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The Fallacy around Inserting Brackets to Evaluate Expressions Involving Multiplication and Division

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***Abstract*—**Evaluating expressions involving multiplication and division can give rise to the fallacy that brackets can be arbitrarily inserted into expressions involving multiplication and division. The aim of this article was to draw upon mathematical theory to prove that brackets cannot be arbitrarily inserted into expressions involving multiplication and division and in particular in expressions where division precedes multiplication. In doing so, it demonstrates that the notion that two different answers are possible, when evaluating expressions involving multiplication and division, is indeed a false one. Searches conducted in a number of scholarly databases unearthed the rules to be applied when removing brackets from expressions, which revealed that consideration needs to be given to sign changes when brackets are removed. The rule pertaining to expressions involving multiplication and division was then extended upon, in its reverse format, to prove that brackets cannot be arbitrarily inserted into expressions involving multiplication and division. The application of the rule demonstrates that an expression involving multiplication and division can have only one correct answer. It is recommended that both the rule and its reverse be included in the curriculum, preferably at the juncture when manipulation with brackets is introduced.

***Keywords*—**Brackets, multiplication, division, operations, order.

I. Introduction

EVALUATING expressions involving multiplication and division, like 8 ÷ 2 x 4 = ?, surfaced in many media platforms with the online community being divided on whether the correct answer should be 16 or 1 [1]. This paper problematizes the notion of the expression having two possible answers, resulting from differences in preference of the order in which the operations are executed. The rationale for the contestation of the duality of answers is underpinned by articulating the conventions regarding the order of operations and the use of brackets. The aim of this paper was to draw upon rules in mathematics to resolve the perceived anomaly by demonstrating that brackets cannot be arbitrarily inserted into expressions, in particular where division is followed by multiplication (or division). This article is theoretical in nature and does not involve the collection of primary data. The article brings to the fore the mathematical rule pertaining to the removal of brackets [2] and extends upon it to contribute to the development of theory. In doing so, it demonstrates that the notion that brackets can be arbitrarily inserted into expressions involving multiplication and division is a fallacy. This paper closes by proposing a rule for the insertion of brackets into expressions involving multiplication and division. The significance of the rule is that it makes possible for expressions involving multiplication and division to be executed in any order of preference of the operations. The full effect of the application of the rule is that it clearly demonstrates that the expression 8 ÷ 2 x 4 having two possible correct answers is indeed a false claim.

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II. Context of the Study

The impetus for this article was the controversy that was stirred after an online posting of the equation: 8 ÷ 2(2+2) = ? on social media [3]. The online postings headlines that featured therein include: “The Math Equation That Tried to Stump the Internet” [3]; “Everyone’s arguing over this very simple math equation…” [4]; “That Vexing Math Equation?” [5]); “Can you solve this math equation stumping the internet?” [1]; “This simple math problem drove our entire staff insane. Can you solve it?” [6]. The equation 8 ÷ 2(2+2) =? was drilled down to being 8 ÷ 2 x 4. One school of thought held that the correct answer, by executing the division first, was 16 and another group claimed, by executing the multiplication first, the correct answer was 1. The online discussions finally simmered down on the note that both answers could be considered as being correct.

III. Problem Statement

Given the inherent exactness of mathematics, where there is explication of right and wrong, it was questioned whether two different answers could be accepted as being correct when evaluating expressions involving division followed by multiplication/division. It was noted that for the equation 8 ÷ 2 x 4 =? most modern calculators, cellphone calculators, Wolfram Alpha, Excel and Google produces the answer of 16. Furthermore, when the multiplicative inverse was used, as in 8 x ½ x 4, the answer was also 16. When the left to right convention was used, the answer was also 16. The approach that claimed 1 as being the correct answer entailed the multiplication being done first. The key question that therefore arose was whether the multiplication can be effected by arbitrarily inserting brackets (mental or physical) as in 8 ÷ 2 x 4 = 8 ÷ (2 x 4) = 1. In this article, it was hypothesized that the brackets cannot be arbitrarily inserted and posited that additional mathematical manipulation is required, to accompany the insertion of the brackets. The aim of the study was to identify an appropriate rule under which brackets could be inserted into expressions involving multiplication and division.

IV. Methodology

In order to fully understand the context of the study the discussion threads on the different social media platforms were read to saturation (when no more new ideas surfaced), culminating in the problem statement. The first objective that was set was to bring into perspective the conventions used to evaluate expressions with regards to the order of operations. The second objective that was set was to identify mathematical rules that could be drawn upon to contribute to resolving the anomaly of duality of answers to the evaluation of an expression involving multiplication and division where the division preceded the multiplication. In order to achieve the objectives, the search string: multiplication; division; order; precedence; brackets, using Boolean search parameters, were deployed in Google Scholar and the ProQuest databases. Title and Abstract searches were undertaken to identify possible journal articles and books and depending on how closely the title or abstract associated with the research question, a decision was made as to whether to read the full text or not. After an exhaustive search through thousands of articles, the early works of [2] was located, which was deemed to constitute the appropriate theory against which the key question posed in this article could be answered.

V. Literature Review

An important aspect of arithmetic is the order of operations, a convention that is commonly remembered by mnemonics, such as BODMAS, PEDMAS, PEMDAS, BEDMAS or BIDMAS, where B stands for 'brackets', O for 'orders', E for 'exponents', D for 'division', M for 'multiplication', A for 'addition', S for 'subtraction' and I for Indices [7]. Despite there being a number of different conventions, there is a common understanding that, when applying any of the conventions, multiplication and/or division is executed before additions and/or subtraction, with multiplication and division having equal priority and addition and subtraction having equal priority [7]. There is however a lack of common understanding with regards to the order in which operations should be executed, for expressions involving multiplication and division only, with some conventions stipulating that the multiplication needs to be executed first and division is to follow in the order in which they appear, as noted very early on by [8]. While such an approach to giving precedence to multiplication over division may be in conflict with the common understanding that multiplication and division has equal priority, such an approach is not necessarily inherently flawed in itself, but can be described as being incomplete. The author attributes the element of incompleteness to the absence of theory on exactly how the manipulation (inserting brackets) is to be effected. It is this gap that this article serves to close.

Alongside conventions, relating to the order of operations, the concept of brackets is typically taught [9]. Brackets can be used with different intentions in mathematics. Firstly, brackets may be used to define structure in mathematical expressions [10]. For instance, when brackets are already given as part of an expression, it constitutes a vital component and signals a higher priority structural element, denoting that operations within brackets should be performed first [7], [11]. In order to fully evaluate expressions containing the brackets, brackets need to be eventually removed. However, the brackets cannot be arbitrarily removed. The late 18th century work of [2], a professor of mathematics at the University of Edinburgh, highlights two rules regarding the removal of brackets. The first is discussed here within the context of addition and subtraction and next is discussed thereafter within the context of multiplication and division:

> Rule (i) for addition and subtraction: "If any number of quantities with the signs + or – occur in a bracket, the bracket may be removed, all the signs remaining the same if + precede the bracket, each + being changed into a – and each – into a + if – precede the bracket." [2].

The above rule has been clearly embedded in the distributive law to evaluate the product of two expressions as in the distributive law for addition and subtraction which states that:

> "The product of two expressions, each of which contains a chain of addition and subtractions, is equal to the chain of additions and subtractions obtained by multiplying each constituent of the first expression by each constituent of the second, setting down all the partial products thus obtained, and prefixing the + sign if the two constituents previously had like signs, the – sign if the constituent previously had unlike signs." [2].

The second rule pertaining to multiplication and division is:

> Rule (ii) for division and multiplication: "when a bracket contains a chain of multiplications and divisions, the brackets may be removed, every sign being unchanged if x precede the bracket, and every sign being reversed if ÷ precede the bracket." [2].

The discussion around brackets, so far, pertained to expressions that already have brackets as part of their structure. The discussion now shifts to the second use of brackets where brackets may be inserted into an expression to emphasize which operation in the expression is to be executed first. The decision around which operation is to be executed first is most often guided by a particular convention with regards to order of operations [12]. With regards to expressions involving addition and subtraction, the insertion of the brackets should be done in such a manner that it is not in conflict with rule (i) or by implication the reverse thereof. Errors made while inserting brackets into expressions has been the subject of much research. In their study exploring brackets, [9] found that students do not seem to understand that brackets cannot be inserted arbitrarily in expressions with them frequently and incorrectly evaluating, for example, the expression $19 - 3 + 6$ as being equal to $19 - (3 + 6) = 10$ instead of $19 - (3 - 6) = 22$.

The insertion of brackets in expressions involving multiplication and division, however, is not well researched [13]. One study [14], that explored three termed expressions involving multiplication followed by division, found that students simplified expressions like $3 \times 8 \div 4$ to equate to 6 by either working left to right, or multiplication before division or

3 x (8÷4) = 3 x 2 = 6 (division before multiplication). The fact that the single answer of 6 is arrived at, irrespective of the convention used lies in the format of the expression. The format of the expression is of the form where multiplication is followed by division.

It should be noted that in the current study, the expression is of the form where multiplication is preceded by division. In a study involving a similar type of expression, [15] found that college students who use the mnemonic PEMDAS tended to work out the multiplication before the division because of the order in which the letters of the acronym are written. The example that was discussed was 6 ÷ 2 x 3, where many students came up with the answer being 1, by undertaking the multiplication first, which was deemed to be incorrect. The shortcoming in that study was that no reason was given as to why the answer of 1 was considered to be incorrect, as it was just mentioned that the correct answer expected was 9. The study would have made a more valuable contribution if it put into perspective the mathematical rule that was transgressed or incorrectly applied, that offers an explanation as to why 1 was considered to be an incorrect answer.

Having reviewed the extant literature, it is noted that for expressions involving multiplication and division there is a gap in the literature regarding the rules that need to be applied when inserting brackets into expressions involving multiplication and division.

VI. Discussion

For the sake of argument, let's assume that brackets can be arbitrarily inserted into expressions involving multiplication and division and we have the following two expressions to be evaluated: the original expression 8 ÷ 2 x 4 and another unrelated expression 8 ÷ 2 ÷ 4. It was already discussed that the evaluation of the original expression could be perceived to have two possible correct answers, 1 and 16. Similarly, the latter expression could be perceived to also have two possible answers, depending on whether the first division or second division operation is executed first, i.e. (8 ÷ 2) ÷ 4 = 1 or 8 ÷ (2 ÷ 4) = 16.

Since both 8 ÷ 2 x 4 and 8 ÷ 2 ÷ 4 could have the same two possible answers, then both expression should be equivalent, which is clearly not the case as expounded in the discussion that follows. If both expressions are evaluated by starting with the division operation then:

$$8 \div 2 \text{ x } 4 = (8 \div 2) \text{ x } 4 = 16 \quad (1)$$

and

$$8 \div 2 \div 4 = (8 \div 2) \div 4 = 1 \quad (2)$$

Alternatively, if both expressions are evaluated by starting with the multiplication operation then:

$$8 \div 2 \text{ x } 4 = 8 \div (2 \text{ x } 4) = 1\text{: and} \quad (3)$$

$$8 \div 2 \div 4 = 8 \div (2 \div 4) = 16 \quad (4)$$

Clearly, (1) ≠ (2) and (3) ≠ (4), therefore the expressions are not equivalent. By implication, it would mean that the expressions must therefore have different answers. In other words if the answer to one expression is 1 then the answer to the other has to be 16, and vice versa. So the answer to the original expression 8 ÷ 2 x 4 has to be either 1 or 16, but not both.

The foregoing discussions shifts the understanding from the claim that both 1 and 16 are the correct answers for the evaluation of the expression 8 ÷ 2 x 4 to the understanding that only of the two possible answers could be considered as being correct. However, it falls short of determining exactly which one of the two is definitely the correct answer. In order to address this shortcoming, the discussion now focusses on rules pertaining to the insertion of brackets into expressions.

When evaluating expressions involving addition and subtraction, brackets are not arbitrarily inserted into the expression and therefore the same should apply when evaluating expressions involving multiplication and division. For example, 8−2+4 can be evaluated by inserting brackets in combination with the reverse of rule (i) as follows: 8 − 2 + 4 = 8 − (2 − 4) = 8 – (− 2) = 8 + 2 =10, with addition be executed first. Similarly, for the case of 8 ÷ 2 x 4, if the multiplication operation is to be executed first, brackets can be inserted to denote the multiplication is being done first. However, that which is lacking in theoretical development, is exactly how the brackets can be inserted. If one had to use the reverse of rule (ii), then: 8 ÷ 2 x 4 could be evaluated as: 8 ÷ 2 x 4 = 8 ÷ (2 ÷ 4) = 8 ÷ ½ = 16. Thus, 16 would be the answer even if the multiplication is done first. This now demonstrates that when the multiplication is done first the answer is also 16 and not 1 as claimed earlier.

While rule (i) is frequently used in the form of the distributive law or reverse distributive law, with regards to addition and subtraction, the same has not been the case for rule (ii) with regards to multiplication and division. A case therefore needs to be presented, that extends beyond a convincing argument for the application of the reverse of rule (ii). The proof of the reverse of rule (ii) for three termed expressions involving multiplication and division is as follows:

$$\mathbf{a \div b \text{ x } c} = a \text{ x } 1/b \text{ x } c = a \text{ x } c \text{ x } 1/b = a \text{ x } (c/b) = a \div (b/c) = \mathbf{a \div (b \div c)} \quad (5)$$

$$\mathbf{a \div b \div c} = a \text{ x } 1/b \text{ x } 1/c = a \text{ x } 1/(b \text{ x } c) = \mathbf{a \div (b \text{ x } c)} \quad (6)$$

$$\mathbf{a \text{ x } b \div c} = a \text{ x } b \text{ x } 1/c = a \text{ x } (b \text{ x } 1/c) = \mathbf{a \text{ x } (b \div c)} \quad (7)$$

where a, b and c are elements of real numbers, (b, c ≠ 0)

The foregoing proofs thus can be generalized as the rule:

> "When an expression contains a chain of multiplications and divisions, brackets may be inserted, with every sign being unchanged if x precede the inserted bracket, and every sign being reversed if ÷ precede the inserted bracket".

The law formulated above is considered to be an extension of the work of [2] and is therefore named: "Chrystal's reverse law for inserting brackets in multiplication and division expressions"

To avert the over proliferation of rules, caution needs to be exercised over the balance of the problems that the rule can solve versus the problems that it can create. Hence, when a rule is proposed there needs to be appropriate justification for the rule. Table I depicts how the proposed rule measures against criteria that [16] suggest for rule justification.

TABLE I
RULE JUSTIFICATION

Justification	How does the rule measure
A rule should facilitate productivity and effectiveness	The law would do the same as the reverse distributive law does for multiplication over addition.
A rule should ensure a richer system with it than one without it.	The law provides the mechanism that facilitates the different conventions to be correctly applied.
A rule must be proved to be true within the rules of the system within which it is meant to operate.	The proofs of the laws have been provided in (5)-(7).

With regards to the justification regarding productivity and effectiveness, the rule scores well on both points. It is more productive to have a clear rule the supports a common methodology. It would be more effective (doing the right thing) as the correct methodology would be employed. With regards to the justification relating to a richer system, the rule does not violate any of the existing conventions, but is rather synergetic facilitating the different conventions to be applied to arrive at the correct answer. In other words, it does matter which one of the conventions (BODMAS, BIDMAS, BEDMAS PEDMAS, PEMDAS) or left to right is used, but when applied together with the rule will result in only the one correct answer. Furthermore, the answer arrived at would be the same even if the multiplication is done before the division and irrespective of whether the expression is of the type where division precedes multiplication or vice versa.

VII. CONCLUSION

This article problematized the notion that more than one correct answer is possible in an expression involving multiplication and division, by virtue of whether the multiplication or division is executed first. It was proved that the perception that two different answers can be possible when different conventions are used was incorrect. The false claim of the duality of answers was attributed to brackets being arbitrarily inserted into the expression. It was demonstrated that the insertion of brackets into expressions involving multiplication and division needs to be done in combination with the reverse of Chrystal's rule regarding multiplication and division. Chrystal's reverse law therefore makes it possible to preserve the equal precedence of multiplication and division irrespective of the conventions used. It is recommended that rule (ii) and its reverse be taught alongside rule (i) when order of operations is being taught. Further research is required to ascertain the extent of prevalence of the arbitrary insertion of brackets into expression involving multiplication and division, among both teachers and students.

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