Multi-Criteria Based Robust Markowitz Model under Box Uncertainty

Pulak Swain, A. K. Ojha

Abstract—Portfolio optimization is based on dealing with the problems of efficient asset allocation. Risk and Expected return are two conflicting criteria in such problems, where the investor prefers the return to be high and the risk to be low. Using multi-objective approach we can solve those type of problems. However the information which we have for the input parameters are generally ambiguous and the input values can fluctuate around some nominal values. We can not ignore the uncertainty in input values, as they can affect the asset allocation drastically. So we use Robust Optimization approach to the problems where the input parameters comes under box uncertainty. In this paper, we solve the multi criteria robust problem with the help of ϵ - constraint method.

Keywords—Portfolio optimization, multi-objective optimization, ϵ - constraint method, box uncertainty, robust optimization.

I. Introduction

WITH the introduction of Markowitz Model [1], several research direction has opened up in the field of finance. Markowitz model is a quadratic programming problem, which maximizes the expected return of the portfolio and minimizes the the variance of the portfolio return [2]. However, later on some other constraints like cardinality, budget and quantity constraints are also included in the portfolio model in order to minimize the transaction cost [3]. The mean-variance model relies on the assumption that the returns of the assets are normally distributed, which is not always the case. Some downside risk measures have been also considered in case of non-normal distribution of asset return [4], [5]. In last few years, the problems under uncertainty have become a challenging research topic in several fields including optimization. Traditionally Stochastic Optimization and Sensitivity Analysis were being used to tackle the uncertain optimization problems. But in the past two decades Robust Optimization [6]-[8] has come into the picture with its ability to find a solution that is completely immunized against uncertainty. Various uncertainty sets in the form of interval, box, ellipsoid, paraboloid, polyhedral have been taken into account to solve the uncertain problems in minmax approach. As in portfolio optimization, the historical data are used to evaluate the future return rates so there is a high chance of the solution to be influenced by uncertainty. So there is a necessity of using robust approach for portfolio problems. Many studies are there in the literature based on the robust approach of portfolio problems [9], [10]. But those studies are based on single objective robust portfolio selection problems. So the goal of this paper is to form a

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robust multi-criteria-based Markowitz model and to apply ϵ - constraint method for solving this.

The paper is organized as follows: Section II presents a preliminary discussion on Markowitz portfolio model, ϵ - constraint method and robust counterpart of uncertain problems. In Section III we derive the robust counterpart of the multi-criteria-based Makowitz model under box uncertainty. In Section IV, an uncertain multi-criteria based portfolio problem has been solved by ϵ - constraint method. And finally, some concluding remarks have been incorporated in Section V.

II. Preliminaries

A. Markowitz Portfolio Model

We suppose a portfolio containing n number of assets with their returns at time t are given by r_{it} $(i = 1, 2, \ldots, n)$. Markowitz portfolio model was based on by taking the mean of return as the reward and the variance of portfolio return as the risk factor. To calculate these, first we need to find expected return of each asset and covariance of return between each pair of assets, which are given by:

$$\mu_i = E(r_i) = \frac{1}{T} \sum_{t=1}^{T} r_{it} \quad \text{for } i = 1, 2, \ldots, n$$

$$\sigma_{ij} = E[(r_i - \mu_i)(r_j - \mu_j)] = \frac{1}{T} \sum_{t=1}^{T} (r_{it} - \mu_i)(r_{jt} - \mu_j)$$

for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n$

The aim is to form a portfolio which will give our desired return with a minimum risk associated with it. Let the weight given to i^{th} asset be x_i. Then the expected return and variance of the portfolio are respectively given by,

$$\mu_P = \sum_i \mu_i x_i, \quad \sigma_P^2 = \sum_{i,j} \sigma_{ij} x_i x_j$$

Markowitz Mean-Variance Model is given by:

$$
\begin{aligned}
\min \quad & \frac{1}{2} \sum_{i,j} \sigma_{ij} x_i x_j \\
\text{s.t.:} \quad & \sum_i \mu_i x_i \geq \tau, \quad \sum_i x_i = 1, \quad x_i \geq 0
\end{aligned}
\tag{1}
$$

Here we minimize the variance of portfolio return at a fixed lower level of expected return (say τ).

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B. Multi-Criteria-Based Optimization and ϵ-Constraint Method

Multi-Criteria Optimization problems are useful when there is more than one objective, which are conflicting in nature. The general form of multi-criteria-based optimization problem is given as [11]-[13]:

$$
\begin{aligned}
\min \text{ or } \max \quad & \{f_i(\boldsymbol{x}), \quad i = 1, 2, \ldots n\} \\
\text{s.t.:} \quad & \Omega
\end{aligned}
\tag{2}
$$

where the $f_i(\boldsymbol{x})$'s are the conflicting objective functions and Ω represents the constraints. ϵ-constraint method is one of the primary methods which is used for the multi-criteria problems. The method is given as:

1) The optimal solution of each f_i subject to Ω is calculated and is denoted as $x^{(i)}$.
2) A payoff table is constructed for each objective f_i with respect to all the points $x^{(1)}, x^{(2)}, \ldots, x^{(n)}$. The pay off table is given as:

TABLE I
PAYOFF TABLE

$\boldsymbol{x}^{(i)}$	f_1	f_2	$\ldots$	f_k
$\boldsymbol{x}^{(1)}$	$f_1(\boldsymbol{x}^{(1)})$	$f_2(\boldsymbol{x}^{(1)})$	$\ldots$	$f_k(\boldsymbol{x}^{(1)})$
$\boldsymbol{x}^{(2)}$	$f_1(\boldsymbol{x}^{(2)})$	$f_2(\boldsymbol{x}^{(2)})$	$\ldots$	$f_k(\boldsymbol{x}^{(2)})$
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
$\boldsymbol{x}^{(n)}$	$f_1(\boldsymbol{x}^{(n)})$	$f_2(\boldsymbol{x}^{(n)})$	$\ldots$	$f_k(\boldsymbol{x}^{(n)})$

3) The lower bound (L_i) and upper bound (U_i) of each f_i is obtained from the payoff table.
4) Then n number of single objective problems are constructed by taking one of the f_i's as its objective and others as the constraints. Those problems are in the form:

$$
\begin{aligned}
\min \quad & f_i, \quad i = 1, 2, \ldots n \\
\text{s.t.:} \quad & \Omega \\
& f_j \le \epsilon_j \quad j \ne i, \quad \epsilon_j \in [L_j, U_j]
\end{aligned}
\tag{3}
$$

5) The problems are solved by changing ϵ_j from L_j to U_j and a set of solutions are obtained.
6) Finally, the most optimal solution (x^*) is selected from all the generated solutions.

C. Uncertain Optimization Problems and Robust Optimization Approach

The general form of an uncertain optimization problem is given by,

$$
\begin{aligned}
\min_{\boldsymbol{x}} \quad & f(\boldsymbol{x}, \boldsymbol{u}) \\
\text{s.t.:} \quad & c(\boldsymbol{x}, \boldsymbol{u}) \le 0, \quad \forall u \in \mathscr{U}(\boldsymbol{x}) = \{u : g(\boldsymbol{x}, \boldsymbol{u}) \le 0\}
\end{aligned}
\tag{4}
$$

where $\boldsymbol{x}$ is the vector of decision variables and $\boldsymbol{u}$ is the vector of uncertain parameters lying in the uncertainty set $\mathscr{U}(\boldsymbol{x})$.

Let $\boldsymbol{u}^0$ be the vector of nominal values (u_i^0) of the uncertain parameters. Then the box uncertainty set can be interpreted mathematically as:

$$
\mathscr{U}_{box} = \{\boldsymbol{u} : ||\boldsymbol{u} - \boldsymbol{u}^0||_\infty \le \boldsymbol{\delta}\}
$$

where $||.||_\infty$ is the supremum norm and $\boldsymbol{\delta}$ is the vector of perturbations (δ_i). That means each component u_i of the vector $\boldsymbol{u}$ perturbs around its nominal value u_i^0 with a radius δ_i.

In Robust Optimization approach, we get the completely "immunized against uncertainty" solutions. That means the solution of the worst case realization problem is considered as the robust solution, so that it will be feasible for any realization of the uncertain parameters. Now the robust counterpart of the problem (4) under box uncertainty is given by,

$$
\begin{aligned}
\min_{\boldsymbol{x}} \quad & \left\{ \max_{\boldsymbol{u} \in \mathscr{U}(\boldsymbol{x})} f(\boldsymbol{x}, \boldsymbol{u}) \right\} \\
\text{s.t.:} \quad & \left\{ \max_{\boldsymbol{u}} c(\boldsymbol{x}, \boldsymbol{u}) : ||\boldsymbol{u} - \boldsymbol{u}^0||_\infty \le \boldsymbol{\delta} \right\} \le 0
\end{aligned}
\tag{5}
$$

III. MULTI-CRITERIA-BASED MARKOWITZ MODEL UNDER BOX UNCERTAINTY

Considering the expected portfolio return and the portfolio risk as two conflicting objectives for our portfolio model, we can transform Markowitz's portfolio model as:

$$
\begin{aligned}
\min \quad & \frac{1}{2} \sum_{i,j} \sigma_{ij} x_i x_j \\
\max \quad & \sum_i \mu_i x_i \\
\text{s.t.:} \quad & \sum_i x_i = 1, \quad x_i \ge 0
\end{aligned}
\tag{6}
$$

Let the expected returns of each asset (μ_i) and the covariance between each pair of assets (σ_{ij}) lie in some box uncertainty set as given by,

$$
\mathscr{U}_\mu = \{\mu_i : |\mu_i - \mu_i^0| \le \delta_i, \quad i = 1, 2, \ldots n\}
$$

and

$$
\mathscr{U}_\sigma = \{\sigma_{ij} : |\sigma_{ij} - \sigma_{ij}^0| \le \delta_{ij}, \quad i = 1, 2, \ldots n, \ j = 1, 2, \ldots n\}
$$

where, μ_i^0 and σ_{ij}^0 are the nominal values for the expected return and covariance of return, respectively. That means each μ_i varies within the range $[\mu_i^0 - \delta_i, \mu_i^0 + \delta_i]$ and σ_{ij} varies within $[\sigma_{ij}^0 - \delta_{ij}, \sigma_{ij}^0 + \delta_{ij}]$. Now the robust counterpart of the multi-criteria problem (6) is formed by taking the worst case realization of μ_i and σ_{ij} and it is given by,

$$
\begin{aligned}
\min_{x_i} \quad & \left\{ \max_{\sigma_{ij} \in \mathscr{U}_\sigma} \frac{1}{2} \sum_{i,j} \sigma_{ij} x_i x_j \right\} \\
\max_{x_i} \quad & \left\{ \min_{\mu_i \in \mathscr{U}_\mu} \sum_i \mu_i x_i \right\} \\
\text{s.t.:} \quad & \sum_i x_i = 1, \quad x_i \ge 0
\end{aligned}
\tag{7}
$$

Since all the x_i's are non-negative, so the robust counterpart of the problem can be obtained by putting the lower bound $(\mu_i^0 + \delta_i)$ from the set $\mathscr{U}_\mu$ and the upper bound $(\sigma_{ij}^0 + \delta_{ij})$ from the set $\mathscr{U}_\sigma$.

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Mathematically it can be written as:

$$\min_{x_i} \quad \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(\sigma_{ij}^0 + \delta_{ij})x_i x_j$$

$$\max_{x_i} \quad \sum_{i=1}^{n}(\mu_i^0 - \delta_i)x_i \qquad (8)$$

$$\text{s.t.:} \quad \sum_i x_i = 1, \quad x_i \geq 0$$

We can transform the maximization objective into minimization form as:

$$\min_{x_i} \quad \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(\sigma_{ij}^0 + \delta_{ij})x_i x_j$$

$$\min_{x_i} \quad -\sum_{i=1}^{n}(\mu_i^0 - \delta_i)x_i \qquad (9)$$

$$\text{s.t.:} \quad \sum_i x_i = 1, \quad x_i \geq 0$$

IV. NUMERICAL EXAMPLE

We solve a portfolio problem by taking the daily return data of three stocks AAPL (Apple Inc.), BAC (Bank of America Corp), TEVA (Teva Pharmaceutical Industries) for the year 2018 from www.kaggle.com. We calculate the expected returns of each asset and the covariance of returns between each pair of assets. Now those nominal values for expected return and covariance returns are given in matrix form as:

$$\boldsymbol{\mu}^0 = \begin{bmatrix} 0.001456 \\ 0.000184 \\ 0.000685 \end{bmatrix}$$

$$\boldsymbol{\Sigma}^0 = \begin{bmatrix} 0.0002090 & 0.0000973 & 0.0000863 \\ 0.0000973 & 0.0002140 & 0.0001540 \\ 0.0000863 & 0.0001540 & 0.0006570 \end{bmatrix}$$

Our aim is to find the optimal weight of each asset and the optimal risk associated with the whole portfolio to achieve the given target rate of return. Let the perturbation vector associated with $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ respectively be given as:

$$\boldsymbol{\delta_{\mu}} = \begin{bmatrix} 0.00001 \\ 0.00001 \\ 0.00001 \end{bmatrix}$$

$$\boldsymbol{\delta_{\Sigma}} = \begin{bmatrix} 0.000001 & 0.000001 & 0.000001 \\ 0.000001 & 0.000001 & 0.000001 \\ 0.000001 & 0.000001 & 0.000001 \end{bmatrix}$$

So the Robust Multi-criteria-based Markowitz model can be formed as given in (9):

$$\min \quad f_1 : \frac{1}{2}[0.0002190x_1^2 + 0.0002150x_2^2 + 0.0006580x_3^2 +$$
$$2 \cdot 0.0000983x_1x_2 + 2 \cdot 0.0000863x_1x_3 +$$
$$2 \cdot 0.0001540x_2x_3]$$

$$\min \quad f_2 : -0.001446x_1 - 0.000174x_2 - 0.000675x_3$$

$$\text{s.t.:} \quad \sum_{i=1}^{3} x_i = 1, \quad x_1, x_2, x_3 \geq 0$$

$$(10)$$

Now ϵ - constraint method can be applied to solve this problem. Here the optimal solutions for the objective f_1 subject to the constraints is given by, $\boldsymbol{x}^{(1)} = [0.4778343 \quad 0.4555913 \quad 0.0665744]^\top$ and that of the objective f_2 subject to the constraints is given by, $\boldsymbol{x}^{(2)} = [1.0000000 \quad 0.0000000 \quad 0.0000000]^\top$. Now, the Payoff table for both the objectives at points $\boldsymbol{x}^{(1)}$ and $\boldsymbol{x}^{(2)}$ is given by,

TABLE II
PAYOFF TABLE FOR BOTH THE OBJECTIVES AT POINTS $\boldsymbol{x}^{(1)}$ AND $\boldsymbol{x}^{(2)}$

$\boldsymbol{x}^{(i)}$	f_1	f_2
$\boldsymbol{x}^{(1)}$	0.00007759	-0.00081516
$\boldsymbol{x}^{(2)}$	0.00001095	-0.00144600

Then, the lower and upper bounds of both the objectives are respectively given as: $[L_1, U_1] = [0.00001095, 0.00007759]$ and $[L_2, U_2] = [-0.00144600, -0.00081516]$. Now, the ϵ - constraint problems for both the objectives are given in (11) and (12) as:

ϵ **- constraint problem I:**

$$\min_{x_i} \quad f_1 : \frac{1}{2}[0.0002190x_1^2 + 0.0002150x_2^2 + 0.0006580x_3^2$$
$$+ 2 \cdot 0.0000983x_1x_2 + 2 \cdot 0.0000863x_1x_3$$
$$+ 2 \cdot 0.0001540x_2x_3]$$

s.t.:

$$f_2 : -0.001446x_1 - 0.000174x_2 - 0.000675x_3 \leq \epsilon_2$$

$$\sum_{i=1}^{3} x_i = 1, \quad x_1, x_2, x_3 \geq 0, \quad \epsilon_2 \in [L_2, U_2]$$

$$(11)$$

ϵ **- constraint problem II:**

$$\min_{x_i} \quad f_2 : \quad -0.001446x_1 - 0.000174x_2 - 0.000675x_3$$

s.t.:

$$f_1 : \frac{1}{2}[0.0002190x_1^2 + 0.0002150x_2^2 + 0.0006580x_3^2$$
$$+ 2 \cdot 0.0000983x_1x_2 + 2 \cdot 0.0000863x_1x_3$$
$$+ 2 \cdot 0.0001540x_2x_3] \leq \epsilon_1$$

$$\sum_{i=1}^{3} x_i = 1, \quad x_1, x_2, x_3 \geq 0, \quad \epsilon_1 \in [L_1, U_1]$$

$$(12)$$

On solving these problems for different values of ϵ_1 and ϵ_2, we obtain the optimal solutions for both the ϵ - constraint problems. Then, we calculate the values of objectives f_1 and f_2 at those two optimal solutions. The obtained results are as follows:

V. CONCLUDING REMARKS

From the optimal solutions given in Table III, it can be clearly seen that the solution of ϵ - constraint problem II is more preferable. So, our obtained optimal weights for the three assets are, $x_1 = 0.4830315$, $x_2 = 0.4500184$, $x_3 = 0.0669501$. And the optimal portfolio gives the return of 0.00082196 with a risk 0.00007759. This robust solution under box uncertainty set is feasible for any realization of the expected returns and covariance of returns within the given

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TABLE III
OPTIMAL SOLUTIONS OF TWO ϵ - CONSTRAINT PROBLEMS

	Problem I	Problem II
Optimal Solution $(\boldsymbol{x}^*)$	$\begin{bmatrix} 0.4807172 \\ 0.4525213 \\ 0.0667616 \end{bmatrix}$	$\begin{bmatrix} 0.4830315 \\ 0.4500184 \\ 0.0669501 \end{bmatrix}$
$f_1(\boldsymbol{x}^*)$	0.00007759	0.00007759
$f_2(\boldsymbol{x}^*)$	-0.00081892	-0.00082196

uncertainty set. However the solution under box uncertainty is slightly more conservative. Therefore, there is a scope of improvement by considering some other uncertainty sets like ellipsoidal, paraboloid etc. instead of box.

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