Integral Domains and Their Algebras: Topological Aspects

Shai Sarussi

Abstract—Let S be an integral domain with field of fractions F and let A be an F-algebra. An S-subalgebra R of A is called S-nice if $R \cap F = S$ and the localization of R with respect to $S \setminus \{0\}$ is A. Denoting by $\mathbb{W}$ the set of all S-nice subalgebras of A, and defining a notion of open sets on $\mathbb{W}$, one can view $\mathbb{W}$ as a T_0-Alexandroff space. Thus, the algebraic structure of $\mathbb{W}$ can be viewed from the point of view of topology. It is shown that every nonempty open subset of $\mathbb{W}$ has a maximal element in it, which is also a maximal element of $\mathbb{W}$. Moreover, a supremum of an irreducible subset of $\mathbb{W}$ always exists. As a notable connection with valuation theory, one considers the case in which S is a valuation domain and A is an algebraic field extension of F; if S is indecomposed in A, then $\mathbb{W}$ is an irreducible topological space, and $\mathbb{W}$ contains a greatest element.

Keywords—Algebras over integral domains, Alexandroff topology, valuation domains, integral domains.

I. Introduction

The study of topological spaces of algebraic objects such as rings or ideals has proven to be very useful in the last century. The first one to conduct such a research was Stone (cf. [15], [16]), who studied topological spaces of prime ideals in the context of distributive lattices and Boolean algebras. Later, important researchers followed Stone: Zariski studied topological spaces of valuation domains (what is now known as the Zariski-Riemann space, cf. [17]); Zariski also studied topological spaces of prime ideals of commutative rings with unity; Hochster (cf. [7]) studied the notion of a spectral space using purely topological properties, and showed that those properties characterize the topological spaces that are homeomorphic to the prime spectrum of a commutative ring with unity, endowed with the Zariski topology; many more researchers continued the study of such and similar spaces. As a notable such study, we note that it was proved in 1986 (cf. [2]) that the Zariski-Riemann space is in fact a spectral space; later, this fact was proved again using different approaches (see [3], [5], [6], and [8]).

In this paper, as the title suggests, we discuss topological (and algebraic) aspects of algebras over an integral domain; the topological aspects we discuss are with respect to an Alexandroff topology. The purpose of this paper is to present a short view on such spaces, which in a way, is broader than the points of view that were given in [13] and [14]; additional aspects of these spaces are also presented.

In this paper the symbol $\subset$ means proper inclusion, and the symbol $\subseteq$ means inclusion or equality.

The motivation for studying algebras over an integral domain initiated with the study of quasi-valuations. A quasi-valuation is a generalization of the notion of valuation.

S. Sarussi is with the Department of Mathematics, SCE College, 84100 Be'er Sheva, Israel (e-mail: sarusss1@gmail.com).

We recall that a valuation on a field F is a function

$$v : F \to \Gamma \cup \{\infty\},$$

where Γ is a totally ordered abelian group, to which we adjoin an element ∞, which is greater than all elements of Γ, and v satisfies the following conditions:

(A1) $v(x) \neq \infty$ iff $x \neq 0$, for all $x \in F$;
(A2) $v(xy) = v(x) + v(y)$ for all $x, y \in F$;
(A3) $v(x + y) \geq \min\{v(x), v(y)\}$ for all $x, y \in F$.

Recall (cf. [9, Introduction]) that a quasi-valuation on a ring R is a function

$$w : R \to M \cup \{\infty\},$$

where M is a totally ordered abelian monoid, to which we adjoin an element ∞, which is greater than all elements of M, and where w satisfies the following properties:

(B1) $w(0) = \infty$;
(B2) $w(xy) \geq w(x) + w(y)$ for all $x, y \in R$;
(B3) $w(x + y) \geq \min\{w(x), w(y)\}$ for all $x, y \in R$.

Let v be a valuation on a field F; the valuation domain of v is the integral domain ,whose field of fractions is F, defined by

$$O_v = \{x \in F \mid v(x) \geq 0\}.$$

Likewise, let w be a quasi-valuation on a ring R; the quasi-valuation ring is the subring of R defined by

$$O_w = \{x \in R \mid w(x) \geq 0\}.$$

In [9] the theory of quasi-valuations that extend a given valuation was developed. Explicitly, for a given valuation v on a field F, a corresponding valuation domain O_v, and a finite field extension E/F, we studied quasi-valuations on E extending v on F. We showed that every such quasi-valuation is dominated by some valuation extending v; more precisely, we showed that there exists a valuation u on E extending v on F such that for every $x \in E$, we have $w(x) \leq u(x)$ (see [9, Section 6]). In addition, a one-to-one correspondence was obtained between exponential quasi-valuations and integrally closed quasi-valuation rings. The most important result in [9] was the construction of the filter quasi-valuation, for any algebra over a valuation domain. We showed that if A is an F-algebra and R is an O_v-subalgebra of A lying over O_v then there exists a quasi-valuation on $R \otimes_{O_v} F$ (called the filter quasi-valuation) extending v on F such that the quasi-valuation ring is equal to R (under the identification of R with $R \otimes_{O_v} 1$). In particular, if R is an O_v-subalgebra of A lying over O_v such that $RF = A$ then there exists a quasi-valuation on A extending v on F.

Some of the notable properties concerning a valuation domain and an algebra over it are the connections between their prime spectra; these connections are studied via the classical lifting conditions of prime ideals.

We recall now these five lifting conditions. We assume that R is an algebra over S. For subsets $I \subseteq R$ and $J \subseteq S$ we say that I is lying over J if

$$J = \{s \in S \mid s \cdot 1_R \in I\}.$$

By abuse of notation, we write $J = I \cap S$ (even when R is not faithful over S).

We say that R satisfies LO (lying over) over S if for all $P \in \text{Spec}(S)$ there exists $Q \in \text{Spec}(R)$ lying over P.

We say that R satisfies GD (going down) over S if for any $P_1 \subset P_2$ in $\text{Spec}(S)$ and for every $Q_2 \in \text{Spec}(R)$ lying over P_2, there exists $Q_1 \subset Q_2$ in $\text{Spec}(R)$ lying over P_1.

We say that R satisfies GU (going up) over S if for any $P_1 \subset P_2$ in $\text{Spec}(S)$ and for every $Q_1 \in \text{Spec}(R)$ lying over P_1, there exists $Q_1 \subset Q_2$ in $\text{Spec}(R)$ lying over P_2.

We say that R satisfies SGB (strong going between) over S if for any $P_1 \subset P_2 \subset P_3$ in $\text{Spec}(S)$ and for every $Q_1 \subset Q_3$ in $\text{Spec}(R)$ such that Q_1 is lying over P_1 and Q_3 is lying over P_3, there exists Q_2, with $Q_1 \subset Q_2 \subset Q_3$ in $\text{Spec}(R)$, lying over P_2.

We say that R satisfies INC (incomparability) over S if whenever $Q_1 \subset Q_2$ in $\text{Spec}(R)$, we have $Q_1 \cap S \subset Q_2 \cap S$.

Another (more general) property that was considered in [12] is the GGD property. It is defined as follows: We say that R satisfies GGD (generalized going down) over S if, for every chain of prime ideals $\mathcal{D}$ of S with a final element P_0 and Q_0 a prime ideal of R lying over P_0, there exists a chain of prime ideals $\mathcal{C}$ of R covering $\mathcal{D}$ (namely, for every $P \in \mathcal{D}$ there exists $Q \in \mathcal{C}$ lying over P), whose final element is Q_0.

In [12, Section 2] we studied the notions LO and INC with respect to algebras over a valuation domain. In [12, Theorem 2.2] we gave sufficient conditions (in terms of the algebra R as well as in terms of the filter quasi-valuation defined on it) for an O_v-algebra to satisfy LO over O_v; we also presented a necessary and sufficient condition for an O_v-algebra to satisfy LO over O_v. In [12, Theorem 2.6] it was shown that if R is a torsion-free algebra over O_v such that

$$[R \otimes_{O_v} F : F] < \infty,$$

then R satisfies INC over O_v; we also showed that one cannot omit any of these assumptions. Using INC, we showed that the number of prime ideals in R lying over a given prime ideal in O_v is less than or equal to the dimension of $R \otimes_{O_v} F$ over F. In particular, we got an upper bound on the size of the prime spectrum of R. Then, combining the properties LO and INC, we obtained in [12, Theorem 2.13] an upper and a lower bound on the size of the prime spectrum of R (in terms of the Krull dimension of O_v).

In [12, Proposition 3.2] we proved that if R is a torsion-free algebra over O_v then R satisfies GD over O_v. It is easy to see that not every algebra over O_v satisfies GD; however, the torsion-free assumption is not a necessary condition for the GD property, as shown in [12, Example 3.5]. We also proved (cf. [12, Theorem 3.8]) that any algebra over a commutative valuation ring satisfies SGB over it; and deduced that a torsion-free algebra over O_v satisfies GGD (generalized going down) over O_v (cf. [12, Corollary 3.9]).

In [12, Section 4] we discussed the property GU. We proved in [12, Theorem 4.12] that if R is a right (or left) Artinian algebra over O_v and w is a v-quasi-valuation on R such that $w(1_R) = 0$, $w(x) \neq \infty$ for all nonzero $x \in R$, and M_w (the value monoid of the quasi-valuation) is cancellative, then O_w satisfies GU over O_v. We also noted that the GU property is not necessarily valid without the assumption on the quasi-valuation; in fact, in a subsequent paper we present equivalent conditions for an O_v subalgebra of E to satisfy GU over O_v, whenever E is a finite dimensional field extension of F. Moreover, if R is a torsion-free algebra over O_v and finitely generated as a module over O_v, we constructed in [12, Theorem 4.29] a v-quasi-valuation w on R such that $w(1_R) = 0$, $w(x) \neq \infty$ for all nonzero $x \in R$, the value monoid of w is equal to the value group of v, and $O_w = R$; for the definition of a v-quasi-valuation see [12, Definition 4.1]. So, in [9] and in some parts of [12] we assumed that a quasi-valuation extending the valuation v exists (or equivalently by [9, Theorem 9.19], we assumed the existence of an appropriate O_v-algebra). In [13] we partially answered the following natural and central question: when does such a quasi-valuation exist? We showed that quasi-valuations extending a valuation v on F exists on any finite dimensional F-algebras, and even more generally, on any F-algebra having an O_v-stable basis. In fact, we proved a more general result and applied it to quasi-valuation theory. We shall discuss this theorem in more detail in Section II. For more information on quasi-valuations see [10] and [11]. So, as explained above, there are some interesting connections between a valuation domain and an algebra over it; in particular, with respect to their prime spectra. It can be shown that some connections can be generalized to the study of arbitrary algebras over an integral domain; In fact, some interesting properties of algebras over an integral domain can be deduced from the study of quasi-valuation rings over a valuation domain.

We recall that a topological space whose set of open sets is closed under arbitrary intersections is called an Alexandroff space, after P. Alexandroff who first introduced such topological spaces in his paper [1] from 1937. Equivalently, A topological space is called an Alexandroff space if every element has a minimal open set containing it.

Let (T, τ) be a topological space. For $X \subseteq T$ we denote by clX the closure of X. It is well known, that if one defines $x \leq_\tau y$ whenever $x \in cl\{y\}$, then $\leq_\tau$ is a quasi-order; i.e., a reflexive and transitive relation. $\leq_\tau$ is called the specialization order. Recall that (T, τ) is called T_0 if for every two distinct elements in T, there exists an open set containing one of them but not the other. It is known that (T, τ) is T_0 if and only if $\leq_\tau$ is a partial order.

We review now some of the common definitions we use from order theory.

Let P be a set with a preorder $\leq$. Let $M \subseteq P$. We say that $a \in M$ is minimal (resp. maximal) in M if for all $x \in M$, $x \leq a$ (resp. $x \geq a$) implies $x = a$. We say that $a \in M$ is a smallest (resp. greatest) element in M if for all $x \in M$ we have $a \leq x$ (resp. $a \geq x$). As mentioned above, if the relation $\leq$ is also antisymmetric then it is called a partial order and P is called a partially ordered set, or a poset. Let $L \subseteq P$. We say that $a \in P$ is a lower (resp. upper) bound of L if $a \leq x$ (resp. $a \geq x$) for all $x \in L$. If the set of lower (resp. upper) bounds of L has a unique greatest (resp. smallest) element, this element is called the greatest lower (resp. least upper) bound of L, and is denoted by inf$L$ (resp. sup$L$). We say that L is a lower set if

$$L = \{y \in P \mid y \leq x \text{ for some } x \in L\}.$$

We say that L is an upper set if

$$L = \{y \in P \mid y \geq x \text{ for some } x \in L\}.$$

II. The Topological Space $\mathbb{W}$

The basic framework of this paper is as follows: let S be an integral domain with field of fractions F and let A be an F-algebra. An S-subalgebra R of A is called S-nice if R is lying over S and the localization of R with respect to $S \setminus \{0\}$ is A. We denote by $\mathbb{W}$ the set of all S-nice subalgebras of A. The notion of an S-stable basis is also important to our study: let B be a basis of A over F. We say that B is S-stable if there exists a basis C of A over F such that for all $c \in C$ and $b \in B$, one has

$$cb \in \sum_{y \in B} Sy.$$

An important observation regarding S-stable bases is the fact that whenever A is finite dimensional over F, there exists an S-stable basis of A over F; more precisely, we have the following proposition.

Proposition 1. *(cf. [13, Proposition 3.12]) If A is finite dimensional over F, then every basis of A over F is S-stable.*

It is worth mentioning that it is still not known whether an S-stable basis of A over F exists for an arbitrary algebra A over F. The existence of an S-stable basis is important to our study due to the following existence theorem.

Theorem 1. *(cf. [13, Theorem 3.14]) If there exists an S-stable basis of A over F, then there exists an S-nice subalgebra of A.*

The property "going-down" is well-known as part of the classical lifting conditions of prime ideals. The following Lemma is a going-down lemma for S-nice subalgebras.

Lemma 1. *(cf. [13, Lemma 3.20]) Let $S_1 \subseteq S_2$ be integral domains with field of fractions F such that $S_2 \neq F$. Assume that there exists an S_1-stable basis of A over F. Let R be an S_2-nice subalgebra of A. Then there exists an S_1-nice subalgebra of A, which is contained in R.*

The idea of the proof is to consider an S_1-nice subalgebra of A, say R_1 (there exists such an algebra by the existence theorem). Then, the intersection of R_1 and R is the required S_1-nice subalgebra of A.

One can take $S_1 = S_2$ in the previous Lemma, and conclude the existence of an infinite decreasing chain of S-nice subalgebras of A; more precisely, we have the following proposition.

Proposition 2. *(cf. [13, Proposition 3.21]) Assume that there exists an S-stable basis of A over F. Let R be an S-nice subalgebra of A. Then there exists an infinite decreasing chain of S-nice subalgebras of A starting from R. In particular, a minimal S-nice subalgebra of A does not exist.*

We conclude,

Proposition 3. *If A is finite dimensional over F then $\mathbb{W}$ is infinite.*

Proof: By Proposition 1 and Proposition 2. ■

We assume throughout this paper that $\mathbb{W}$ is not empty.

For every $M \subseteq A$ we denote by $V(M)$ the set of all S-nice subalgebras of A containing M. It is easy to see that $V(\{0\}) = \mathbb{S}$, $V(F) = \emptyset$, and for every $M_1, M_2 \subseteq A$, we have

$$V(M_1 \cup M_2) = V(M_1) \cap V(M_2).$$

Thus, the set B=$\{V(M)\}_{M \subseteq A}$ is a basis for a topology on $\mathbb{S}$. Moreover, for every set $\{M_i\}_{i \in I}$ of subsets of A, we have

$$V(\cup_{i \in I} M_i) = \cap_{i \in I} V(M_i).$$

Thus, $\mathbb{W}$ is an Alexandroff topological space with respect to the topology whose basis is B.

It is not difficult to see that, for $R \in \mathbb{W}$, the minimal open set containing R is actually $V(R)$. Also, for $R_1, R_2 \in \mathbb{W}$, the specialization order on $\mathbb{W}$ is defined by

$$R_1 \leq R_2 \text{ whenever } R_1 \in cl\{R_2\};$$

i.e., every open set containing R_1 also contains R_2; since $\mathbb{W}$ is Alexandroff,

$$R_1 \in cl\{R_2\} \text{ iff } R_2 \in V(R_1),$$

i.e., $R_1 \subseteq R_2$. In other words, the specialization order on $\mathbb{W}$ is the order of containment; thus, in particular, the topology on $\mathbb{W}$ is T_0. Moreover, as in any Alexandroff topological space, $U \subseteq \mathbb{W}$ is open iff U is an upper set of $\mathbb{W}$; namely,

$$U = \{R \in \mathbb{W} \mid R_1 \subseteq R \text{ for some } R_1 \in U\}.$$

In general order theory, upper sets do not have to contain a maximal element; however, in $\mathbb{W}$ we have the following theorem, which is a generalization of [14, Lemma 2.6].

Theorem 2. *Let U be a nonempty open subset of $\mathbb{W}$. Then there exists a maximal element in U, which is also a maximal element of $\mathbb{W}$. In fact, for every $R \in U$, there exists a maximal $R_0 \in U$ containing R.*

Proof: It is enough to prove the second assertion. So, let $R \in U$, and consider $V(R)$, which is obviously contained in U, with the specialization order. Let C be a nonempty chain of elements in $V(R)$. Consider

$$R_1 = \cup_{R \in C} R;$$

since C is a chain, R_1 is an S-nice subalgebra of A. Clearly, $R_1 \in V(R)$. Hence, every nonempty chain in $V(R)$ has an upper bound. Thus, by Zorn's Lemma, $V(R)$ has a maximal element; in particular, this maximal element contains R. Since $V(R)$ is an upper set, this maximal element is also a maximal element of U and a maximal element of $\mathbb{W}$. ∎

Remark 1. In view of the previous Theorem, note that every subset of the set of all maximal elements of $\mathbb{W}$ is an open subset of $\mathbb{W}$.

Recall that a topological space is called Noetherian if it satisfies the descending chain condition for closed subsets. We have the following observation.

Proposition 4. *If A is finite dimensional over F, then $\mathbb{W}$ is not Noetherian.*

Proof: By Proposition 1 every basis of A over F is S-stable; A obviously contains a basis over F. Now, let $R \in \mathbb{W}$. By Proposition 2 there exists an infinite decreasing chain, say ω, of S-nice subalgebras of A starting from R. By considering the infinite decreasing chain of the closures of the algebras in ω, we conclude that $\mathbb{W}$ is not Noetherian. ∎

Recall that a topological space is called Hausdorff if for any two points x, y in it, there exist two disjoint open sets U and V such that U contains x and V contains y. It is well-known that an Hausdorff topological space which is also Alexandroff must be finite. Thus, in view of Proposition 3, we have,

Lemma 2. *If A is finite dimensional over F, then $\mathbb{W}$ is not Hausdorff.*

In fact, we can say even more. Without any assumptions on the dimension of A over F, whenever $R_1 \subset R_2$ in $\mathbb{W}$, it is clear that $R_2 \in V(R_1)$. Since $V(R_1)$ is the smallest open set containing R_1, there exists no open set containing R_1 and not containing R_2.

The following Lemma is important for the understanding of irreducible subsets of $\mathbb{W}$.

Lemma 3. *Let I be an irreducible subset of $\mathbb{W}$ and let $R_1, R_2 \in I$. Then there exists $R_3 \in I$ such that $R_1 \cup R_2 \subseteq R_3$.*

Proof: Assume to the contrary that there exists no such R_3; in particular,

$$R_1 \nsubseteq R_2 \text{ and } R_2 \nsubseteq R_1.$$

We consider the closed sets in $\mathbb{W}$ not containing R_1, and let C_1 denote the union of all those sets. Similarly, consider the closed sets in $\mathbb{W}$ not containing R_2, and let C_2 denote the union of all those sets. Since $\mathbb{W}$ is Alexandroff, both C_1 and C_2 are closed. Then, by our assumption,

$$I \subseteq C_1 \cup C_2.$$

However, since $R_1 \notin C_1$ and $R_2 \notin C_2$, we have

$$I \nsubseteq C_1 \text{ and } I \nsubseteq C_2,$$

a contradiction. ∎

In [14, Theorem 2.13] it is proved that an irreducible subset of $\mathbb{W}$ has a supremum in $\mathbb{W}$. Using Lemma 3 we obtain a simpler and more informative proof for the theorem.

Theorem 3. *Let I be an irreducible subset of $\mathbb{W}$. Then $\cup_{R\in I} R$ is an S-nice subalgebra of A; in particular,*

$$supI = \cup_{R\in I} R.$$

Proof: By Lemma 3, it is clear that $\cup_{R\in I} R$ is an S-subalgebra of A. It is easy to conclude now that $\cup_{R\in I} R$ is S-nice. ∎

Let K be a field and let L be an algebraic field extension of K. Let T be a valuation domain of K. Recall (cf. [4, Corollary 13.5]) that T is indecomposed in L if there exists a unique valuation domain of L lying over T. Moreover, by [4, Corollary 13.7]), whenever the separable degree of K over L is finite, the number of valuation domains of L that are lying over T is less than or equal to the separable degree of K over L.

Now, whenever S is a valuation domain of F, and A is an algebraic field extension of F, it is well known (cf. [4, Proposition 13.2]) that there exists a valuation domain of A lying over S; in particular, there exists an S-nice subalgebra of A. Moreover, every such valuation domain of A is a maximal S-nice subalgebra of A and vice versa. Indeed, any subalgebra of A containing such a valuation domain lies over a proper overring of S, and every S-nice subalgebra of A has a valuation domain lying over S that contains it, by [4, Theorem 9.11].

In [14, Theorem 2.15] the irreducible open subsets of $\mathbb{W}$ are characterized, among other characterizations, as the open subsets that have a greatest element. Thus, in view of this characterization we have the following observation.

Proposition 5. *Assume that S is a valuation domain of F and A is an algebraic field extension of F. If S is indecomposed in A, then $\mathbb{W}$ is an irreducible Alexandroff topological space.*

In [14, Proposition 2.16] the irreducible components of $\mathbb{W}$ are characterized as the closed sets of the form $cl\{R\}$, where R is a maximal element of $\mathbb{W}$.

Finally, by [4, Corollary 13.7] and the characterization of the irreducible components mentioned above, we conclude the following generalization of the previous Proposition,

Corollary 1. *Assume that S is a valuation domain of F and A is an algebraic field extension of F. If the separable degree of A over F is finite, then $\mathbb{W}$ contains finitely many irreducible components.*

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