size and the transaction cost increasing functions of the optimal investment strategies.

REFERENCES

[1] J. Xiao, Z. Hong, C. Qin. (2007). The constant elasticity of variance (CEV) model and the Legendre transform-dual solution for annuity contracts, Insurance, 40(2), 302–310.

[2] J. Gao (2009). Optimal portfolios for DC pension plan under a CEV model. Insurance Mathematics and Economics 44(3): 479-490.

[3] B. O. Osu, K. N. C. Njoku, B. I. Oruh. (2020). On the Investment Strategy, Effect of Inflation and Impact of Hedging on Pension Wealth during Accumulation and Distribution Phases. *Journal of the Nigerian Society of Physical Sciences*, *2*(3), 170-179. https://doi.org/10.46481/jnsps.2020.62

[4] D. Li, X. Rong, H. Zhao, (2013). Optimal investment problem with taxes, dividends and transaction costs under the constant elasticity of variance model. Transaction on Mathematics, 12(3), 243-255.

[5] M. Gu, Y. Yang, S. Li, J. Zhang, (2010). Constant elasticity of variance model for proportional reinsurance and investment strategies, Insurance: Mathematics and Economics. 46(3), 580-587

[6] D. Li, X. Rong, H. Zhao, B. Yi. Equilibrium investment strategy for DC pension plan with default risk and return of premiums clauses under CEV model, Insurance 72(2017), 6-20.

[7] B. O.Osu, E. E. Akpanibah, C. Olunkwa (2018). Mean-Variance Optimization of portfolios with return of premium clauses in a DC pension plan with multiple contributors under constant elasticity of variance model, Int. J. Math. Comput. Sci., 12, 85–90.

[8] J. F. Boulier, S. Huang, G. Taillard G. Optimal management under stochastic interest rates: the case of a protected defined contribution pension fund, Insurance 28(2) (2001), 173–189.

[9] G. Deelstra, M. Grasselli and. P. F. Koehl (2003). Optimal investment strategies in the presence of a minimum guarantee. *Insurance*, 33(1): 189–207.

[10] J. Gao (2008) "Stochastic optimal control of DC pension funds, *Insurance*, vol. 42(3), 1159–1164.

[11] P. Battocchio, F. Menoncin. Optimal pension management in a stochastic framework, Insurance34(1) (2004) 79–95.

[12] A. J. G. Cairns, D. Blake, K. Dowd. Stochastic life styling: optimal dynamic asset allocation for defined contribution pension plans, Journal of Economic Dynamics &Control 30(5) (2006) 843–877.

[13] C. Zhang, X Rong. (2013). Optimal investment strategies for DC pension with a stochastic salary under affine interest rate model. Hindawi Publishing Corporation, vol 2013 http://dx.doi.org/10.1155/2013/297875,

[14] E. E. Akpanibah, B. O. Osu, K.N. C. Njoku, E. O. Akak. (2017) Optimization of Wealth Investment Strategies for a DC Pension Fund with Stochastic Salary and Extra Contributions. International Journal of Partial Diff. Equations and Application, 5(1): 33-41.

[15] E. E. Akpanibah, U. O. Ini (2020). Portfolio Strategy for an Investor with Logarithm Utility and Stochastic Interest Rate under Constant Elasticity of Variance Model. *Journal of the Nigerian Society of Physical Sciences*, *2*(3), 186-196.

[16] S. A. Ihedioha, N. T. Danat, A. Buba. (2020). Investor's Optimal Strategy with and Without Transaction Cost Under Ornstein-Uhlenbeck and Constant Elasticity of Variance (CEV) Models via Exponential Utility Maximization. Pure and Applied MathematicsJournal,9(3): 55-63.

[17] X. Xiao, K, Yonggui Kao. (2020). The optimal investment strategy of a DC pension plan under deposit loan spread and the O-U process. Preprint submitted to Elsevier.

[18] Jose, Luis, Menaldi. (2006). Controlled Markov processes and viscosity solutions 25.

[19] H, Zhao, X, Rong. (2012). Portfolio selection problem with multiple risky assets under the constant elasticity of variance model, Mathematics and Economics, 50, 179-190.

[20] M. Jonsson, R. Sircar (2002). Optimal investment problems and volatility homogenization approximations. in *Modern Methods in Scientific Computing and Applications*, 75(2): 255–281.