World Academy of Science, Engineering and Technology
International Journal of Mathematical and Computational Sciences
Vol:15, No:3, 2021

Hermite–Hadamard Type Integral Inequalities Involving k–Riemann–Liouville Fractional Integrals and Their Applications

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Abstract—In this paper, some generalization integral inequalities of Hermite–Hadamard type for functions whose derivatives are s–convex in modulus are given by using k–fractional integrals. Some applications to special means are obtained as well. Some known versions are recovered as special cases from our results. We note that our inequalities can be viewed as new refinements of the previous results. Finally, our results have a deep connection with various fractional integral operators and interested readers can find new interesting results using our idea and technique as well.

Keywords—Hermite–Hadamard's inequalities, k–Riemann–Liouville fractional integral, Hölder's inequality, Special means.

I. Introduction and Preliminaries

The theory of convexity presents an amazing, fascinating and captivating field of research and also played significant role in the development of the theory of inequalities. Due to a lot of applications the theory of convexity has become a rich source of motivation and absorbing field for researchers. Many researchers endeavor and attempt to define and introduce new ideas and concepts about convex functions and extend and generalize its variant forms in different ways using innovative ideas and fruitful techniques. Using the theory of convexity, mathematicians provide an amazing tool, numerical techniques to tackle and to solve a wide class of problems which arise in pure and applied sciences. In diverse and opponent research, inequalities have a lot of applications in statistical problems, probability and numerical quadrature formulas. Many famously known results in inequalities theory can be obtained using the convexity property of the functions. In 1994, first time Hudzik and Maligranda [1] introduced the class of s–convex functions in second sense. Further in this direction Dragomir and Fitzpatrick [2] put efforts and established new integral inequalities via s–convex functions. Recently İşcan [3] asserted that some new Hermite-Hadamard type inequalities for s–convex functions and their applications with the help of well known and remarkable inequalities improved power–mean integral inequality and Hölder–İşcan integral inequality. By the time Muddassar [4] adds some contributions via s–convex functions in this dynamic and captivating field. Noor [5] keeps his work on generalizations, introduced and proved new versions of Hermite–Hadamard inequality for exponentially s–convex function via the Katugampola fractional integral.

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Integral inequalities are generally applicable in many branches of mathematics such as mathematical analysis, fractional calculus, discrete fractional calculus and abstract spaces; for an overview, the reader should see the literature on integral inequalities, e.g., [6]–[12] and the references therein.

Nowadays, the study of convexity is considered as an original icon in the investigation of theoretical behavior of mathematical inequalities, e.g., [13]–[15]. Recently, several works on integral inequalities for convex functions were conducted. In particular, much attention has been given to the theoretical studies of inequalities on different types of convex functions such as s–geometrically convex functions [16], GA–convex functions [17], MT–convex function [18], [19], (α, m)–convex functions [20], [21], F–convex functions [22], λ_ψ–convex functions [23], a new class of convex functions [24], and many other types can be found in [25].

Let us recall some basic definitions that we will used in sequel.

Definition 1: Let $\psi : \mathcal{I} \to \Re$ be a real valued function. A function ψ is said to be convex, if

$$\psi\left(t\mu_1 + (1-t)\mu_2\right) \le t\psi\left(\mu_1\right) + (1-t)\psi\left(\mu_2\right) \tag{1}$$

holds for all $\mu_1, \mu_2 \in \mathcal{I}$ and $t \in [0,1]$.

Definition 2: Let $\psi : \mathcal{I} \to \Re$ be a real valued function and $s \in (0,1]$ be fixed. A function ψ is said to be s–convex, if

$$\psi\left(t\mu_1 + (1-t)\mu_2\right) \le t^s\psi\left(\mu_1\right) + (1-t)^s\psi\left(\mu_2\right) \tag{2}$$

holds for all $\mu_1, \mu_2 \in \mathcal{I}$ and $t \in [0,1]$.

Many generalizations, variants and extensions for the convexity have attracted the attention of many researchers, see [26]–[29]. Any paper on Hermite inequalities seems to be incomplete without mentioning the well–known Hermite–Hadamard inequality.

Theorem 1: If $\psi : \mathcal{I} \to \Re$ is a convex function for all $\mu_1, \mu_2 \in \mathcal{I}$, then

$$\psi\left(\frac{\mu_1+\mu_2}{2}\right) \le \frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}\psi(x)dx \le \frac{\psi(\mu_1)+\psi(\mu_2)}{2}. \tag{3}$$

Interested readers can refer to [1]–[6], [8]–[31].

Definition 3: [30] Let $\psi \in \mathcal{L}[\mu_1, \mu_2]$. Then k–fractional integrals of order $\alpha, k > 0$ with $\mu_1 \ge 0$ are defined by

$$\mathcal{I}^{\alpha,k}_{\mu_1^+}\psi(x) = \frac{1}{k\Gamma_k(\alpha)}\int_{\mu_1}^{x}(x-t)^{\frac{\alpha}{k}-1}\psi(t)dt, \quad x > \mu_1 \tag{4}$$

and

$$\mathcal{I}^{\alpha,k}_{\mu_2^-}\psi(x)=\frac{1}{k\Gamma_k(\alpha)}\int_x^{\mu_2}(t-x)^{\frac{\alpha}{k}-1}\psi(t)dt,\quad \mu_2>x, \quad (5)$$

where $\Gamma_k(\alpha+k)=\alpha\Gamma_k(\alpha)$ is k–Gamma function.

For $k = 1$, k–fractional integrals become Riemann–Liouville integrals. For $\alpha = k = 1$, k–fractional integrals become classical integrals. Motivated by the above results and literatures, we will give in Section II, some generalization integral inequalities of Hermite–Hadamard type for functions whose derivatives are s–convex in modulus by using k–fractional integrals. Some known versions will be recovered as special cases from our results. We will note that our inequalities can be viewed as new refinements of the previous results. In Section III, some applications to special means will be obtained. In Section IV, a brief conclusion will be provided as well.

II. Main Results

In order to obtain some results using s–convex functions, we need the following Lemma 1:

Lemma 1: Let $\psi:[\mu_1,\mu_2]\to\Re$ be a differentiable function on (μ_1,μ_2) with $0\le\mu_1<\mu_2$. If $\psi'\in\mathcal{L}[\mu_1,\mu_2]$, then for $\lambda\in(0,1)$ and $\alpha,k>0$, the following equality for fractional integrals hold:

$$\lambda^{\frac{\alpha}{k}}(1-\lambda)^{\frac{\alpha}{k}}\psi(\lambda\mu_1+(1-\lambda)\mu_2)-\frac{\Gamma_k(\alpha+k)}{(\mu_2-\mu_1)^{\frac{\alpha}{k}}} \quad (6)$$

$$\times\Big[\lambda^{\frac{\alpha}{k}+1}\mathcal{I}^{\alpha,k}_{(\lambda\mu_1+(1-\lambda)\mu_2)^-}\psi(\mu_1) \quad (7)$$

$$+(1-\lambda)^{\frac{\alpha}{k}+1}\mathcal{I}^{\alpha,k}_{(\lambda\mu_1+(1-\lambda)\mu_2)^+}\psi(\mu_2)\Big] \quad (8)$$

$$=\lambda^{\frac{\alpha}{k}+1}(1-\lambda)^{\frac{\alpha}{k}+1}(\mu_2-\mu_1) \quad (9)$$

$$\times\left[\int_0^1 t^{\frac{\alpha}{k}}\psi'\big[t(\lambda\mu_1+(1-\lambda)\mu_2)+(1-t)\mu_1\big]dt \quad (10)\right.$$

$$\left.-\int_0^1(1-t)^{\frac{\alpha}{k}}\psi'\big[t\mu_2+(1-t)(\lambda\mu_1+(1-\lambda)\mu_2)\big]dt\right]. \quad (11)$$

Proof: Let us denote, respectively,

$$\mathcal{I}_1=\int_0^1 t^{\frac{\alpha}{k}}\psi'\big[t(\lambda\mu_1+(1-\lambda)\mu_2)+(1-t)\mu_1\big]dt \quad (12)$$

and

$$\mathcal{I}_2=-\int_0^1(1-t)^{\frac{\alpha}{k}}\psi'\big[t\mu_2+(1-t)(\lambda\mu_1+(1-\lambda)\mu_2)\big]dt. \quad (13)$$

Integrating by parts and changing the variables, we have

$$\mathcal{I}_1=\frac{t^{\frac{\alpha}{k}}\psi\big[t(\lambda\mu_1+(1-\lambda)\mu_2)+(1-t)\mu_1\big]}{(1-\lambda)(\mu_2-\mu_1)}\Big|_0^1 \quad (14)$$

$$-\frac{\frac{\alpha}{k}}{(1-\lambda)(\mu_2-\mu_1)} \quad (15)$$

$$\times\int_0^1 t^{\frac{\alpha}{k}-1}\psi\big[t(\lambda\mu_1+(1-\lambda)\mu_2)+(1-t)\mu_1\big]dt \quad (16)$$

$$=\frac{\psi(\lambda\mu_1+(1-\lambda)\mu_2)}{(1-\lambda)(\mu_2-\mu_1)} \quad (17)$$

$$-\frac{\Gamma_k(\alpha+k)}{(1-\lambda)^{\frac{\alpha}{k}+1}(\mu_2-\mu_1)^{\frac{\alpha}{k}+1}}\mathcal{I}^{\alpha,k}_{(\lambda\mu_1+(1-\lambda)\mu_2)^-}\psi(\mu_1). \quad (18)$$

Similarly, we get

$$\mathcal{I}_2=\frac{\psi(\lambda\mu_1+(1-\lambda)\mu_2)}{\lambda(\mu_2-\mu_1)} \quad (19)$$

$$-\frac{\Gamma_k(\alpha+k)}{\lambda^{\frac{\alpha}{k}+1}(\mu_2-\mu_1)^{\frac{\alpha}{k}+1}}\mathcal{I}^{\alpha,k}_{(\lambda\mu_1+(1-\lambda)\mu_2)^+}\psi(\mu_2). \quad (20)$$

Adding $\mathcal{I}_1$ and $\mathcal{I}_2$ and multiplying by the factor $\lambda^{\frac{\alpha}{k}+1}(1-\lambda)^{\frac{\alpha}{k}+1}(\mu_2-\mu_1)$, we obtain the desired result. ■

Remark 1: Taking $k = 1$ in Lemma 1, we have ([31], Lemma 2.1).

Remark 2: Choosing $\lambda=\frac{1}{2}$ in Lemma 1, then we get

$$\psi\left(\frac{\mu_1+\mu_2}{2}\right)-\frac{\Gamma_k(\alpha+k)}{2^{1-\frac{\alpha}{k}}(\mu_2-\mu_1)^{\frac{\alpha}{k}}} \quad (21)$$

$$\times\Big[\mathcal{I}^{\alpha,k}_{\left(\frac{\mu_1+\mu_2}{2}\right)^-}\psi(\mu_1)+\mathcal{I}^{\alpha,k}_{\left(\frac{\mu_1+\mu_2}{2}\right)^+}\psi(\mu_2)\Big] \quad (22)$$

$$=\left(\frac{\mu_2-\mu_1}{4}\right) \quad (23)$$

$$\times\left[\int_0^1 t^{\frac{\alpha}{k}}\psi'\left(t\frac{\mu_1+\mu_2}{2}+(1-t)\mu_1\right)dt \quad (24)\right.$$

$$\left.-\int_0^1(1-t)^{\frac{\alpha}{k}}\psi'\left(t\mu_2+(1-t)\frac{\mu_1+\mu_2}{2}\right)dt\right]. \quad (25)$$

For brevity, we denote

$$\mathcal{T}_\psi(\lambda,\alpha,k;\mu_1,\mu_2):=\lambda^{\frac{\alpha}{k}}(1-\lambda)^{\frac{\alpha}{k}}\psi(\lambda\mu_1+(1-\lambda)\mu_2) \quad (26)$$

$$-\frac{\Gamma_k(\alpha+k)}{(\mu_2-\mu_1)^{\frac{\alpha}{k}}}\Big[\lambda^{\frac{\alpha}{k}+1}\mathcal{I}^{\alpha,k}_{(\lambda\mu_1+(1-\lambda)\mu_2)^-}\psi(\mu_1) \quad (27)$$

$$+(1-\lambda)^{\frac{\alpha}{k}+1}\mathcal{I}^{\alpha,k}_{(\lambda\mu_1+(1-\lambda)\mu_2)^+}\psi(\mu_2)\Big]. \quad (28)$$

Theorem 2: Let $\psi:[\mu_1,\mu_2]\to\Re$ be a differentiable function on (μ_1,μ_2) with $0\le\mu_1<\mu_2$. If $|\psi'|^q$ is s–convex on $[\mu_1,\mu_2]$ for $s\in(0,1]$ and $q\ge1$, then for $\lambda\in(0,1)$ and $\alpha,k>0$, the following inequality for fractional integrals hold:

$$|\mathcal{T}_\psi(\lambda,\alpha,k;\mu_1,\mu_2)| \quad (29)$$

$$\le\left(\frac{k}{\alpha+k}\right)^{1-\frac{1}{q}}\lambda^{\frac{\alpha}{k}+1}(1-\lambda)^{\frac{\alpha}{k}+1}(\mu_2-\mu_1) \quad (30)$$

$$\times\left\{\Big[\beta\left(\frac{\alpha}{k}+1,s+1\right)|\psi'(\mu_1)|^q \quad (31)\right.$$

$$+\frac{k}{\alpha+k(s+1)}|\psi'(\lambda\mu_1+(1-\lambda)\mu_2)|^q\Big]^{\frac{1}{q}} \quad (32)$$

$$+\Big[\beta\left(\frac{\alpha}{k}+1,s+1\right)|\psi'(\mu_2)|^q \quad (33)$$

$$\left.+\frac{k}{\alpha+k(s+1)}|\psi'(\lambda\mu_1+(1-\lambda)\mu_2)|^q\Big]^{\frac{1}{q}}\right\}, \quad (34)$$

where $\beta(\cdot,\cdot)$ is the well–known Euler Beta function.

Proof: Using Lemma 1, the well–known power mean inequality, s–convexity of $|\psi'|^q$ and properties of modulus, we have

$$|\mathcal{T}_\psi(\lambda,\alpha,k;\mu_1,\mu_2)| \tag{35}$$
$$\le \lambda^{\frac{\alpha}{k}+1}(1-\lambda)^{\frac{\alpha}{k}+1}(\mu_2-\mu_1) \tag{36}$$
$$\times\Bigg[\int_0^1 t^{\frac{\alpha}{k}}\big|\psi'\big[t(\lambda\mu_1+(1-\lambda)\mu_2)+(1-t)\mu_1\big]\big|dt \tag{37}$$
$$+\int_0^1 (1-t)^{\frac{\alpha}{k}}\big|\psi'\big[t\mu_2+(1-t)(\lambda\mu_1+(1-\lambda)\mu_2)\big]\big|dt\Bigg] \tag{38}$$
$$\le \lambda^{\frac{\alpha}{k}+1}(1-\lambda)^{\frac{\alpha}{k}+1}(\mu_2-\mu_1) \tag{39}$$
$$\times\Bigg\{\left(\int_0^1 t^{\frac{\alpha}{k}}dt\right)^{1-\frac{1}{q}} \tag{40}$$
$$\times\Bigg[\int_0^1 t^{\frac{\alpha}{k}}\big[t^s|\psi'(\lambda\mu_1+(1-\lambda)\mu_2)|^q \tag{41}$$
$$+(1-t)^s|\psi'(\mu_1)|^q\big]dt\Bigg]^{\frac{1}{q}} \tag{42}$$
$$+\left(\int_0^1 (1-t)^{\frac{\alpha}{k}}dt\right)^{1-\frac{1}{q}} \tag{43}$$
$$\times\Bigg[\int_0^1 (1-t)^{\frac{\alpha}{k}}\big[t^s|\psi'(\mu_2)|^q \tag{44}$$
$$+(1-t)^s|\psi'(\lambda\mu_1+(1-\lambda)\mu_2)|^q\big]dt\Bigg]^{\frac{1}{q}}\Bigg\} \tag{45}$$
$$=\left(\frac{k}{\alpha+k}\right)^{1-\frac{1}{q}}\lambda^{\frac{\alpha}{k}+1}(1-\lambda)^{\frac{\alpha}{k}+1}(\mu_2-\mu_1) \tag{46}$$
$$\times\Bigg\{\Big[\beta\left(\frac{\alpha}{k}+1,s+1\right)|\psi'(\mu_1)|^q \tag{47}$$
$$+\frac{k}{\alpha+k(s+1)}|\psi'(\lambda\mu_1+(1-\lambda)\mu_2)|^q\Big]^{\frac{1}{q}} \tag{48}$$
$$+\Big[\beta\left(\frac{\alpha}{k}+1,s+1\right)|\psi'(\mu_2)|^q \tag{49}$$
$$+\frac{k}{\alpha+k(s+1)}|\psi'(\lambda\mu_1+(1-\lambda)\mu_2)|^q\Big]^{\frac{1}{q}}\Bigg\}. \tag{50}$$

The proof of Theorem 2 is completed. ■

Corollary 1: Taking $s=1$ in Theorem 2, we get

$$|\mathcal{T}_\psi(\lambda,\alpha,k;\mu_1,\mu_2)| \tag{51}$$
$$\le\left(\frac{k}{\alpha+k}\right)^{1-\frac{1}{q}}\lambda^{\frac{\alpha}{k}+1}(1-\lambda)^{\frac{\alpha}{k}+1}(\mu_2-\mu_1) \tag{52}$$
$$\times\Bigg\{\Big[\beta\left(\frac{\alpha}{k}+1,2\right)|\psi'(\mu_1)|^q \tag{53}$$
$$+\frac{k}{\alpha+2k}|\psi'(\lambda\mu_1+(1-\lambda)\mu_2)|^q\Big]^{\frac{1}{q}} \tag{54}$$
$$+\Big[\beta\left(\frac{\alpha}{k}+1,2\right)|\psi'(\mu_2)|^q \tag{55}$$
$$+\frac{k}{\alpha+2k}|\psi'(\lambda\mu_1+(1-\lambda)\mu_2)|^q\Big]^{\frac{1}{q}}\Bigg\}. \tag{56}$$

Remark 3: Taking $k=1$ in Corollary 1, we obtain ([31], Theorem 2.1).

Corollary 2: Choosing $\lambda=\frac{1}{2}$ in Theorem 2, we have

$$\Bigg|\psi\left(\frac{\mu_1+\mu_2}{2}\right)-\frac{\Gamma_k(\alpha+k)}{2^{1-\frac{\alpha}{k}}(\mu_2-\mu_1)^{\frac{\alpha}{k}}} \tag{57}$$
$$\times\Big[\mathcal{I}^{\alpha,k}_{\left(\frac{\mu_1+\mu_2}{2}\right)^-}\psi(\mu_1)+\mathcal{I}^{\alpha,k}_{\left(\frac{\mu_1+\mu_2}{2}\right)^+}\psi(\mu_2)\Big]\Bigg| \tag{58}$$
$$\le\left(\frac{k}{\alpha+k}\right)^{1-\frac{1}{q}}\left(\frac{1}{4}\right)^{\frac{\alpha}{k}+1}(\mu_2-\mu_1) \tag{59}$$
$$\times\Bigg\{\Bigg[\beta\left(\frac{\alpha}{k}+1,s+1\right)|\psi'(\mu_1)|^q \tag{60}$$
$$+\frac{k}{\alpha+k(s+1)}\Big|\psi'\left(\frac{\mu_1+\mu_2}{2}\right)\Big|^q\Bigg]^{\frac{1}{q}} \tag{61}$$
$$+\Bigg[\beta\left(\frac{\alpha}{k}+1,s+1\right)|\psi'(\mu_2)|^q \tag{62}$$
$$+\frac{k}{\alpha+k(s+1)}|\Big|\psi'\left(\frac{\mu_1+\mu_2}{2}\right)\Big|^q\Bigg]^{\frac{1}{q}}\Bigg\}. \tag{63}$$

Corollary 3: Taking $|\psi'|\le M$ in Theorem 2, we get

$$|\mathcal{T}_\psi(\lambda,\alpha,k;\mu_1,\mu_2)| \tag{64}$$
$$\le 2M\left(\frac{k}{\alpha+k}\right)^{1-\frac{1}{q}}\lambda^{\frac{\alpha}{k}+1}(1-\lambda)^{\frac{\alpha}{k}+1}(\mu_2-\mu_1) \tag{65}$$
$$\times\left[\beta\left(\frac{\alpha}{k}+1,s+1\right)+\frac{k}{\alpha+k(s+1)}\right]^{\frac{1}{q}}. \tag{66}$$

Theorem 3: Let $\psi:[\mu_1,\mu_2]\to\Re$ be a differentiable function on (μ_1,μ_2) with $0\le\mu_1<\mu_2$. If $|\psi'|^q$ is s–convex on $[\mu_1,\mu_2]$ for $s\in(0,1]$ and $\frac{1}{p}+\frac{1}{q}=1$ with $q>1$, then for $\lambda\in(0,1)$ and $\alpha,k>0$, the following inequality for fractional integrals hold:

$$|\mathcal{T}_\psi(\lambda,\alpha,k;\mu_1,\mu_2)|\le\sqrt[p]{\frac{k}{p\alpha+k}}\frac{1}{\sqrt[q]{s+1}} \tag{67}$$
$$\times\lambda^{\frac{\alpha}{k}+1}(1-\lambda)^{\frac{\alpha}{k}+1}(\mu_2-\mu_1) \tag{68}$$
$$\times\Bigg\{\Big[|\psi'(\mu_1)|^q+|\psi'(\lambda\mu_1+(1-\lambda)\mu_2)|^q\Big]^{\frac{1}{q}} \tag{69}$$
$$+\Big[|\psi'(\mu_2)|^q+|\psi'(\lambda\mu_1+(1-\lambda)\mu_2)|^q\Big]^{\frac{1}{q}}\Bigg\}. \tag{70}$$

Proof: Using Lemma 1, Hölder's inequality, s–convexity of $|\psi'|^q$ and properties of modulus, we have

$$|\mathcal{T}_\psi(\lambda,\alpha,k;\mu_1,\mu_2)| \tag{71}$$
$$\leq \lambda^{\frac{\alpha}{k}+1}(1-\lambda)^{\frac{\alpha}{k}+1}(\mu_2-\mu_1) \tag{72}$$
$$\times\left[\int_0^1 t^{\frac{\alpha}{k}}\left|\psi'\left[t(\lambda\mu_1+(1-\lambda)\mu_2)+(1-t)\mu_1\right]\right|dt\right. \tag{73}$$
$$\left.+\int_0^1 (1-t)^{\frac{\alpha}{k}}\left|\psi'\left[t\mu_2+(1-t)(\lambda\mu_1+(1-\lambda)\mu_2)\right]\right|dt\right] \tag{74}$$
$$\leq \lambda^{\frac{\alpha}{k}+1}(1-\lambda)^{\frac{\alpha}{k}+1}(\mu_2-\mu_1) \tag{75}$$
$$\times\left\{\left(\int_0^1 t^{\frac{p\alpha}{k}}dt\right)^{\frac{1}{p}}\right. \tag{76}$$
$$\times\left(\int_0^1 \left[t^s|\psi'(\lambda\mu_1+(1-\lambda)\mu_2)|^q+(1-t)^s|\psi'(\mu_1)|^q\right]dt\right)^{\frac{1}{q}} \tag{77}$$
$$+\left(\int_0^1 (1-t)^{\frac{p\alpha}{k}}dt\right)^{\frac{1}{p}} \tag{78}$$
$$\times\left[\int_0^1 \left[t^s|\psi'(\mu_2)|^q\right.\right. \tag{79}$$
$$\left.\left.\left.+(1-t)^s|\psi'(\lambda\mu_1+(1-\lambda)\mu_2)|^q\right]dt\right]^{\frac{1}{q}}\right\} \tag{80}$$
$$=\sqrt[p]{\frac{k}{p\alpha+k}}\frac{1}{\sqrt[q]{s+1}} \tag{81}$$
$$\times\lambda^{\frac{\alpha}{k}+1}(1-\lambda)^{\frac{\alpha}{k}+1}(\mu_2-\mu_1) \tag{82}$$
$$\times\left\{\left[|\psi'(\mu_1)|^q+|\psi'(\lambda\mu_1+(1-\lambda)\mu_2)|^q\right]^{\frac{1}{q}}\right. \tag{83}$$
$$\left.+\left[|\psi'(\mu_2)|^q+|\psi'(\lambda\mu_1+(1-\lambda)\mu_2)|^q\right]^{\frac{1}{q}}\right\}. \tag{84}$$

The proof of Theorem 3 is completed. ■

Corollary 4: Taking $s=1$ in Theorem 3, we obtain

$$|\mathcal{T}_\psi(\lambda,\alpha,k;\mu_1,\mu_2)| \leq \frac{1}{\sqrt[q]{2}}\sqrt[p]{\frac{k}{p\alpha+k}} \tag{85}$$
$$\times\lambda^{\frac{\alpha}{k}+1}(1-\lambda)^{\frac{\alpha}{k}+1}(\mu_2-\mu_1) \tag{86}$$
$$\times\left\{\left[|\psi'(\mu_1)|^q+|\psi'(\lambda\mu_1+(1-\lambda)\mu_2)|^q\right]^{\frac{1}{q}}\right. \tag{87}$$
$$\left.+\left[|\psi'(\mu_2)|^q+|\psi'(\lambda\mu_1+(1-\lambda)\mu_2)|^q\right]^{\frac{1}{q}}\right\}. \tag{88}$$

Remark 4: Taking $k=1$ in Corollary 4, we have ([31], Theorem 2.2).

Corollary 5: Choosing $\lambda=\frac{1}{2}$ in Theorem 3, we get

$$\left|\psi\left(\frac{\mu_1+\mu_2}{2}\right)-\frac{\Gamma_k(\alpha+k)}{2^{1-\frac{\alpha}{k}}(\mu_2-\mu_1)^{\frac{\alpha}{k}}}\right. \tag{89}$$
$$\left.\times\left[\mathcal{I}^{\alpha,k}_{\left(\frac{\mu_1+\mu_2}{2}\right)^-}\psi(\mu_1)+\mathcal{I}^{\alpha,k}_{\left(\frac{\mu_1+\mu_2}{2}\right)^+}\psi(\mu_2)\right]\right| \tag{90}$$
$$\leq\sqrt[p]{\frac{k}{p\alpha+k}}\frac{1}{\sqrt[q]{s+1}}\left(\frac{1}{4}\right)^{\frac{\alpha}{k}+1}(\mu_2-\mu_1) \tag{91}$$
$$\times\left\{\left[|\psi'(\mu_1)|^q+\left|\psi'\left(\frac{\mu_1+\mu_2}{2}\right)\right|^q\right]^{\frac{1}{q}}\right. \tag{92}$$
$$\left.+\left[|\psi'(\mu_2)|^q+\left|\psi'\left(\frac{\mu_1+\mu_2}{2}\right)\right|^q\right]^{\frac{1}{q}}\right\}. \tag{93}$$

Corollary 6: Taking $|\psi'|\leq M$ in Theorem 3, we obtain

$$|\mathcal{T}_\psi(\lambda,\alpha,k;\mu_1,\mu_2)|\leq 2M\sqrt[p]{\frac{k}{p\alpha+k}}\sqrt[q]{\frac{2}{s+1}} \tag{94}$$
$$\times\lambda^{\frac{\alpha}{k}+1}(1-\lambda)^{\frac{\alpha}{k}+1}(\mu_2-\mu_1). \tag{95}$$

III. Applications to Special Means

We consider the following two special means for different positive real numbers μ_1 and μ_2, where $\mu_1<\mu_2$:

- The arithmetic mean:

$$\mathcal{A}(\mu_1,\mu_2)=\frac{\mu_1+\mu_2}{2}, \tag{96}$$

- The generalized logarithmic mean:

$$\mathcal{L}_r(\mu_1,\mu_2)=\left[\frac{\mu_2^{r+1}-\mu_1^{r+1}}{(r+1)(\mu_2-\mu_1)}\right]^{\frac{1}{r}},\quad r\in\Re\setminus\{-1,0\}. \tag{97}$$

Proposition 1: Let $0<\mu_1<\mu_2$ and $s\in(0,1]$ be fixed. Then for $q\geq 1$ and $\lambda\in(0,1)$, we have

$$\left|2^s\lambda(1-\lambda)\mathcal{A}^s(\lambda\mu_1,(1-\lambda)\mu_2)\right. \tag{98}$$
$$-\lambda^2(1-\lambda)\mathcal{L}_s^s(\mu_1,\lambda\mu_1+(1-\lambda)\mu_2) \tag{99}$$
$$\left.-\lambda(1-\lambda)^2\mathcal{L}_s^s(\lambda\mu_1+(1-\lambda)\mu_2,\mu_2)\right| \tag{100}$$
$$\leq\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\frac{\lambda^2(1-\lambda)^2 s}{\sqrt[q]{(s+1)(s+2)}}(\mu_2-\mu_1) \tag{101}$$
$$\times\left\{\left[\mu_1^{q(s-1)}+2^{q(s-1)}(s+1)\mathcal{A}^{q(s-1)}(\lambda\mu_1,(1-\lambda)\mu_2)\right]^{\frac{1}{q}}\right. \tag{102}$$
$$\left.+\left[\mu_2^{q(s-1)}+2^{q(s-1)}(s+1)\mathcal{A}^{q(s-1)}(\lambda\mu_1,(1-\lambda)\mu_2)\right]^{\frac{1}{q}}\right\}. \tag{103}$$

Proof: Taking $\psi(x)=x^s,\ x>0$ where $s\in(0,1]$ is fixed and using Theorem 2, the result (98) is evident. ■

Remark 5: Taking $\lambda = \frac{1}{2}$ in Proposition 1, we get

$$\left|\mathcal{A}^s(\mu_1,\mu_2) - \frac{1}{2}\left[\mathcal{L}_s^s\left(\mu_1, \frac{\mu_1+\mu_2}{2}\right) + \mathcal{L}_s^s\left(\frac{\mu_1+\mu_2}{2}, \mu_2\right)\right]\right| \tag{104}$$

$$\leq \left(\frac{1}{2}\right)^{1-\frac{1}{q}} \frac{s}{4\sqrt[q]{(s+1)(s+2)}}(\mu_2-\mu_1) \tag{105}$$

$$\times\left\{\left[\mu_1^{q(s-1)} + (s+1)\mathcal{A}^{q(s-1)}(\mu_1,\mu_2)\right]^{\frac{1}{q}}\right. \tag{106}$$

$$\left.+\left[\mu_2^{q(s-1)} + (s+1)\mathcal{A}^{q(s-1)}(\mu_1,\mu_2)\right]^{\frac{1}{q}}\right\}. \tag{107}$$

Proposition 2: Let $0 < \mu_1 < \mu_2$ and $s \in (0,1]$ be fixed. Then for $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, where $\lambda \in (0,1)$, we obtain

$$\left|2^s\lambda(1-\lambda)\mathcal{A}^s(\lambda\mu_1, (1-\lambda)\mu_2)\right. \tag{108}$$

$$-\lambda^2(1-\lambda)\mathcal{L}_s^s(\mu_1, \lambda\mu_1 + (1-\lambda)\mu_2) \tag{109}$$

$$\left.-\lambda(1-\lambda)^2\mathcal{L}_s^s(\lambda\mu_1 + (1-\lambda)\mu_2, \mu_2)\right| \tag{110}$$

$$\leq s\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\lambda^2(1-\lambda)^2(\mu_2-\mu_1) \tag{111}$$

$$\times\left\{\left[\mu_1^{q(s-1)} + 2^{q(s-1)}\mathcal{A}^{q(s-1)}(\lambda\mu_1,(1-\lambda)\mu_2)\right]^{\frac{1}{q}}\right. \tag{112}$$

$$\left.+\left[\mu_2^{q(s-1)} + 2^{q(s-1)}\mathcal{A}^{q(s-1)}(\lambda\mu_1,(1-\lambda)\mu_2)\right]^{\frac{1}{q}}\right\}. \tag{113}$$

Proof: Taking $\psi(x) = x^s,\ x > 0$ where $s \in (0,1]$ is fixed and applying Theorem 3, the result (108) is obvious. ■

Remark 6: Taking $\lambda = \frac{1}{2}$ in Proposition 2, we have

$$\left|\mathcal{A}^s(\mu_1,\mu_2) - \frac{1}{2}\left[\mathcal{L}_s^s\left(\mu_1, \frac{\mu_1+\mu_2}{2}\right) + \mathcal{L}_s^s\left(\frac{\mu_1+\mu_2}{2}, \mu_2\right)\right]\right| \tag{114}$$

$$\leq s\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\frac{(\mu_2-\mu_1)}{4} \tag{115}$$

$$\times\left\{\left[\mu_1^{q(s-1)} + \mathcal{A}^{q(s-1)}(\mu_1,\mu_2)\right]^{\frac{1}{q}}\right. \tag{116}$$

$$\left.+\left[\mu_2^{q(s-1)} + \mathcal{A}^{q(s-1)}(\mu_1,\mu_2)\right]^{\frac{1}{q}}\right\}. \tag{117}$$

IV. Conclusion

In our study the obtained results can be viewed as refinements of the previous results and also they have a deep connection with various fractional integral operators. We hope that current work using our idea and technique will attract the attention of researchers working in mathematical analysis and other related fields in pure and applied sciences.

Acknowledgment

We thank anonymous referees for valuable suggestions regarding the manuscript.

References

[1] H. Hudzik and L. Maligranda, Some remarks on s–convex functions, *Aequationes Math.*, **48** (1994), 100-111.
[2] S. S. Dragomir and S. Fitzpatrik, The Hadamard's inequality for s–convex functions in the second sense, *Demonstratio Math.*, **32**(4) (1999), 687-696.
[3] S. ́Ozcan and İ. İşcan, Some new Hermite–Hadamard type inequalities for s–convex functions and their applications, *J. Inequal. Appl.*, **2019**(201) (2019), 1–11.
[4] M. Muddassar, M. I. Bhatti and M. Iqbal, Some new s–Hermite-Hadamard type inequalities for differentiable functions and their applications, *Proc. Pak. Acad. Sci.*, **49**(1) (2012), 9–17.
[5] S. Rashid, M. A. Noor, K. I. Noor and A. O. Akdemir, Some New Generalizations for Exponentially s-Convex Functions and Inequalities via Fractional Operators, *Int. J. Sci. Innovation Tech.*, **1**(1) (2014), 1–12.
[6] M. J. Cloud, B. C. Drachman and L. Lebedev, Inequalities, *Springer*, Cham, Second edition, 2014.
[7] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, Theory and Applications of Fractional Differential Equations, *North-Holland Mathematics Studies; Elsevier Sci. B.V.*: Amsterdam, The Netherlands, 2006; Vol. 204.
[8] D. Baleanu, P. O. Mohammed, M. J. Vivas-Cortez, and Y.-R. Oliveros, Some modifications in conformable fractional integral inequalities, *Adv. Differ. Equ.*, **2020**(374), (2020).
[9] T. Abdeljawad, P. O. Mohammed and A. Kashuri, New Modified Conformable Fractional Integral Inequalities of Hermite–Hadamard Type with Applications, *J. Funct. Spaces*, **2020** Article 4352357, (2020).
[10] P. O. Mohammed, Some integral inequalities of fractional quantum type, *Malaya J. Mat.*, **4**(1) (2016), 93–99.
[11] P. O. Mohammed and T. Abdeljawad, Integral inequalities for a fractional operator of a function with respect to another function with nonsingular kernel, *Adv. Differ. Equ.*, **2020**(363) (2020).
[12] P. O. Mohammed, New integral inequalities for preinvex functions via generalized beta function, *J. Interdiscip. Math.*, **22**(4) (2019), 539–549.
[13] M. Chudziak and M. Íołdak, Hermite–Hadamard and Fejér Inequalities for Co-Ordinated (F,G)–Convex Functions on a Rectangle, *Symmetry*, **12**(13) (2020).
[14] P. O. Mohammed and M. Z. Sarikaya, On generalized fractional integral inequalities for twice differentiable convex functions, *J. Comput. Appl. Math.*, **372** Article 112740, (2020).
[15] P. O. Mohammed and I. Brevik, A New Version of the Hermite–Hadamard Inequality for Riemann–Liouville Fractional Integrals, *Symmetry*, **12**(610) (2020).
[16] T.-Y. Zhang, A.-P. Ji and F. Qi, On Integral Inequalities of Hermite–Hadamard Type for s–Geometrically Convex Functions, *Abst. Appl. Anal.*, **2012** Article 560586, (2012).
[17] T.-Y. Zhang, A.-P. Ji and F. Qi, Some inequalities of Hermite–Hadamard type for GA–convex functions with applications to means, *Le Mat.*, **68** (2013), 229–239.
[18] P. O. Mohammed, Some new Hermite–Hadamard type inequalities for MT–convex functions on differentiable coordinates, *J. King Saud Univ. Sci.*, **30** (2018), 258–262.
[19] J. Han, P. O. Mohammed and H. Zeng, Generalized fractional integral inequalities of Hermite–Hadamard-type for a convex function, *Open Math.*, **18** (2020), 794–806.
[20] D.-P. Shi, B.-Y. Xi and F. Qi, Hermite-Hadamard type inequalities for Riemann–Liouville fractional integrals of (α,m)–convex functions, *Fract. Differ. Calc.*, **4** (2014), 31–43.
[21] F. Qi, P. O. Mohammed, J. C. Yao and Y. H. Yao, Generalized fractional integral inequalities of Hermite–Hadamard type for (α,m)–convex functions, *J. Inequal. Appl.*, **2019**(135) (2019).
[22] P. O. Mohammed and M. Z. Sarikaya, Hermite–Hadamard type inequalities for F–convex function involving fractional integrals, *J. Inequal. Appl.*, **2018**(359) (2018).
[23] D. Baleanu, P. O. Mohammed and S. Zeng, Inequalities of trapezoidal type involving generalized fractional integrals, *Alex. Eng. J.*, (2020).
[24] P. O. Mohammed, T. Abdeljawad, S. Zeng and A. Kashuri, Fractional Hermite–Hadamard Integral Inequalities for a New Class of Convex Functions, *Symmetry*, **12**(1485) (2020).
[25] S. S. Dragomir and C. E. M. Pearce, Selected Topics on Hermite–Hadamard Inequalities and Applications, *RGMIA Monographs*; Victoria University: Footscray, Australia, 2000.
[26] G. Farid, X. Qiang and S. B. Akbar, Generalized fractional integrals inequalities for exponentially (s,m)–convex functions, *J. Inequal. Appl.*, (2020).
[27] A. Guessab and G. Schmeisser, Sharp integral inequalities of the Hermite-Hadamard type, *J. Approx. Theory*, **115**(2) (2002), 260–288.

[28] A. Iqbal, M. A. Khan, S. Ullah and Y.-M. Chu, Some new Hermite-Hadamard-type inequalities associated with conformable fractional integrals and their applications, *J. Funct. Spaces*, **2020**, Article ID 9845407 (2020).
[29] Y. Khurshid, M. A. Khan and Y.-M. Chu, Conformable integral inequalities of the Hermite–Hadamard type in terms of GG– and GA–convexities, *J. Funct. Spaces*, **2019**, Article ID 6926107 (2019).
[30] S. Mubeen and G. M. Habibullah, k–Fractional integrals and applications, *Int. J. Contemp. Math. Sci.*, **7** (2012), 89–94.
[31] M. Z. Sarikaya and H. Yaldiz, On generalized Hermite–Hadamard type integral inequalities involving Riemann–Liouville fractional integrals, *Nihonkai Math. J.*, **25** (2014), 93–104.

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