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Derivation of Fractional Black-Scholes Equations Driven by Fractional G-Brownian Motion and Their Application in European Option Pricing

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Abstract—In this paper, fractional Black-Scholes models for the European option pricing were established based on the fractional G-Brownian motion (fGBm), which generalizes the concepts of the classical Brownian motion, fractional Brownian motion and the G-Brownian motion, and that can be used to be a tool for considering the long range dependence and uncertain volatility for the financial markets simultaneously. A generalized fractional Black-Scholes equation (FBSE) was derived by using the Taylor's series of fractional order and the theory of absence of arbitrage. Finally, some explicit option pricing formulas for the European call option and put option under the FBSE were also solved, which extended the classical option pricing formulas given by F. Black and M. Scholes.

Keywords—European option pricing, fractional Black-Scholes equations, fractional G-Brownian motion, Taylor's series of fractional order, uncertain volatility.

I. Introduction

SINCE the celebrated Black-Scholes (BS) model was proposed to price the stock options, the well-known Black-Scholes equation and formulas have become the most popular methods for pricing the options and other financial derivatives [1]. Meanwhile, some fractal structures of the financial markets have been discovered in recent years, and thus some fractional calculus were applied in the fields to extend the mathematical finance theory [2]. Thus combining the classical BS equation, some fractional BS equation were established for the option pricing. Wyss firstly deduced the time-fractional BS equation to price the European call option and then gave the complete solution of the equation by Laplace and Mellin transforms [3]. Cartea and del-Castillo-Negrete obtained some fractional partial differential equation to model the option prices in markets with jumps and priced the barrier option[4]. Later, Jumarie adopted the Maruyama's notation and selected the dynamical equation for the stock value $x(t)$ as

$$dx = \mu x dt + \sigma x db(t,\alpha) = \mu x dt + \sigma x \omega(t)(dt)^{\alpha}, \quad (1)$$

where $x = x(t)$ and $0 < \alpha < 1$, then applied the Taylor's series of fractional order to derive the following fractional BS models for stock exchange dynamics [5],

$$\begin{aligned}\frac{\partial^{\alpha} P}{\partial t^{\alpha}} &= \left[\frac{r}{(1-\alpha)!}P - rx^{\alpha}\frac{\partial^{\alpha} P}{\partial x^{\alpha}}\right]t^{1-\alpha} \\ &\quad -\frac{(\alpha!)^3[(1-\alpha)!]^2}{\Gamma(1+2\alpha)}\sigma^2 x^{2\alpha}\frac{\partial^{2\alpha} P}{\partial x^{2\alpha}}.\end{aligned} \quad (2)$$

After that, Jumarie promoted his previous work and gave out an optimal fractional Mertons portfolio, which has wider applications in the actual financial market [6]. In 2010, Liang, $et\,al.$ [7], [8] derived some bi-fractional Black-Scholes-Merton models for the option pricing with the assumption the stock price $S(t)$ satisfying the fractional exponential equation

$$(dS)^{2H} = \mu S^{2H}(dt)^{2H} + \sigma S^{2H} dB_H(t), \quad 0 < H < 1, \quad (3)$$

and they obtained the analytical solutions for these models with the help of the Laplace transform and Fourier transform technique. Recently, Chen $et\,al.$ also derived European-style spatial-fractional BS equation under the finite moment log stable(FMLS) model and found some explicit closed-form analytical solutions for the model [9]. For other models of the time/spatial-fractional BS equations and their analytical or numerical solutions, we refer readers to [10], [11], [12], [13], and reference therein. However, whether the classical BS equation or the above mentioned fractional BS models, they are under the same assumptions that the dynamical equation for the stock value follows as a stochastic differential equation, where the source of randomness were taken as the classical Brownian motion(Bm) or the fractional Brownian motion(fBm), as seen in (1) and (3). Although compared to the classical Bm, the fBm has more advantages in describing the long-range dependency or persistence in the financial markets[14], they are still not completely consistent with the actual stock movement. One of the incompatibilities is not considering the uncertainty, especially the volatility uncertainty for the stock price. For considering the volatility uncertainty, Peng proposed a formal mathematical approach under the framework of nonlinear expectation and the related G-Brownian motion (GBm) in some sublinear space $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ [15], [16], [17], [18]. Under the nonlinear expectation space framework, Chen and Epstein obtained a time consistent G-expectation bid-ask dynamic pricing mechanism for the European contingent claim in the uncertainty financial market [19]. Epstein and Ji also studied the utility uncertainty application in economics [20]. Thus in order to make full use of the advantages of long-range dependency from the fGm and the volatility uncertainty from GBm, we

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introduced a generalized concept called fractional G-Brownian motion(fGBm), which generalize the concepts of the classical Bm, fBm and GBm [21], [22]. And we believe that the fGBm is more suitable to capture the intrinsic characteristics of the financial markets, especially in considering the long-range dependency and uncertainty simultaneously. Thus in this paper, we are going to consider that the dynamics of the stock prices follows a fractional stochastic differential equation, where the source of randomness was driven by the generalized fGBm, and propose some fractional Black-Scholes models for the European option pricing. The main techniques are the fractional order Taylor's formulas in fractional calculus and the stochastic analysis for the fGBm.

The rest of paper is organized as follows. In Section II, we summarize some results of the related fractional calculus, especially including the Taylor's series of fractional order, and recall some for preliminaries for the fGBm. In Section III, we will derive the time spatial fractional Black-Scholes equations (FBSE)(49) driven by the fGBm, which generalizes the obtained fractional models. In Section IV, we obtain the explicit option pricing formulas for the European call option and put option governed by the FBSE (49), respectively. The last Section V offers the conclusion.

II. Some Preliminaries

In this section, we first recall some preliminaries about the fractional calculus [23], Taylor's series of fractional order[24], [25], [26], and fractional G-Brownian motion [21], [22].

A. Fractional Calculus and Taylor's Series of Fractional Order

Definition 1: (Riemann-Liouville fractional integral) Let $f(x)$ be piecewise continuous on $(0,\infty)$ and integrable on any finite subinterval of $[0,\infty)$. Then for $x>0$ we call

$$I^\alpha f(x)=\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}f(t)dt,\quad \alpha>0, \tag{4}$$

the Riemann-Liouville fractional integral of $f(x)$ of order α, where $\Gamma(\alpha)$ is the Gamma function with $\Gamma(\alpha)=\int_0^\infty x^{\alpha-1}\exp(-x)dx$.

Definition 2: (Riemann-Liouville fractional derivative) Let $f(x)$ be a function in Definition 1 and let $\alpha>0$. Let m be the smallest integer that exceeds α. Then the Riemann-Liouville fractional derivative of $f(x)$ of order α is defined as

$$\begin{aligned}{}_0^R D^\alpha f(x) &= \tfrac{d^m}{dx^m}\left[I^{m-\alpha}f(x)\right]\\ &= \tfrac{1}{\Gamma(m-\alpha)}\tfrac{d^m}{dx^m}\left[\int_0^x (x-t)^{m-\alpha-1}f(t)dt\right].\end{aligned} \tag{5}$$

Definition 3: (Caputo fractional derivative) Let $f(x)$ be a function in Definition 1 and let $\alpha>0$. Let m be the smallest integer that exceeds α. Then the Caputo fractional derivative of $f(x)$ of order α is defined as

$$\begin{aligned}{}_0^C D^\alpha f(x) &= I^{m-\alpha}\left[\tfrac{d^m}{dx^m}f(x)\right]\\ &= \tfrac{1}{\Gamma(m-\alpha)}\int_0^x (x-t)^{m-\alpha-1}f^{(m)}(t)dt.\end{aligned} \tag{6}$$

Here we need to point out that the definitions for the fractional derivative of $f(x)$ in Definition 2 and Definition 3 are not equivalent. Since the Caputo fractional derivative has "good physical properties" as, for example that the derivative of a constant is zero or that Cauchy problems requires initial conditions formulated in terms of integer order derivatives interpreted as initial position, initial velocity, etc[23]. Thus in this paper we mainly adopt the Caputo fractional derivative, and for convenience we denote

$$D^\alpha f(x)=f^{(\alpha)}(x)=\frac{d^\alpha f(x)}{dx^\alpha}={}_0^C D^\alpha f(x), \tag{7}$$

and

$$D_x^\alpha f(x,y)=f_x^{(\alpha)}(x,y)=\frac{d^\alpha f(x,y)}{dx^\alpha}={}_0^C D^\alpha f(x,y), \tag{8}$$

for the fractional derivative and the fractional partial derivative, respectively.

Definition 4: Let $f:\mathbb{R}\to\mathbb{R}, x\to f(x)$ denote a continuous (but not necessarily differentiable) function and let $h>0$ denote a constant discretization span. Define the forward operator $FW(h)$, i. e.

$$FW(h)f(x):=f(x+h),$$

then the fractional difference of order $\alpha, 0<\alpha\le 1$, of $f(x)$ is defined by the expression

$$\begin{aligned}\Delta f(x) &:= (FW-1)^\alpha f(x)\\ &= \sum_{k=0}^{\infty}(-1)^k\tbinom{\alpha}{k}f[x+(\alpha-k)h].\end{aligned} \tag{9}$$

Based on the definition of the forward operator, there holds the following results, whose proofs can be seen in [5], [24], [25], [26], and reference therein.

Lemma 1: The following equality holds,

$$f^\alpha(x)=\lim_{h\downarrow 0}\frac{\Delta f(x)}{h^\alpha}. \tag{10}$$

Lemma 2: We assume that $f(x):\mathbb{R}\to\mathbb{R}$ is the continuous function and has fractional derivative of order α, $0<\alpha\le 1$. Then the following important relations hold,

$$\Delta^\alpha f\approx\Gamma(1+\alpha)\Delta f, \tag{11}$$

$$\Delta^\alpha f=\frac{1}{\Gamma(2-\alpha)}f^{1-\alpha}(\Delta f)^\alpha, \tag{12}$$

or in continuous form

$$d^\alpha f\approx\Gamma(1+\alpha)df, \tag{13}$$

$$d^\alpha f=\frac{1}{\Gamma(2-\alpha)}f^{1-\alpha}(df)^\alpha. \tag{14}$$

Proposition 1: (Taylor's series of fractional order) Assume that the continuous function $f:\mathbb{R}\to\mathbb{R}, x\to f(x)$ has fractional derivative of order $k\alpha$, for any positive integer k and any $\alpha, 0<\alpha\le 1$, then the following equality holds, which is

$$f(x+h)=\sum_{k=0}^{\infty}\frac{h^{\alpha k}}{\Gamma(1+\alpha k)}f^{(\alpha k)}(x),\quad 0<\alpha\le 1, \tag{15}$$

where $f^{(\alpha k)}(x)$ is the derivative of order αk of $f(x)$.

Proposition 2: (Multivariable fractional Taylor's series) Assume that the continuous function $f:\mathbb{R}\times\mathbb{R}\to\mathbb{R},(x,y)\to f(x,y)$ has fractional partial derivative of order $k\alpha$. Then for any $\alpha, 0<\alpha\leq 1$, one has the series

$$\begin{aligned} f(x+h,y+l) &= E_\alpha(h^\alpha D_x^\alpha)E_\alpha(l^\alpha D_y^\alpha)f(x,y) \\ &= E_\alpha(l^\alpha D_y^\alpha)E_\alpha(h^\alpha D_x^\alpha)f(x,y) \\ &= E_\alpha\{(lD_x+lD_y)^\alpha\}f(x,y), \end{aligned} \tag{16}$$

where $E_\alpha(x)$ denotes the Mittag-Leffler functions defined by the expression

$$E_\alpha(x):=\sum_{k=0}^{\infty}\frac{x^k}{\Gamma(1+\alpha k)}. \tag{17}$$

If taking the approximation of order 2α, one has

$$\begin{aligned} &f(x+h,y+l) \\ &\approx f(x,y)+\tfrac{1}{\Gamma(1+\alpha)}\left(f_x^{(\alpha)}(x,y)h^\alpha+f_y^{(\alpha)}(x,y)l^\alpha\right) \\ &\quad+\tfrac{1}{2\Gamma(1+\alpha)}\left(f_x^{(2\alpha)}(x,y)h^{2\alpha}+f_y^{(2\alpha)}(x,y)l^{2\alpha}\right) \\ &\quad+\tfrac{1}{[\Gamma(1+\alpha)]^2}f_{xy}^{(2\alpha)}(x,y)h^\alpha l^\alpha. \end{aligned} \tag{18}$$

Lemma 3: (Integration with respect to $(dt)^\alpha$) Let $f(t)$ denote a continuous $\mathbb{R}\to\mathbb{R}$ function. Then its integral with respect to $(dt)^\alpha, 0<\alpha\leq 1$ is defined by

$$\int_0^t f(\tau)(d\tau)^\alpha=\alpha\int_0^t(t-\tau)^{\alpha-1}f(\tau)d\tau. \tag{19}$$

If taking $f(\tau)=\tau^\gamma$ for special case, one obtains

$$\int_0^t \tau^\gamma(d\tau)^\alpha=\frac{\Gamma(1+\alpha)\Gamma(1+\gamma)}{\Gamma(1+\alpha+\gamma)}t^{\alpha+\gamma}. \tag{20}$$

B. Fractional G-Brownian Motion

In this subsection, we recall some results for the sublinear expectation space and fGBm from [21], [22] that we will need. For other preliminaries about the GBm, we refer readers to [15], [16], [17], [18], and reference therein.

Definition 5: (Sublinear expectation) A sublinear expectation $\widehat{\mathbb{E}}$ is a functional $\widehat{\mathbb{E}}:\mathcal{H}\to\mathbb{R}$ satisfying

(1) [Monotonicity] $\widehat{\mathbb{E}}[X]\geq\widehat{\mathbb{E}}[Y]$ if $X\geq Y$.

(2) [Constant preserving] $\widehat{\mathbb{E}}[c]=c$ for $c\in\mathbb{R}$.

(3) [Sub-additivity] For each $X,Y\in\mathcal{H}$, $\widehat{\mathbb{E}}[X+Y]\leq\widehat{\mathbb{E}}[X]+\widehat{\mathbb{E}}[Y]$.

(4) [Positive homogeneity] $\widehat{\mathbb{E}}[\lambda X]=\lambda\widehat{\mathbb{E}}[X]$ for $\lambda\geq 0$.

Then the triple $(\Omega,\mathcal{H},\widehat{\mathbb{E}})$ is called a sublinear expectation space.

Definition 6: (Distribution) Let $X=(X_1,\cdots,X_n)$ be a given n-dimensional random vector in a sublinear expectation space $(\Omega,\mathcal{H},\widehat{\mathbb{E}})$. We define a functional on $C_{l,Lip}(\mathbb{R}^n)$ by

$$\mathbb{F}_X[\varphi]=\widehat{\mathbb{E}}[\varphi(X)]:\ \varphi\in C_{l,Lip}(\mathbb{R}^n)\to\mathbb{R}.$$

Then $\mathbb{F}_X$ is called the distribution of X under $\widehat{\mathbb{E}}$.

Remark 1: For a random variable X on the sublinear expectation space $(\Omega,\mathcal{H},\widehat{\mathbb{E}})$, the distribution of X has the following four typical parameters

$$\underline{\mu}=-\widehat{\mathbb{E}}[-X],\ \overline{\mu}=\widehat{\mathbb{E}}[X],\ \underline{\sigma}^2=-\widehat{\mathbb{E}}[-X^2],\ \overline{\sigma}^2=\widehat{\mathbb{E}}[X^2],$$

where $[\underline{\mu},\overline{\mu}]$ and $[\underline{\sigma}^2,\overline{\sigma}^2]$ characterize the mean-uncertainty and the variance-uncertainty of X respectively.

Definition 7: (Identically distributed) Let X_1 and X_2 be two random variables defined in sublinear expectation space $(\Omega_1,\mathcal{H}_1,\widehat{\mathbb{E}}_1)$ and $(\Omega_2,\mathcal{H}_2,\widehat{\mathbb{E}}_2)$. They are called identically distributed, denoted by $X_1\overset{d}{=}X_2$, if

$$\widehat{\mathbb{E}}_1[\varphi(X_1)]=\widehat{\mathbb{E}}_2[\varphi(X_2)]\ \text{ for } \varphi\in C_{l,Lip}(\mathbb{R}).$$

Definition 8: (Independent) In the sublinear expectation space $(\Omega,\mathcal{H},\widehat{\mathbb{E}})$, a random variable Y is said to be independent from another random variable $X\in\mathcal{H}$ under $\widehat{\mathbb{E}}$), if for each function $\varphi\in C_{l,Lip}(\mathbb{R}^2)$,

$$\widehat{\mathbb{E}}[\varphi(X,Y)]=\widehat{\mathbb{E}}\left[\widehat{\mathbb{E}}[\varphi(x,Y)]\,|_{x=X}\right].$$

Furthermore, Y is called an independent copy of X if $Y\overset{d}{=}X$ and Y is independent from X.

Definition 9: (G-normal distribution) A random variable X in a sublinear expectation space $(\Omega,\mathcal{H},\widehat{\mathbb{E}})$ is called G-normal distributed if

$$aX+b\overline{X}\overset{d}{=}\sqrt{a^2+b^2}X,\quad \text{for } a,b\geq 0,$$

where $\overline{X}$ is an independent copy of X.

Remark 2: It is easy to find that if X is G-normal distributed, then

$$\underline{\mu}=-\widehat{\mathbb{E}}[-X]=\overline{\mu}=\widehat{\mathbb{E}}[X]=0,$$

and thus we denote the G-normal distribution as $X\sim\mathcal{N}\left(\{0\},[\underline{\sigma}^2,\overline{\sigma}^2]\right)$. And here, we point out that the variance uncertainty of the G-normal distribution is the source of the volatility uncertainty of the stocks' price processes.

Definition 10: (G-Brownian motion)[18] A stochastic process $B_G(t)_{t\in\mathbb{R}^+}$ in a sublinear expectation space $(\Omega,\mathcal{H},\widehat{\mathbb{E}})$ is called a G-Brownian motion (GBm) if the following properties are satisfied:

(1) $B_G(0)=0$,

(2) For each $t,s\geq 0$, the increment $B_G(t+s)-B_G(t)$ is G-normal distributed by $\mathcal{N}\left(\{0\},[s\underline{\sigma}^2,s\overline{\sigma}^2]\right)$, where $\underline{\sigma}^2=-\widehat{\mathbb{E}}[-B_G^2(1)]$ and $\overline{\sigma}^2=\widehat{\mathbb{E}}[B_G^2(1)]$, and is independent from $(B_G(t_1),B_G(t_2),\cdots,B_G(t_n))$, for each $n\in\mathbb{N}$ and $0\leq t_1\leq\cdots\leq t_n\leq t$.

In order to consider the long-range dependency and uncertainty in the financial markets simultaneously, we introduces a generalized concept called fractional G-Brownian motion(fGBm), which was defined as follows.

Definition 11: (Fractional G-Brownian motion)[21], [22] Let $H\in(0,1)$, a continuous stochastic process $B_{FG}(t)_{t\in\mathbb{R}^+}$ in a sublinear expectation space $(\Omega,\mathcal{H},\widehat{\mathbb{E}})$ is called fractional G-Brownian motion (fGBm) with Hurst index H if

(1) $B_{FG}(0)=0$, and for all $t\geq 0$

$$-\widehat{\mathbb{E}}[-B_{FG}(t)]=\widehat{\mathbb{E}}[B_{FG}(t)]=0. \tag{21}$$

(2) For all $s,t\geq 0$, there holds

$$\begin{aligned} &\widehat{\mathbb{E}}[B_{FG}(t)B_{FG}(s)]=\tfrac{1}{2}\overline{\sigma}^2\left(t^{2H}+s^{2H}-|t-s|^{2H}\right), \\ &-\widehat{\mathbb{E}}[-B_{FG}(t)B_{FG}(s)]=\tfrac{1}{2}\underline{\sigma}^2\left(t^{2H}+s^{2H}-|t-s|^{2H}\right), \end{aligned} \tag{22}$$

where $\underline{\sigma}^2=-\widehat{\mathbb{E}}[-B_{FG}^2(1)]$ and $\overline{\sigma}^2=\widehat{\mathbb{E}}[B_{FG}^2(1)]$.

(3) For each $t, s \geq 0$, the increment $B_{FG}(t+s) - B_{FG}(s)$ is identically distributed with $B_{FG}(t)$.

Remark 3: The fGBm $B_{FG}(t)_{t\in\mathbb{R}^+}$ generalizes the concepts of the classical Brownian motion $B(t)_{t\in\mathbb{R}^+}$ [27], fractional Brownian motion $B_H(t)_{t\in\mathbb{R}^+}$ [28] and the G-Brownian motion $B_G(t)_{t\in\mathbb{R}^+}$ [18]. And it can posses the long-range dependence property and consider the volatility uncertainty simultaneously.

Proposition 3: [21], [22] For the fGBm $B_{FG}(t)_{t\in\mathbb{R}^+}$, there holds

$$-\widehat{\mathbb{E}}\left[-B_{FG}(t)\right] = \widehat{\mathbb{E}}\left[B_{FG}(t)\right] = 0, \tag{23}$$

$$-\widehat{\mathbb{E}}\left[-B_{FG}^2(t)\right] = \underline{\sigma}^2 t^{2H}, \quad \widehat{\mathbb{E}}\left[B_{FG}^2(t)\right] = \overline{\sigma}^2 t^{2H}. \tag{24}$$

And also the fGBm has the self-similarity property as

$$a^{-H}B_{FG}(at) \overset{d}{=} B_{FG}(t), \text{ for } a > 0. \tag{25}$$

Based on the Definition 11 and Proposition 3, we can extend the Maruyama's notation for the fGBm as follows, which plays an important role in the derivation of the fractional Black-Scholes models.

Proposition 4: Extending the Maruyama's notation for fGBm $B_{FG}(t)$, we introduce and use the expression

$$dB_{FG}(t) = \widetilde{\omega}(t)(dt)^H, \tag{26}$$

where $0 < H < 1$ and $\widetilde{\omega}(t)$ is G-normal distributed as $\widetilde{\omega}(t) \sim \mathcal{N}\left(\{0\}, [\underline{\sigma}^2, \overline{\sigma}^2]\right)$, with $\underline{\sigma}^2 = -\widehat{\mathbb{E}}[-\widetilde{\omega}(t)^2]$ and $\overline{\sigma}^2 = \widehat{\mathbb{E}}[\widetilde{\omega}(t)^2]$. This also provides the mathematical expectation and the variance as

$$\widehat{\mathbb{E}}[dB_{FG}(t)] = 0, \quad \mathrm{Var}[dB_{FG}(t)] = \widehat{\mathbb{E}}[\widetilde{\omega}(t)^2](dt)^{2H}. \tag{27}$$

Remark 4: This expression (26) is considered to be as a formal definition, and moreover it is an approximation only.

III. Derivation of the Fractional BS Equations

In this section, we will derive the time spatial fractional Black-Scholes equations for the European option pricing driven by the fGBm. First we suppose that the dynamics of stock price $S(t)$ follows as a fractional stochastic differential equation

$$d^\alpha S = \mu(t,S)dt^\alpha + \sigma(t,S)dB_{FG}(t), \tag{28}$$

where $0 < \alpha \leq 1, 0 < H < 1$, $\mu(t,S)$ is the drift term, $\sigma(t,S)$ is the diffusion term, $dB_{FG}(t)$ is the fractional G-Brownian process, as $dB_{FG}(t) = \widetilde{\omega}(t)(dt)^H$. For a special case that $\mu(t,S) = \mu S$ and $\sigma(t,S) = \sigma S$, then (28) becomes as

$$d^\alpha S = \mu S dt^\alpha + \sigma S dB_{FG}(t). \tag{29}$$

First from Lemma 2, there arrives directly

$$dS \approx \frac{1}{\Gamma(1+\alpha)} d^\alpha S, \tag{30}$$

and

$$(dS)^\alpha = \Gamma(2-\alpha)S^{\alpha-1}d^\alpha S. \tag{31}$$

Now let $P = P(S,t)$ be the value of an option on a stock at the time t, and according to the multivariable fractional Taylor's series (18) in Proposition 2, we have

$$\begin{aligned}\Delta P &= P(S+\Delta S, t+\Delta S) - P(S,t)\\ &= \tfrac{1}{\Gamma(1+\alpha)}\left[P_S^{(\alpha)}(\Delta S)^\alpha + P_t^{(\alpha)}(\Delta t)^\alpha\right]\\ &\quad + \tfrac{1}{\Gamma(1+2\alpha)}\left[P_S^{(2\alpha)}(\Delta S)^{2\alpha} + P_t^{(2\alpha)}(\Delta t)^{2\alpha}\right]\\ &\quad + \tfrac{1}{[\Gamma(1+\alpha)]^2}P_{St}^{(2\alpha)}(\Delta S)^\alpha(\Delta t)^\alpha + \cdots,\end{aligned} \tag{32}$$

which can be approximated as

$$\begin{aligned}dP &\approx \tfrac{1}{\Gamma(1+\alpha)}\left[P_S^{(\alpha)}(dS)^\alpha + P_t^{(\alpha)}(dt)^\alpha\right]\\ &\quad + \tfrac{1}{\Gamma(1+2\alpha)}\left[P_S^{(2\alpha)}(dS)^{2\alpha} + P_t^{(2\alpha)}(dt)^{2\alpha}\right]\\ &\quad + \tfrac{1}{[\Gamma(1+\alpha)]^2}P_{St}^{(2\alpha)}(dS)^\alpha(dt)^\alpha.\end{aligned} \tag{33}$$

Now from (29)-(31), there holds

$$\begin{aligned}(dS)^{2\alpha} &= [(dS)^\alpha]^2\\ &= \left[\Gamma(2-\alpha)S^{\alpha-1}d^\alpha S\right]^2\\ &= [\Gamma(2-\alpha)]^2 S^{2\alpha-2}\left[\mu S dt^\alpha + \sigma S dB_{FG}(t)\right]^2\\ &= [\Gamma(2-\alpha)]^2 S^{2\alpha-2}\\ &\quad \cdot\left[\sigma^2 S^2 \widehat{\mathbb{E}}[\widetilde{\omega}^2(t)](dt)^{2H} + O((dt)^\alpha)\right]\\ &\approx [\Gamma(2-\alpha)]^2 S^{2\alpha}\sigma^2\widehat{\mathbb{E}}[\widetilde{\omega}^2(t)](dt)^{2H}.\end{aligned} \tag{34}$$

Substituting (34) into (33) and taking the approximation of order $(dt)^{2\alpha}$, we obtain that if $2H > \alpha$,

$$dP = \frac{1}{\Gamma(1+\alpha)}P_S^{(\alpha)}(dS)^\alpha + \frac{1}{\Gamma(1+\alpha)}P_t^{(\alpha)}(dt)^\alpha, \tag{35}$$

and if $0 < 2H \leq \alpha$,

$$\begin{aligned}dP &= \tfrac{1}{\Gamma(1+\alpha)}P_S^{(\alpha)}(dS)^\alpha + \tfrac{1}{\Gamma(1+\alpha)}P_t^{(\alpha)}(dt)^\alpha\\ &\quad + \tfrac{[\Gamma(2-\alpha)]^2}{\Gamma(1+2\alpha)}P_S^{(2\alpha)}S^{2\alpha}\sigma^2\widehat{\mathbb{E}}[\widetilde{\omega}(t)^2](dt)^{2H}.\end{aligned} \tag{36}$$

In what follows, we mainly discuss the case $0 < 2H \leq \alpha$. First from (13) and (14), we have

$$(dt)^\alpha = \Gamma(2-\alpha)t^{\alpha-1}d^\alpha t \approx \Gamma(1+\alpha)\Gamma(2-\alpha)t^{\alpha-1}dt, \tag{37}$$

and substitute this equation into (36), we have

$$\begin{aligned}dP &= \tfrac{1}{\Gamma(1+\alpha)}P_S^{(\alpha)}(dS)^\alpha\\ &\quad + \tfrac{1}{\Gamma(1+\alpha)}\Gamma(1+\alpha)\Gamma(2-\alpha)t^{\alpha-1}P_t^{(\alpha)}dt\\ &\quad + \tfrac{[\Gamma(2-\alpha)]^2}{\Gamma(1+2\alpha)}S^{2\alpha}\sigma^2\widehat{\mathbb{E}}[\widetilde{\omega}(t)^2]\Gamma(1+2H)\Gamma(2-2H)\\ &\quad \cdot t^{2H-1}P_S^{(2\alpha)}dt.\end{aligned} \tag{38}$$

Multiplying both side of (38) by $\Gamma(1+\alpha)$ yields that

$$\begin{aligned}d^\alpha P &= P_S^{(\alpha)}(dS)^\alpha + \Big[\Gamma(1+\alpha)\Gamma(2-\alpha)t^{\alpha-1}P_t^{(\alpha)}\\ &\quad + \tfrac{\Gamma(1+\alpha)[\Gamma(2-\alpha)]^2\Gamma(1+2H)\Gamma(2-2H)\sigma^2}{\Gamma(1+2\alpha)}\widehat{\mathbb{E}}[\widetilde{\omega}(t)^2]\\ &\quad \cdot t^{2H-1}S^{2\alpha}P_S^{(2\alpha)}\Big]dt.\end{aligned} \tag{39}$$

Meanwhile, it is possible to form a portfolio of stock and options to offset the randomness, and here suppose that is *long* λ shares and *short* 1 option. Thus the value V of the portfolio at the time t is

$$V(t) = \lambda S(t) - P(t), \tag{40}$$

where the minus sign is from the fact that the option is owed rather than owned. Furthermore, the value of the portfolio changes as

$$dV = \lambda dS - dP. \tag{41}$$

Multiplying both side of (41) by $\Gamma(1+\alpha)$ and combining (31) and (39), we have

$$\begin{aligned} d^\alpha V &= \lambda d^\alpha S - d^\alpha P \\ &= \left[\lambda \tfrac{1}{\Gamma(2-\alpha)} S^{1-\alpha} - P_S^{(\alpha)}\right] (dS)^\alpha \\ &\quad - \Big[\Gamma(1+\alpha)\Gamma(2-\alpha)t^{\alpha-1}P_t^{(\alpha)} \\ &\quad + \tfrac{\Gamma(1+\alpha)[\Gamma(2-\alpha)]^2\Gamma(1+2H)\Gamma(2-2H)\sigma^2}{\Gamma(1+2\alpha)} \widehat{\mathbb{E}}[\widetilde{\omega}(t)^2] \\ &\quad \cdot t^{2H-1} S^{2\alpha} P_S^{(2\alpha)}\Big] dt. \end{aligned} \tag{42}$$

As one can see, the random term included in $(dS)^\alpha$ can be disappeared by choosing some suitable λ such that the coefficient becomes to zero, that is

$$\frac{\lambda}{\Gamma(2-\alpha)} S^{1-\alpha} - P_S^{(\alpha)} = 0, \tag{43}$$

which implies

$$\lambda = \Gamma(2-\alpha) S^{\alpha-1} P_S^{(\alpha)}. \tag{44}$$

And this leaves

$$\begin{aligned} d^\alpha V &= -\Big[\Gamma(1+\alpha)\Gamma(2-\alpha)t^{\alpha-1}P_t^{(\alpha)} \\ &\quad + \tfrac{\Gamma(1+\alpha)[\Gamma(2-\alpha)]^2\Gamma(1+2H)\Gamma(2-2H)\sigma^2}{\Gamma(1+2\alpha)} \widehat{\mathbb{E}}[\widetilde{\omega}(t)^2] \\ &\quad \cdot t^{2H-1} S^{2\alpha} P_S^{(2\alpha)}\Big] dt. \end{aligned} \tag{45}$$

On the other hand, the fundamental condition for establishing the price of an option is absence of arbitrage. Thus the portfolio in riskless and its value must increase in accordance with the risk-free interest rate. The interest accrued on 1 unit of money over a time interval of length dt is $1rdt$. So there holds

$$dV = rVdt = r(\lambda S - P)dt. \tag{46}$$

Multiplying both side of (46) by $\Gamma(1+\alpha)$, we obtain

$$\begin{aligned} d^\alpha V &= \Gamma(1+\alpha) r(\lambda S - P) \\ &= \Gamma(1+\alpha) r \left[\Gamma(2-\alpha) S^\alpha P_S^{(\alpha)} - P\right] dt. \end{aligned} \tag{47}$$

Equating (45) and (47) yields that

$$\begin{aligned} &-\Gamma(1+\alpha)\Gamma(2-\alpha)t^{\alpha-1}P_t^{(\alpha)} \\ &- \tfrac{\Gamma(1+\alpha)[\Gamma(2-\alpha)]^2\Gamma(1+2H)\Gamma(2-2H)\sigma^2}{\Gamma(1+2\alpha)} \widehat{\mathbb{E}}[\widetilde{\omega}(t)^2] t^{2H-1} S^{2\alpha} P_S^{(2\alpha)} \\ &= \Gamma(1+\alpha) r \left[\Gamma(2-\alpha) S^\alpha P_S^{(\alpha)} - P\right], \end{aligned} \tag{48}$$

which implies

$$\begin{aligned} P_t^{(\alpha)} &= \left[\tfrac{r}{\Gamma(2-\alpha)} P - rS^\alpha P_S^{(\alpha)}\right] t^{1-\alpha} \\ &\quad - \tfrac{[\Gamma(2-\alpha)]^2\Gamma(1+2H)\Gamma(2-2H)}{\Gamma(1+2\alpha)} \widehat{\mathbb{E}}[\widetilde{\omega}(t)^2] \\ &\quad \cdot t^{2H-\alpha}\sigma^2 S^{2\alpha} P_S^{(2\alpha)}. \end{aligned} \tag{49}$$

This is the fractional Black-Scholes equation (FBSE) driven by fractional G-Brownian motion fGBm.

Remark 5: The term $\widehat{\mathbb{E}}[\widetilde{\omega}(t)^2]\sigma^2$ in the FBSE (49) reflects the effects of the volatility uncertainty of the stocks price processes.

Remark 6: As we can see, the fractional Black-Scholes equation (49) generalizes some well-known BS equations, for some cases as follows

Case (1): if taking $\widehat{\mathbb{E}}[\widetilde{\omega}(t)^2] = \overline{\sigma}^2 \neq 1$ and $\alpha = 1$, then (49) will become

$$\begin{aligned} P_t &= rP - rSP_S \\ &\quad - \tfrac{\Gamma(1+2H)\Gamma(2-2H)\sigma^2}{2} \widehat{\mathbb{E}}[\widetilde{\omega}(t)^2] t^{2H-1} S^2 P_{SS}, \end{aligned} \tag{50}$$

which is the same as fractional G-Black-Scholes equation [22]

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + H\widehat{\mathbb{E}}\left[B_G^2(1)\right]\sigma^2 S^2 t^{2H-1}\frac{\partial^2 V}{\partial S^2} = rV. \tag{51}$$

The only difference is the coefficient of the second order derivative term, since we adopt the the Maruyama's notation for fractional G-Brownian motion $B_{FG}(t)$ in Proposition 4.

Case (2): If we take $\widehat{\mathbb{E}}[\widetilde{\omega}(t)^2] = \overline{\sigma}^2 = 1$, $\alpha = 2H$ and "σ" $= \Gamma(1+\alpha)\sigma$, then we get

$$\begin{aligned} P_t^{(\alpha)} &= \left[\tfrac{r}{\Gamma(2-\alpha)} P - rS^\alpha P_S^{(\alpha)}\right] t^{1-\alpha} \\ &\quad - \tfrac{[\Gamma(2-\alpha)]^2(\Gamma(1+\alpha))^3}{\Gamma(1+2\alpha)} \sigma^2 S^{2\alpha} P_S^{(2\alpha)}, \end{aligned} \tag{52}$$

which is the exact fractional Black-Scholes equation (2) obtained by Jumarie in [5].

Case (3): if taking $\widehat{\mathbb{E}}[\widetilde{\omega}(t)^2] = \overline{\sigma}^2 = 1$ and $\alpha = 1$, then (49) will be reduced as

$$\begin{aligned} P_t &= rP - rSP_S \\ &\quad - \tfrac{\Gamma(1+2H)\Gamma(2-2H)\sigma^2}{2} t^{2H-1} S^2 P_{SS}, \end{aligned} \tag{53}$$

which is the fractional BS equation driven by the fBm $B_H(t)$ [29]. (The only difference is also the coefficient):

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + H\sigma^2 S^2 t^{2H-1}\frac{\partial^2 V}{\partial S^2} = rV. \tag{54}$$

Case (4): for the most especial case as taking $\widehat{\mathbb{E}}[\widetilde{\omega}(t)^2] = 1$, $\alpha = 1$ and $H = \frac{1}{2}$, then it is simplified as

$$P_t + rSP_S + \frac{\sigma^2}{2} S^2 P_{SS} = rP, \tag{55}$$

which is exactly the classical BS equation obtained by Black and Scholes [1].

Remark 7: Under the condition $0 < \alpha < 2H$, from (33) and following the same way of the derivation for the fractional Black-Scholes equation (49), we can obtain another fractional Black-Scholes equation as

$$P_t^{(\alpha)} + rt^{1-\alpha} S^\alpha P_S^{(\alpha)} = \frac{r}{\Gamma(2-\alpha)} t^{1-\alpha} P. \tag{56}$$

IV. Solutions of the Fractional BS Equations

In this section, we will study the explicit option pricing formulas for the European call option and put option governed by the FBSE (49), respectively. The key terminal boundary conditions for the European call option and put option are

$$P(S,T) = \max(S-K, 0), \tag{57}$$

and

$$P(S,T) = \max(K-S, 0), \tag{58}$$

when $t = T$, where T is the maturity date of the option, and K is the exercise price of the option. For simplicity, here we consider the European call option for example, and for the European put option, one can process the deduction in the same way. Thus first from the FBSE (49), we have

$$\begin{cases} P_t^{(\alpha)} &= \left[\frac{r}{\Gamma(2-\alpha)}P - rS^\alpha P_S^{(\alpha)}\right] t^{1-\alpha} \\ &\quad + At^{2H-\alpha}S^{2\alpha}P_S^{(2\alpha)}, \\ P(S,T) = \max(S-K,0), \end{cases} \tag{59}$$

where we denote

$$A = -\frac{[\Gamma(2-\alpha)]^2\Gamma(1+2H)\Gamma(2-2H)}{\Gamma(1+2\alpha)}\widehat{\mathbb{E}}[\widetilde{\omega}(t)^2]\sigma^2. \tag{60}$$

Now we need to solve the boundary value problem (59). First letting

$$P(S,t) = \mathrm{e}^{-r(T-t)}\widetilde{P}(S,t), \tag{61}$$

then there holds

$$\begin{aligned} &P_t^{(\alpha)}(S,t) \\ &= D_t^\alpha\left(\mathrm{e}^{-r(T-t)}\right)\widetilde{P}(S,t) + \mathrm{e}^{-r(T-t)}\widetilde{P}_t^{(\alpha)}(S,t) \\ &= r\mathrm{e}^{-r(T-t)}\tfrac{t^{1-\alpha}}{\Gamma(2-\alpha)}\widetilde{P}(S,t) + \mathrm{e}^{-r(T-t)}\widetilde{P}_t^{(\alpha)}(S,t). \end{aligned} \tag{62}$$

Substituting (62) into (59), we have

$$\begin{cases} \widetilde{P}_t^{(\alpha)} = -rt^{1-\alpha}S^\alpha\widetilde{P}_S^{(\alpha)} + At^{2H-\alpha}S^{2\alpha}\widetilde{P}_S^{(2\alpha)}, \\ \widetilde{P}(S,T) = P(S,T) = \max(S-K,0). \end{cases} \tag{63}$$

Now we make the change of variable as

$$\widetilde{P}(S,t) = Q(y,t), \quad y = \ln(S) + a, \tag{64}$$

where a is a constant. Then there arrives

$$\begin{aligned} &\widetilde{P}_S^{(\alpha)} = Q_y\left(\tfrac{1}{S}\right)^\alpha, \\ &\widetilde{P}_S^{(2\alpha)} = Q_y\left(\tfrac{\Gamma(1-\alpha)}{\Gamma(1-2\alpha)}\left(\tfrac{1}{S}\right)^{2\alpha}\right) + Q_{yy}\left(\tfrac{1}{S}\right)^{2\alpha}, \end{aligned} \tag{65}$$

which implies

$$S^\alpha\widetilde{P}_S^{(\alpha)} = Q_y, \quad S^{2\alpha}\widetilde{P}_S^{(2\alpha)} = \frac{\Gamma(1-\alpha)}{\Gamma(1-2\alpha)}Q_y + Q_{yy}. \tag{66}$$

Substituting (66) into (63), we obtain

$$\begin{cases} Q_t^{(\alpha)}(y,t) = \left[-rt^{1-\alpha} + \frac{A\Gamma(1-\alpha)}{\Gamma(1-2\alpha)}t^{2H-\alpha}\right]Q_y(y,t) \\ \qquad + At^{2H-\alpha}Q_{yy}(y,t), \\ Q(y,T) = \widetilde{P}(S,T) = P(S,T) \\ \qquad = \max(S-K,0) = \max(\mathrm{e}^y - K, 0), \end{cases} \tag{67}$$

Again, we make the transformation as

$$\begin{aligned} &Q(y,t) = R(z,t), \\ &z = y - \ln(K) + E(t-T) + F\left(t^{2H} - T^{2H}\right), \end{aligned} \tag{68}$$

where E and F are defined as

$$E = \frac{r\Gamma(2-\alpha)}{\alpha}, \quad F = -\frac{A\Gamma(1-\alpha)\Gamma(1+2H-\alpha)}{\Gamma(1-2\alpha)\Gamma(1+2H)}. \tag{69}$$

Then we can compute that

$$\begin{aligned} &R_t^{(\alpha)}(z,t) \\ &= \tfrac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}R_s^{(1)}(z,s)ds \\ &= \tfrac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}\Big[Q_z^{(1)}(z,s)(\alpha E + 2HFt^{2H-1}) \\ &\qquad + Q_s^{(1)}(y,s)\Big]ds \\ &= \tfrac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}Q_s^{(1)}(y,s)ds \\ &\quad + Q_y\Big[\tfrac{\alpha E}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}ds \\ &\qquad + \tfrac{2HF}{\Gamma(1-\alpha)}\textstyle\int_0^t (t-s)^{-\alpha}t^{2H-1}ds\Big] \\ &= Q_t^{(\alpha)}(y,t) \\ &\quad + Q_y\left[\tfrac{\alpha E}{\Gamma(1-\alpha)}\tfrac{t^{1-\alpha}}{(1-\alpha)} + \tfrac{2HF}{\Gamma(1-\alpha)}\tfrac{\Gamma(2H)\Gamma(1-\alpha)}{\Gamma(1+2H-\alpha)}t^{2H-\alpha}\right] \\ &= Q_t^{(\alpha)}(y,t) + Q_y\left[\tfrac{\alpha E}{\Gamma(2-\alpha)}t^{1-\alpha} + \tfrac{\Gamma(1+2H)F}{\Gamma(1+2H-\alpha)}t^{2H-\alpha}\right] \\ &= Q_t^{(\alpha)}(y,t) + \left[rt^{1-\alpha} - \tfrac{A\Gamma(1-\alpha)}{\Gamma(1-2\alpha)}t^{2H-\alpha}\right]Q_y(y,t). \end{aligned} \tag{70}$$

Thus from (67), there holds

$$\begin{cases} R_t^{(\alpha)}(z,t) = At^{2H-\alpha}R_{zz}(z,t), \\ R(z,T) = \max\left(K(\mathrm{e}^z - 1), 0\right) = R_T(z). \end{cases} \tag{71}$$

If we denote the Fourier transform of $f(x)$ as

$$\widehat{f}(\xi) = \mathcal{F}(f(x)) = \int_{-\infty}^{\infty}\mathrm{e}^{-ix\cdot\xi}f(x)dx,$$

and the inverse Fourier transform as

$$f(x) = \mathcal{F}^{-1}(\widehat{f}(\xi)) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{ix\cdot\xi}\widehat{f}(\xi)d\xi.$$

Then taking the Fourier transform of (71), we have

$$\begin{cases} \widehat{R}_t^{(\alpha)}(\xi,t) = A\xi^2t^{2H-\alpha}\widehat{R}(\xi,t), \\ \widehat{R}(\xi,T) = \mathcal{F}\left(R_T(z)\right). \end{cases} \tag{72}$$

On the other hand, as we know, the following homogeneous equation

$$y^{(\alpha)}(t) = a(t)y(t), \quad y(T) = Y_T, \tag{73}$$

has the explicit solution, which can be expressed as

$$y(t) = y(T)E_\alpha\left\{-\int_t^T a(\tau)(d\tau)^\alpha\right\}, \tag{74}$$

where $E_\alpha(x)$ denotes the Mittag-Leffler functions defined in (17). Thus the solution of (72) can be obtained as

$$\begin{aligned} \widehat{R}(\xi,t) &= \widehat{R}(\xi,T)E_\alpha\left\{-A\xi^2\int_t^T \tau^{2H-\alpha}(d\tau)^\alpha\right\} \\ &= \widehat{R}(\xi,T)E_\alpha\left\{-\tfrac{\Gamma(1+\alpha)\Gamma(1+2H-\alpha)A\xi^2}{\Gamma(1+2H)}\left(T^{2H} - t^{2H}\right)\right\} \\ &= \widehat{R}(\xi,T)G(\xi,t), \end{aligned}$$

where $G(\xi,t) = E_\alpha\left\{-\frac{\Gamma(1+\alpha)\Gamma(1+2H-\alpha)A\xi^2}{\Gamma(1+2H)}\left(T^{2H} - t^{2H}\right)\right\}$.

If we denote $g(z,t)$ be the inverse Fourier transform of $G(\xi,t)$, we have

$$\begin{aligned} g(z,t) &= \mathcal{F}^{-1}\Big[E_\alpha\Big\{-\tfrac{\Gamma(1+\alpha)\Gamma(1+2H-\alpha)A\xi^2}{\Gamma(1+2H)} \\ &\qquad \left(T^{2H} - t^{2H}\right)\Big\}\Big], \end{aligned} \tag{75}$$

and furthermore

$$\begin{aligned} R(z,t) &= \mathcal{F}^{-1}\left[\widehat{R}(\xi,T)G(\xi,t)\right] \\ &= F^{-1}\left(\widehat{R}(\xi,T)\right) * F^{-1}\left(G(\xi,T)\right) \\ &= R_T(z) * g(z,t). \end{aligned} \tag{76}$$

This is the exact solution of the problem (71). Going back the original of variables, we can obtain the explicit solution for the problem (59) and thus get the option pricing formulas for the European call option. For the European put option, one can obtain the similar explicit formulas.

Remark 8: For the special case $\widehat{\mathbb{E}}[\widetilde{\omega}(t)^2] = 1$, $\alpha = 1$ and $H = \frac{1}{2}$, the explicit solution from (61), (64), (68), and (76) is simplified to the classical Black-Scholes formulas for the European call option pricing[1].

V. Conclusion

In this paper, we introduced a concept for some stochastic processes called fractional G-Brownian motion (fGBm), which generalizes the concepts of the classical Bm, fBm and GBm. Since the fGBm can be used to be a tool for considering the long range dependence and uncertain volatility simultaneously, it is more suitable to capture the intrinsic characteristics of the financial markets, so we employ the fGBm to the mathematical finance, especially for the option pricing. Combining the fractional calculus, we derived some fractional Black-Scholes models for the European option pricing driven by the fGBm with the help of the Taylor's series of fractional order and the theory of absence of arbitrage, which generalizes the obtained fractional models. And furthermore, the explicit option pricing formulas for the European call option and put option under the fractional model were also obtained, which also generalized the classical Black-Scholes formulas given by Black and Scholes [1]. Since the fractional Black-Scholes models were established based on the fGBm, we do believe it is better to describe the dynamics of the stock prices and give better valuation for the option pricing. For other researches about the fractional Black-Scholes models driven by the fGBm and their applications in other financial derivatives, we are working on it and will be seen in our forthcoming paper.

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