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Algebras over an Integral Domain and Immediate Neighbors

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Abstract—Let S be an integral domain with field of fractions F and let A be an F-algebra. An S-subalgebra R of A is called S-nice if $R \cap F = S$ and the localization of R with respect to $S \setminus \{0\}$ is A. Denoting by $\mathbb{W}$ the set of all S-nice subalgebras of A, and defining a notion of open sets on $\mathbb{W}$, one can view $\mathbb{W}$ as a T_0-Alexandroff space. A characterization of the property of immediate neighbors in an Alexandroff topological space is given, in terms of closed and open subsets of appropriate subspaces. Moreover, two special subspaces of $\mathbb{W}$ are introduced, and a way in which their closed and open subsets induce $\mathbb{W}$ is presented.

Keywords—Algebras over integral domains, Alexandroff topology, immediate neighbors, integral domains.

I. Introduction

TOPOLOGICAL spaces of rings or ideals have been studied for almost a century now. The study of such spaces and the reciprocity between their topological properties and their algebraic properties is a fruitful and important topic. The pioneer of this research was Stone, who studied in 1936 topological spaces of prime ideals in the context of distributive lattices and Boolean algebras [16], [17].

Let K be a field and let D be a subring of K. Let Zar(K/D) denote the set of all valuation domains having quotient field K and containing D. In 1944, Zariski [18] defined a topology on Zar(K/D); the basic open sets of this space are the sets of the form $V(M)$, where $V(M)$ denotes the elements of Zar(K/D) containing a finite subset M of K. Zar(K/D) is called the Zariski-Riemann space. Zariski proved a general result implying that this space is quasi-compact.

In 1969 Hochster [6] studied the notion of a spectral space using purely topological properties. Recall that a topological space T is called a spectral space if the following conditions are satisfied:

(a) T is quasi-compact and T_0.
(b) The set of open and quasi-compact subsets of T is closed under finite intersections, and is a basis of the topology.
(c) T is sober.

By Hochster's characterization, these spaces are precisely the topological spaces which arise as the prime spectrum of a commutative unitary ring, equipped with the Zariski topology - the topology whose closed sets are the sets of the form

$$V(I) = \{P \in \text{Spec}(R) \mid I \subseteq P\},$$

where I is an ideal of R.

In 1986 Dobbs and Fontana explicitly provided a Bézout domain whose prime spectrum is canonically homeomorphic to Zar(K/D) in the special case where K is the quotient field of D (see [2, Theorem 2]). Other proofs, using a variety of different techniques, can be found in [3], [5] and [8]. In 2013 Finocchiaro, Fontana, and Loper [4] extended this result and proved that for any subring D of K, the space Zar(K/D) is spectral with respect to the constructible (and to the Zariski) topology, by explicitly constructing a ring whose prime spectrum is canonically homeomorphic to Zar(K/D).

We note that in this paper the symbol $\subset$ means proper inclusion and the symbol $\subseteq$ means inclusion or equality. Let R be a commutative ring with unity and let Spec(R) denote the prime spectrum of R. Two basic properties concerning Spec(R) were presented by Irving Kapalansky in his book "Commutative Rings". The first property, denoted by (K1), is that every nonempty totally ordered subset of Spec(R) has an infimum and a supremum (cf. [7, Theorem 9]). The second property, denoted by (K2), is that given two prime ideals $P_1 \subset P_2$, there exist prime ideals

$$P_1 \subseteq P_3 \subset P_4 \subseteq P_2$$

such that P_3 and P_4 are immediate neighbors; i.e., there exists no prime ideal between P_3 and P_4 (cf. [7, Theorem 11]). Kaplansky conjectured that every partially ordered set satisfying these two properties is isomorphic to the prime spectrum of some commutative ring. However, in 1969 Hochster showed that the conjecture was false. As mentioned above, Hochster proved that the spectral spaces (that were defined using purely topological properties) are precisely the topological spaces which arise as the prime spectrum of a commutative ring (cf. [6, Theorem 6 and Proposition 10]). In 1972, Speed (cf. [15, Corollary 1]) noted that the following characterization of partially ordered sets appearing as the prime spectra of commutative rings can be deduced from Hochster's result: a partially ordered set P is isomorphic to the prime spectrum of some commutative ring if and only if P is an inverse limit of finite partially ordered sets in the category of partially ordered sets.

In this paper, we discuss topological and algebraic aspects of algebras over an integral domain, with emphasis on immediate neighbors.

Recall that a topological space whose set of open sets is closed under arbitrary intersections is called an Alexandroff space, after P. Alexandroff who first introduced such topological spaces in his paper [1] from 1937. Equivalently, A topological space is called an Alexandroff space if every element has a minimal open set containing it.

Let (T, τ) be a topological space. For $X \subseteq T$ we denote by clX the closure of X. It is well known, that if one defines $x \leq_\tau y$ whenever $x \in cl\{y\}$, then $\leq_\tau$ is a quasi-order (also called a preorder); i.e., a reflexive and transitive relation. $\leq_\tau$

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is called the specialization order. Recall that (T, τ) is called T_0 if for every two distinct elements in T, there exists an open set containing one of them but not the other. It is known that (T, τ) is T_0 if and only if $\leq_\tau$ is a partial order; i.e., a preorder which is also antisymmetric.

We discuss now some of the common definitions we use from order theory.

Let P be a set with a preorder $\leq$. Let $M \subseteq P$. We say that $a \in M$ is minimal (resp. maximal) in M if for all $x \in M$, $x \leq a$ (resp. $x \geq a$) implies $x = a$. We say that $a \in M$ is a smallest (resp. greatest) element in M if for all $x \in M$ we have $a \leq x$ (resp. $a \geq x$). As mentioned above, if the relation $\leq$ is also antisymmetric then it is called a partial order and P is called a partially ordered set, or a poset. Let $L \subseteq P$. We say that $a \in P$ is a lower (resp. upper) bound of L if $a \leq x$ (resp. $a \geq x$) for all $x \in L$. If the set of lower (resp. upper) bounds of L has a unique greatest (resp. smallest) element, this element is called the greatest lower (resp. least upper) bound of L, and is denoted by inf$L$ (resp. sup$L$). We say that L is a lower set if

$$L = \{y \in P \mid y \leq x \text{ for some } x \in L\}.$$

We say that L is an upper set if

$$L = \{y \in P \mid y \geq x \text{ for some } x \in L\}.$$

Let $(P, \leq)$ be a poset and let $a, b \in P$. We write $a < b$ if $a \leq b$ and $a \neq b$. We say that a is a *predecessor* of b and that b is a *successor* of a whenever $a < b$. If $a < b$ and there is no $c \in P$ such that $a < c < b$ we say that a and b are *immediate neighbors* in P; in this case, we also say that a is an *immediate predecessor* of b and that b is an *immediate successor* of a.

We say that P satisfies (K1) if every nonempty totally ordered subset of P has an infimum and a supremum. We say that P satisfies (K2) if for all $a < b$ in P there exist $a \leq c < d \leq b$ in P such that c and d are immediate neighbors. let $E \subseteq P$ be a chain and let $a, b \in P$. We say that E is a *maximal chain between* a *and* b if a is the smallest member of E, b is the greatest member of E, and for any $x \in P \setminus E$ such that $a < x < b$, one has $E \cup \{x\}$ is not a chain (i.e., one cannot "insert" an element of P between the elements of E). We say that E is a *maximal chain in* P if for any $x \in P \setminus E$, one has $E \cup \{x\}$ is not a chain. We define now two special subsets of P; the set of all immediate predecessors in P, denoted P^-, is the set

$$P^- = \{a \in P \mid a \text{ is an immediate predecessor in } P\};$$

and the set of all immediate successors in P, denoted P^+, is the set

$$P^+ = \{a \in P \mid a \text{ is an immediate successor in } P\}.$$

Finally, P is called strictly inductive if every nonempty chain in P has a least upper bound.

Throughout this paper S denotes an integral domain with field of fractions F and A is an F-algebra. An S-subalgebra R of A is called S-nice if R is lying over S and the localization of R with respect to $S \setminus \{0\}$ is A. We denote by $\mathbb{W}$ the set of all S-nice subalgebras of A. It should be noted that the study of S-nice subalgebras of A was initiated with the study of quasi-valuations; a quasi-valuation is a generalization of the famous and most applicable notion of valuation. For extensive and detailed research on quasi-valuations see [9]- [12].

For every $M \subseteq A$ we denote by $V(M)$ the set of all S-nice subalgebras of A containing M. It is easy to see that $V(\{0\}) = \mathbb{W}$, $V(F) = \emptyset$, and for every $M_1, M_2 \subseteq A$, we have

$$V(M_1 \cup M_2) = V(M_1) \cap V(M_2).$$

Thus, the set B=$\{V(M)\}_{M \subseteq A}$ is a basis for a topology on $\mathbb{W}$. Moreover, for every set $\{M_i\}_{i \in I}$ of subsets of A, we have

$$V(\cup_{i \in I} M_i) = \cap_{i \in I} V(M_i).$$

Thus, $\mathbb{W}$ is an Alexandroff topological space with respect to the topology whose basis is B.

It is not difficult to see that, for $R \in \mathbb{W}$, the minimal open set containing R is actually $V(R)$. Also, for $R_1, R_2 \in \mathbb{W}$, the specialization order on $\mathbb{W}$ is defined by $R_1 \leq R_2$ whenever $R_1 \in cl\{R_2\}$; i.e., every open set containing R_1 also contains R_2; since $\mathbb{W}$ is Alexandroff,

$$R_1 \in cl\{R_2\} \text{ iff } R_2 \in V(R_1); \text{ i.e., } R_1 \subseteq R_2.$$

In other words, the specialization order on $\mathbb{W}$ is the order of containment; thus, in particular, the topology on $\mathbb{W}$ is T_0. Moreover, as in any Alexandroff topological space, $C \subseteq \mathbb{W}$ is closed iff C is a lower set of $\mathbb{W}$; namely,

$$C = \{R \in \mathbb{W} \mid R \subseteq R_1 \text{ for some } R_1 \in C\}.$$

Dually, $U \subseteq \mathbb{W}$ is open iff U is an upper set of $\mathbb{W}$; namely,

$$U = \{R \in \mathbb{W} \mid R_1 \subseteq R \text{ for some } R_1 \in U\}.$$

II. Immediate Neighbors in $\mathbb{W}$

In this section we discuss some known results regarding the topological space $\mathbb{W}$, as well as presenting several key results concerning immediate neighbors in $\mathbb{W}$.

The notion of an S-stable basis is important to the study of $\mathbb{W}$; it is defined as follows: let B be a basis of A over F. B is called S-stable if there exists a basis C of A over F such that for all $c \in C$ and $b \in B$, one has

$$cb \in \sum_{y \in B} Sy.$$

An important observation regarding S-stable bases is the fact that whenever A is finite dimensional over F, there exists an S-stable basis of A over F; more precisely, we have the following proposition.

Proposition 1. *(cf. [13, Proposition 3.12]) If A is finite dimensional over F, then every basis of A over F is S-stable.*

Remark 1. It is still not known whether an S-stable basis of A over F exists for an arbitrary algebra A over F.

The existence of an S-stable basis is important to our study because of the following existence theorem.

Theorem 1. *(cf. [13, Proposition 3.14]) If there exists an S-stable basis of A over F, then there exists an S-nice subalgebra of A.*

The property "going-down" is well-known as part of the classical lifting conditions of prime ideals. The following Lemma is a going-down lemma for S-nice subalgebras.

Lemma 1. *(cf. [13, Proposition 3.20]) Let $S_1 \subseteq S_2$ be integral domains with field of fractions F such that $S_2 \neq F$. Assume that there exists an S_1-stable basis of A over F. Let R be an S_2-nice subalgebra of A. Then there exists an S_1-nice subalgebra of A, which is contained in R.*

The proof relies on the fact that there exists an S_1-nice subalgebra of A by the existence theorem, and an intersection of an S_2-nice subalgebra of A and an S_1-nice subalgebra of A is again an S_1-nice subalgebra of A.

Taking $S_1 = S_2$ in the previous lemma, we deduce the existence of an infinite decreasing chain of S-nice subalgebras of A; more precisely, the following proposition was proved in [13].

Proposition 2. *Assume that there exists an S-stable basis of A over F. Let R be an S-nice subalgebra of A. Then there exists an infinite decreasing chain of S-nice subalgebras of A starting from R. In particular, a minimal S-nice subalgebra of A does not exist.*

We conclude,

Proposition 3. *If A is finite dimensional over F then $\mathbb{W}$ is infinite.*

Proof: By Proposition 1 and Proposition 2. ■

We assume throughout this section that there exists an S-stable basis of A over F; in particular, $\mathbb{W}$ is not empty, and $\mathbb{W}$ does not contain minimal elements.

The following four basic lemmas were proved in [14].

Lemma 2. *Let R_1 and R_2 be two elements of $\mathbb{W}$ and let R be an S-algebra satisfying $R_1 \subseteq R \subseteq R_2$; then R is an S-nice subalgebra of A.*

Lemma 3. *Let $\{R_i\}_{1\leq i\leq n}$ be a finite subset of $\mathbb{W}$; then $\cap_{1\leq i\leq n} R_i \in \mathbb{W}$.*

For subsets $M \subseteq A$ and $T \subseteq F$ one defines

$$T \cdot M = \{ \sum_{1\leq i\leq n} t_i m_i \mid t_i \in T, m_i \in M\}.$$

Lemma 4. *Let $K = \{R_i\}_{i\in I}$ be a nonempty subset of $\mathbb{W}$ and denote $R_0 = \cup_{i\in I} R_i$. Then the following three properties are valid:*

(a) $R_0 \cap F = S$;
(b) $F \cdot R_0 = A$; and
(c) If R_0 is closed under addition then $S \cdot R_0 \subseteq R_0$.

In particular, if R_0 is a ring then it is an S-nice subalgebra of A.

Lemma 5. $(\mathbb{W}, \subseteq)$ *is strictly inductive.*

The following more delicate property of $\mathbb{W}$ can be deduced from the proof of the previous lemma:

Lemma 6. *Let E be a nonempty chain in $\mathbb{W}$. Then*

$$R_0 = \cup_{R\in E} R$$

is an element of $\mathbb{W}$.

In [14] the following general property regarding T_0-topological spaces is proved: let T is a T_0-topological space such that every nonempty chain in T has an upper bound; then any nonempty open subset of T has a maximal element, which is also a maximal element of T. Using this property it is easy to deduce that $\mathbb{W}$ contains a maximal element; more precisely, we have:

Lemma 7. *(cf. [14, Lemma 2.6]) Let U be a nonempty open subset of $\mathbb{W}$. Then there exists a maximal element in U, which is also a maximal element of $\mathbb{W}$.*

The existence of an infimum (resp. supremum) of a nonempty subset of $\mathbb{W}$ was characterized in [14, Proposition 2.11] by the existence of a lower (resp. upper) bound. Explicitly, the following was proved.

Proposition 4. *Let H be a nonempty subset of $\mathbb{W}$; then*

1. *There exists a lower bound for H iff the infimum of H exists; in this case inf$H = \cap_{R\in H} R$.*
2. *There exists an upper bound for H iff the supremum of H exists.*

Remark 2. Note that by Lemma 6, if H is a chain then

$$\text{sup}H = \cup_{R\in H} R.$$

The irreducible subsets of $\mathbb{W}$ play an important role in the study of the topological structure of $\mathbb{W}$, as discussed in [14]. It is well known that in an Alexandroff topological space a set is irreducible if and only if it is directed under the specialization order.

Remark 3. In [14, Theorem 2.13] it is proved that an irreducible subset of $\mathbb{W}$ has a supremum in $\mathbb{W}$. In view of the characterization of irreducible subsets of an Alexandroff topological space, this fact can be considered as a generalization of Lemma 6.

Inspired by [11], we characterize now immediate neighbors in an Alexandroff topological space.

Proposition 5. *Let T be an Alexandroff topological space and let $x_1, x_2 \in T$ such that $x_1 < x_2$. Then there exist immediate neighbors in T between x_1 and x_2 iff there exist a maximal chain $E \subseteq T$ between x_1 and x_2, and a partition $\{C, U\}$ of E such that C is closed in E and U is open in E (with respect to the induced topology of T on E), sup$C$ and inf$U$ exist in T, and sup$C \neq$inf$U$.*

Proof: $(\Rightarrow)$ Let $y_1 < y_2$ be immediate neighbors in T between x_1 and x_2. Using Zorn's Lemma, let $C \subseteq T$ (resp. $U \subseteq T$) be a maximal chain between x_1 and y_1 (resp. between y_2 and x_2). Since $y_1 < y_2$ are immediate neighbors in T, we deduce that the disjoint union $E = C \cup U$ is a maximal chain in T between x_1 and x_2. Also, sup$C = y_1$ and inf$U = y_2$, where C is a lower set of E and U is a upper set of E.

$(\Leftarrow)$ By assumption, the chain E between x_1 and x_2 is the disjoint union of the closed subset C and the open subset U; thus, for all $c \in C$ and for all $u \in U$, we have

$$x_1 \leq c < u \leq x_2.$$

Since sup$C$ and inf$U$ exist in T and are different, we get

$$x_1 \leq \text{sup}C < \text{inf}U \leq x_2.$$

By the assumption that E is a maximal chain between x_1 and x_2, we get that there exists no $z \in T$ such that

$$\text{sup}C < z < \text{inf}U.$$

So, sup$C$ and inf$U$ are immediate neighbors in T between x_1 and x_2.

■

Remark 4. Note that the C and U in the previous proposition are also irreducible subsets of T (and of E), because they are chains. Also note that sup$C$ is actually the greatest element of C, and likewise inf$U$ is the smallest element of U.

It is easy to see that not every Alexandroff topological space satisfies (K2); in the following proposition we prove that $\mathbb{W}$ indeed satisfies (K2).

Proposition 6. $\mathbb{W}$ *satisfies (K2).*

Proof: Let $R_1 \subset R_2$ in $\mathbb{W}$. Let E be a maximal chain in $\mathbb{W}$ between R_1 and R_2 and let $x \in R_2 \setminus R_1$. Let

$$C = \{R \in E \mid x \notin R\}$$

and let

$$U = \{R \in E \mid x \in R\}.$$

Viewing E as a subspace of $\mathbb{W}$, it is clear that C is a lower set of E, and U is an upper set of E. It is also clear that E is the disjoint union of C and U. By Lemma 6

$$\text{sup}C = \cup_{R\in C} R \in \mathbb{W},$$

and, since U has R_1 as a lower bound, by Proposition 4,

$$\text{inf}U = \cap_{R\in U} R \in \mathbb{W}.$$

Since $x \notin$ sup$C$ and $x \in$ inf$U$, we get sup$C \neq$ inf$U$. Thus, by Lemma 5, $\mathbb{W}$ satisfies (K2).

■

Using the notation defined earlier, $\mathbb{W}^-$ denotes the subspace of $\mathbb{W}$ of all immediate predecessors in $\mathbb{W}$, and $\mathbb{W}^+$ denotes the subspace of $\mathbb{W}$ of all immediate successors in $\mathbb{W}$.

We prove now that the subspace $\mathbb{W}^+$ generates $\mathbb{W}$, i.e., every element of $\mathbb{W}$ can be presented by a union of elements of $\mathbb{W}^+$.

Theorem 2. *Let $R_0 \in \mathbb{W}$. Then there exists a unique maximal nonempty closed subset C of $\mathbb{W}^+$ (maximal with respect to containment) such that $R_0 = \cup_{R\in C} R$. Furthermore, $R_0 = \cup_{R\in E} R$ for every maximal chain $E \subseteq C$.*

Proof: Let $C = \{R \in \mathbb{W}^+ \mid R \subseteq R_0\}$. By Lemma 2, R_0 is not minimal in $\mathbb{W}$; thus, there exists $R_1 \in \mathbb{W}$ such that $R_1 \subset R_0$. Therefore, by Proposition 6, $C \neq \emptyset$. Clearly, C is a closed subset of $\mathbb{W}^+$. Now, let E be any maximal chain contained in C. Since E is bounded from above by R_0, by Proposition 4, $\cup_{R\in E} R \in \mathbb{W}$. Assume to the contrary that $\cup_{R\in E} R \subset R_0$. Then, by Lemma 6, there exists

$$\cup_{R\in E} R \subset R_2 \in C.$$

Hence, $E \cup \{R_2\} \subseteq C$ is a chain strictly containing E, a contradiction. Thus,

$$\cup_{R\in C} R = \cup_{R\in E} R = R_0.$$

Note that by the definition of C, it is clear that any subset C_1 of $\mathbb{W}^+$ such that $R_0 = \cup_{R\in C_1} R$ satisfies $C_1 \subseteq C$.

■

Dually to Theorem 2, we prove now that the subspace $\mathbb{W}^-$ is rich enough to produce every non-maximal element in $\mathbb{W}$; in other words, every non-maximal element of $\mathbb{W}$ can be presented as an intersection of elements of $\mathbb{W}^-$. Note that, unlike Theorem 2, in which it was proved that every element of $\mathbb{W}$ is generated by $\mathbb{W}^+$ (because minimal elements do not exist in $\mathbb{W}$, in view of Lemma 2), $\mathbb{W}^-$ cannot generate $\mathbb{W}$, because maximal elements exist in $\mathbb{W}$, as shown in Lemma 7. Thus, $\mathbb{W}^-$ can only generate $\mathbb{W} \setminus \{\text{maximal elements in } \mathbb{W}\}$. The proof of the following theorem is quite similar to the proof of Theorem 2. We prove it here for the reader's convenience.

Theorem 3. *Let R_0 be a non-maximal element in $\mathbb{W}$. Then there exists a unique maximal nonempty open subset U of $\mathbb{W}^-$ such that $R_0 = \cap_{R\in\mathcal{U}} R$. Furthermore, $R_0 = \cap_{R\in E} R$ for every maximal chain $E \subseteq U$.*

Proof: Let $U = \{R \in \mathbb{W}^- \mid R_0 \subseteq R\}$. By assumption, R_0 is not maximal in $\mathbb{W}$; thus, there exists $R_1 \in \mathbb{W}$ such that $R_0 \subset R_1$. Therefore, by Proposition 6, $U \neq \emptyset$. Clearly, U is an open subset of $\mathbb{W}^-$. Now, let E be any maximal chain contained in U. Since E is bounded from below by R_0, by Proposition 4, $\cap_{R\in E} R \in \mathbb{W}$. Assume to the contrary that $R_0 \subset \cap_{R\in E} R$. Then, by Lemma 6, there exists $R_2 \in U$ such that

$$R_0 \subseteq R_2 \subset \cap_{R\in E} R.$$

Hence, $E \cup \{R_2\} \subseteq U$ is a chain strictly containing E, a contradiction. Thus,

$$\cap_{R\in U} R = \cap_{R\in E} R = R_0.$$

Clearly, any subset U_1 of $\mathbb{W}^-$ such that $R_0 = \cap_{R\in U_1} R$ satisfies $U_1 \subseteq U$.

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