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Comparison of Two Maintenance Policies for a Two-Unit Series System Considering General Repair

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***Abstract*—**In recent years, maintenance optimization has attracted special attention due to the growth of industrial systems complexity. Maintenance costs are high for many systems, and preventive maintenance is effective when it increases operations' reliability and safety at a reduced cost. The novelty of this research is to consider general repair in the modeling of multi-unit series systems and solve the maintenance problem for such systems using the semi-Markov decision process (SMDP) framework. We propose an opportunistic maintenance policy for a series system composed of two main units. Unit 1, which is more expensive than unit 2, is subjected to condition monitoring, and its deterioration is modeled using a gamma process. Unit 1 hazard rate is estimated by the proportional hazards model (PHM), and two hazard rate control limits are considered as the thresholds of maintenance interventions for unit 1. Maintenance is performed on unit 2, considering an age control limit. The objective is to find the optimal control limits and minimize the long-run expected average cost per unit time. The proposed algorithm is applied to a numerical example to compare the effectiveness of the proposed policy (policy I) with policy II, which is similar to policy I, but instead of general repair, replacement is performed. Results show that policy I leads to lower average cost compared with policy II.

***Keywords*—**Condition-based maintenance, proportional hazards model, semi-Markov decision process, two-unit series systems.

I. Introduction

THE reliability of operations has always been a concern for managers of complex industrial systems, where failures can lead to production losses, maintenance costs, and even endanger the safety of operators. Preventive maintenance is performed to prevent unexpected failures. However, keeping a balance between available resources and preventive activities is of considerable significance [1].

It has been possible to monitor the exact state of systems due to technological advances in recent decades. CBM proposes actions to be performed when the degradation of the system reaches a certain threshold [2]. CBM reduces the risk of unscheduled downtime due to failures, utilizing actual data obtained from regular or continuous inspections. The hazard rate, i.e., the conditional risk of failure, can be used as a health indicator of systems. The PHM integrates both age and deterioration of the system to estimate the risk of failure and is composed of a baseline function $h_0(t)$ and a positive link function $\psi(X_t)$. The general form of the PHM is represented as [3]:

$$h(t, X_t) = h_0(t)\psi(X_t) \tag{1}$$

where $h_0(t)$ and $\psi(X_t)$ depend on age and covariates (i.e., condition data), respectively. Weibull distributed hazard function is widely used to estimate the hazard rate of systems which is expressed as:

$$h(t, X_t) = \frac{kt^{k-1}}{\lambda^k} \cdot \exp(\theta X_t) \tag{2}$$

where λ is the scale parameter, k is the shape parameter, and θ is the regression coefficient vector of covariates.

Over the useful life of systems, they are not entirely replaced but repaired when maintenance is required, despite minor components. Repair activities can be classified into three categories considering the degree of rejuvenation: replacement, general repair, and minimal repair. Replacement completely restores a system to statistically as-new condition. General repair improves the condition of a system to a state between as-new and as-old [4]. Minimal repair is performed to return a failed unit to the same operational condition just before failure (as-old) [5].

Several studies proposed models to maintain single-unit systems. However, complex systems are composed of several interacting components. Jafari et al. developed a CBM model in the SMDP framework for a multi-component system with two main units. The core unit is subject to condition monitoring, and the PHM is used to describe the hazard function of this unit, and the age information of the second important unit is available. A preventive maintenance level (U) and an opportunistic maintenance level (W) are considered for the core unit. When the hazard rate of the core unit crosses U or the age of the second unit exceeds the predetermined maximum useful operating age, an opportunity arises for the replacement of other units. While the second unit is not operating due to failure or maintenance, if the hazard rate of the core unit exceeds W, it is opportunistically replaced. The results confirmed that maintenance policy with opportunistic activities results in lower average cost compared to policies with only corrective maintenance. In this policy, two actions are allowed: replacement and doing nothing.

Shuai et al. [6] studied a system subjected to random soft and hard failures. The gamma process is applied to describe the deterioration process of the system, and a PHM is used to estimate the hazard rate of the system. An algorithm is proposed to assess the health of the system. The system is modeled as a whole, without considering the relationship between the units.

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Moghaddam [7] studied a multi-unit series system and proposed two non-linear programming models. A cost minimization model and a reliability maximization model are proposed to find an optimal preventive policy for the system. The system is inspected at equidistant decision epochs, and each unit can be repaired, replaced, or left operational.

In this paper, a maintenance policy is suggested for a two-unit series system where actions can be performed with different quality levels, ranging from minimal to perfect repair. The hazard rate of the main unit is estimated by the PHM. The problem is formulated using SMDP to achieve the optimal multi-level control policy.

This paper proposes an algorithm for the maintenance optimization of two-unit systems in the SMDP framework when general repair is allowed.

II. Model Description

The system is composed of two units: unit 1 is more expensive than unit 2 and subjected to condition monitoring, and only age information for unit 2 is available. The system is regularly inspected to monitor the hazard rate of unit 1. The deterioration of unit 1 follows a gamma process $\{X_t | t \geq 0\}$, which is represented as follows:

$$f(x) = Ga(x \mid \alpha(t), \beta) = \frac{\beta^{\alpha(t)} x^{\alpha(t)-1} e^{-\beta x}}{\Gamma(\alpha(t))}, x \geq 0 \quad (3)$$

where the shape and scale parameters of the gamma process are $\alpha(t) > 0$ and $\beta > 0$, respectively, and the gamma function is:

$$\Gamma(\alpha(t)) = \int_0^\infty z^{\alpha(t)-1} e^{-z} dz \quad (4)$$

A. Discretization of the Deterioration Process

The continuous gamma process $\{X_t | t \geq 0\}$ is discretized to represent the deterioration of unit 1 as a Markov process with finite state space $\Omega = \{0,1,2,\ldots,D\}$. The deterioration of unit 1 ranges from 0 (i.e., as-good-as-new) to D (absorbing state). Also, unit 1 age is discretized as $0, d, 2d, \ldots, N\Delta$ where N is the maximum useful life of the system, and the length of inspection interval is assumed to be $\Delta = md$ for $m \in N^+$.

Table I shows the relationship between the deterioration of unit 1 and respective discrete states, where w is the discretization interval.

TABLE I
DETERIORATION PROCESS

X_t	State space
$X_t = 0$	$z = 0$
$X_t \in ((z-1)w, zw]$	$z \in \{1, \ldots, D\}$
$X_t \in ((D-1)w, +\infty]$	$z = D$

The midpoint of each interval is considered as the representative of the whole interval. Transition probabilities from state z to z' over interval d, where $z, z' \in \Omega$ are calculated using the following equations:

1) When unit 1 is as-good-as-new and $z' = 0$:

$$P_{zz'}(id,(i+1)d) = 0 \quad (5)$$

2) When the deterioration process transits from $z = 0$ to $z' = D$, then:

$$P_{zz'}(id,(i+1)d) = \frac{\Gamma(\alpha((i+1)d) - \alpha(id), (D-1)w\beta)}{\Gamma(\alpha((i+1)d) - \alpha(id))} \quad (6)$$

3) If $z = 0$ and $z' \in \{1, \ldots, D-1\}$, then we have:

$$P_{zz'}(id,(i+1)d) = \frac{\Gamma(\alpha((i+1)d) - \alpha(id), (z'-1)w\beta)}{\Gamma(\alpha((i+1)d) - \alpha(id))} - \frac{\Gamma(\alpha((i+1)d) - \alpha(id), z'w\beta)}{\Gamma(\alpha((i+1)d) - \alpha(id))} \quad (7)$$

4) When $z = z' \in \{1, \ldots, D\}$:

$$P_{zz'}(id,(i+1)d) = 1 - \sum_{z \neq z'} P(id,(i+1)d) \quad (8)$$

5) If there is a transition from z to z' and $0 < z < z' < D$:

$$P_{zz'}(id,(i+1)d) = \frac{\Gamma(\alpha((i+1)d) - \alpha(id), (z'-z-0.5)w\beta)}{\Gamma(\alpha((i+1)d) - \alpha(id))} - \frac{\Gamma(\alpha((i+1)d) - \alpha(id), (z'-z+0.5)w\beta)}{\Gamma(\alpha((i+1)d) - \alpha(id))} \quad (9)$$

6) When $z \in \{1, \ldots, D-1\}$ and $z' = D$:

$$P_{zz'}(id,(i+1)d) = \frac{\Gamma(\alpha((i+1)d) - \alpha(id), (z'-z-0.5)w\beta)}{\Gamma(\alpha((i+1)d) - \alpha(id))} \quad (10)$$

The deterioration process cannot move into a healthier state without performing a maintenance action, and the state $z = D$ is an absorbing state.

A matrix-based approximation method developed by [8] is employed to discretize the joint age and deterioration process of unit 1, which is represented by $Y_t = (t, X_t)$. The probability of a successful transition of Y_t over an interval d is:

$$\begin{aligned} \Lambda_{z,z'}(id) &= P(\xi_1 \geq (i+1)d, x_{(i+1)d} = z' \mid \xi_1 \geq id, x_{id} = z) \\ &= P(x_{(i+1)d} = z' \mid \xi_1 \geq (i+1)d, x_{id} = z) \\ &.P(\xi_1 \geq (i+1)d \mid \xi_1 \geq id, x_{id} = z) \\ &= P_{zz'}(id,(i+1)d).R_1(d \mid id, z) \end{aligned} \quad (11)$$

In the above equation, the failure time of unit 1 is denoted by ζ_1, and id is the age of unit 1, where $i = 0, \ldots, Nm-1$. If the discretization interval is selected sufficiently small and $z > 0$, the conditional reliability of unit 1 is calculated as:

$$\begin{aligned} R_1(d \mid id, z) &= P(\xi_1 \geq (i+1)d \mid \xi_1 \geq id, x_{id} = z) \\ &= E[\exp\left(-\int_{id}^{(i+1)d} h_0(s)\psi(x_s)ds\right) \mid x_{id} = (z-0.5)w] \\ &\approx \exp\left(-\int_{id}^{(i+1)d} h_0(s)\psi((z-0.5)w)ds\right) \end{aligned} \tag{12}$$

When the unit is as-new, then we have:

$$R_1(d \mid id, z) \approx \exp\left(-\int_{id}^{(i+1)d} h_0(s)\psi(0)ds\right) \tag{13}$$

The transition probabilities of the joint process Y_t from state z to z' over an interval $(id,(i+1)d)$, where $z, z' \in \Omega$ and $i = 0, \ldots, Nm-1$, are represented by a $[D+1] \times [D+1]$ matrix $\Lambda(id)$. $(\mathbf{I} - \boldsymbol{\Lambda}(id))\mathbf{1}$ is a $[D+1] \times [D+1]$ matrix that calculates the probability of failure, where **I** is a $[D+1] \times [D+1]$ identity matrix and $\mathbf{1}$ is a $[\mathrm{D}+1]$ vector of 1. The following matrix represents the transition probability of the joint process Y_t:

$$\mathbf{P} = \begin{bmatrix} \mathbf{0} & \boldsymbol{\Lambda}(0) & \mathbf{0} & \ldots & \mathbf{0} & \ldots & \mathbf{0} & (\mathbf{I}-\boldsymbol{\Lambda}(0))\mathbf{1} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{\Lambda}(d) & \ldots & \mathbf{0} & \ldots & \mathbf{0} & (\mathbf{I}-\boldsymbol{\Lambda}(d))\mathbf{1} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \ldots & \boldsymbol{\Lambda}(id) & \ldots & \mathbf{0} & (\mathbf{I}-\boldsymbol{\Lambda}(id))\mathbf{1} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} & \ldots & \boldsymbol{\Lambda}((Nm-2)d) & (\mathbf{I}-\boldsymbol{\Lambda}((Nm-2)d)\mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} & \ldots & \boldsymbol{\Lambda}((Nm-1)d) & (\mathbf{I}-\boldsymbol{\Lambda}((Nm-1)d)\mathbf{1} \\ 0 & 0 & 0 & \ldots & 0 & \ldots & 0 & 1 \end{bmatrix}$$

where $\mathbf{0}$ is a $[D+1] \times [D+1]$ matrix of 0. For ease of calculations, matrix **P** is partitioned as follows:

$$\mathbf{P} = \left[\begin{array}{c|c} \mathbf{R} & \mathbf{P_F} \\ \hline \mathbf{0}^{\mathrm{T}} & 1 \end{array}\right] \tag{14}$$

where **R** is an $[Nm(D+1)] \times [Nm(D+1)]$ matrix and $\mathbf{P_F}$ is an $Nm(D+1)$ column vector that represents the failure probabilities:

$$\mathbf{P} = \left[\begin{array}{c|c} \mathbf{R} & (\mathbf{I}-\mathbf{R})\mathbf{1} \\ \hline \mathbf{0}^{\mathrm{T}} & 1 \end{array}\right] \tag{15}$$

where **I** is an $[Nm(D+1)] \times [Nm(D+1)]$ identity matrix, and $\mathbf{1}$ is an $Nm(D+1)$ vector of element 1. Considering the stochastic property of matrix **P**, we can calculate $\mathbf{P}^r$ as:

$$\mathbf{P}^r = \left[\begin{array}{c|c} \mathbf{R}^r & (\mathbf{I}-\mathbf{R}^r)\mathbf{1} \\ \hline \mathbf{0}^{\mathrm{T}} & 1 \end{array}\right] \tag{16}$$

where $r \in Z$. The probability of surviving over $(i+r)d$, given that the unit has survived after id with deterioration $z \in \Omega$ is:

$$\begin{aligned} R_1(rd \mid (id, z)) &= P(\xi_1 \geq (i+r)d \mid \xi_1 \geq id, x_{id} = z) \\ &= 1 - \pi_z(i)(\mathbf{I} - \mathbf{R}^r)\mathbf{1} \end{aligned} \tag{17}$$

which represents the reliability function of unit 1. $\pi_z(i)$ is a vector of $Nm(D+1)$ elements equal to zero except for the $(i(D+1)+z+1)_{th}$ element, which equals to 1.

B. The Structure of Policy I

Maintenance decisions are made considering the hazard rate of unit 1 and the age of unit 2 at each decision epoch. Perfect inspections are performed after each Δ time regularly. The hazard rate of unit 1 is estimated using (2). Two hazard control limits, L_1 and L_2, where $L_1 < L_2$ and an age control limit L_3 are considered as the thresholds for the maintenance of unit 1 and unit 2, respectively.

Preventive maintenance (PM) for unit 1 is as follows: If the hazard rate is less than L_1, unit 1 is left operational. If the hazard rate is between levels L_1 and L_2, general repair is performed, and if it exceeds L_3, unit 1 is replaced. PM is performed on unit 2 considering its age: if its age does not exceed L_3, no action is taken; otherwise, it is replaced.

When a unit fails, the whole system fails, and corrective maintenance (CM) is performed. When unit 1 fails, and the age of the system does not exceed $(N-1)\Delta$, and the deterioration of unit 1 is more than zero, general repair rectifies it to operation. However, if the deterioration of unit 1 equals to zero, minimal repair is performed. If unit 2 fails and the age of the system does not cross $(N-1)\Delta$, and the age of unit 2 does not exceed L_3, minimal repair is performed on unit 2; otherwise, it is replaced. If the system fails and system age reaches $(N-1)\Delta$, then both units are replaced.

When a unit fails, opportunistic maintenance (OM) is performed to improve the other unit. When unit 2 fails, and the deterioration of unit 1 is more than zero, general repair is performed on unit 1; otherwise, unit 1 is left operational. When unit 1 fails, if the age of unit 2 is less than L_3, unit 2 is left operational; otherwise, it is replaced.

III. Problem Formulation

The system state space can be expressed as $S_1 \cup S_2 \cup S_3$:
If both units are operational:

$$S_1 = \{(z, i, j, 0) \mid z \in \Omega, 0 \leq i, j \leq N\} \tag{18}$$

If unit 1 fails:

$$S_2 = \{(z, i, j, F_1) \mid z \in \Omega, 0 \leq i, j \leq N\} \tag{19}$$

If unit 2 fails:

$$S_3 = \{(z, i, j, F_2) \mid z \in \Omega, 0 \leq i, j \leq N\} \tag{20}$$

where $z \in \{0, \ldots, D\}$ represents the deterioration of unit 1, $i\Delta$ and $j\Delta$ represents the age of unit 1 and unit 2, respectively. N is the maximum lifetime of the system. At each decision epoch, actions $a_1, a_2 \in A = \{0,1,2,3\}$ are performed on unit 1 and unit 2, respectively, where 0,1,2 and 3 represent do nothing, minimal repair, general repair, and replacement, respectively. In policy I, general repair reduces the deterioration of unit 1 to zero without affecting the age of the system.

The problem is formulated in the SMDP framework, and the first step of the policy iteration algorithm is applied to find the optimal solution [9], which can be achieved by solving the following linear equations:

$$v_l = c_l(a_1,a_2) - g(L_1,L_2,L_3).\tau_l(a_1,a_2) + \sum_{m\in S} P_{l,m}(a_1,a_2)v_m \quad (21)$$
$$l \in S \text{ and } v_n = 0, \text{for an arbitrarily selected state } n \in S$$

where $g(L_1, L_2, L_3)$ is the optimal long-run expected average cost per unit time, and the other elements of the SMDP are defined as follows:

- $P_{l,m}(a_1, a_2)$: the probability of transition from state l to m, given that actions a_1 and a_2 are selected.
- $C_l(a_1, a_2)$: the expected cost of the system which is in state l and actions a_1 and a_2 are performed.
- $\tau_l(a_1, a_2)$: the expected sojourn time, i.e., the expected time until the next decision epoch, when the current state of the system is l, and actions a_1 and a_2 are taken.

A. Transition Probabilities

Transition probabilities are calculated using the following equations:

When both units are operational, the system is in the state $(z, i, j, 0)$, $j\Delta < L_3$, $h(i\Delta, z) < L_1$, and $Max(i, j) < N$, then no PM is performed, and the system transits to state $(z', i+1, j+1, 0)$ with the following probability:

$$P_{(z,i,j,0),(z',i+1,j+1,0)}(0,0) = P_{zz'}(i\Delta,(i+1)\Delta).R_1(\Delta \mid i\Delta, z).\frac{R_2((j+1)\Delta)}{R_2(j\Delta)} \quad (22)$$

where the first term is calculated by (5)-(10), the second term is obtained by (17), and R_2 is the reliability function for unit 2.

The failure probability of unit 1 is:

$$P_{(z,i,j,0),(z,i,j,F_1)}(0,0) = \sum_{k=1}^{m}(R_1((k-1)d \mid i\Delta,z) - R_1(kd \mid i\Delta,z)).\frac{R_2(j\Delta+kd)}{R_2(j\Delta)} \quad (23)$$

The failure probability of unit 2 is:

$$P_{(z,i,j,0),(z,i,j,F_2)}(0,0) = \sum_{k=1}^{m}\frac{R_2(j\Delta+(k-1)d) - R_2(j\Delta+kd))}{R_2(j\Delta)}.R_1(kd \mid i\Delta,z) \quad (24)$$

In cases that a maintenance action is performed, the system will deterministically transit from its current state to a healthier one.

If unit 1 fails, $z > 0$, $j\Delta < L_3$ and $Max(i, j) < N - 1$, then we have:

$$P_{(z,i,j,F_1),(0,i,j,0)}(2,0) = 1 \quad (25)$$

If the age of the system exceeds N, then both units are replaced preventively:

$$P_{(z,N,j,0),(0,0,0,0)}(3,3) = 1 \quad (26)$$

If unit 1 fails and the age of the system exceeds $N - 1$, then both units are replaced:

$$P_{(z,N-1,j,F_1),(0,0,0,0)}(3,3) = 1 \quad (27)$$

If unit 2 fails and the age of the system exceeds $N - 1$, then both units are replaced:

$$P_{(z,N-1,j,F_2),(0,0,0,0)}(3,3) = 1 \quad (28)$$

B. Expected Sojourn Times

Maintenance actions can be performed on both units simultaneously, and inspection time is negligible. The following notation is used to calculate the expected sojourn times.

TABLE II
TIME NOTATION

Notation	Definition
T_{PG1}	time of preventive/opportunistic general repair of unit 1
T_{PR1}	time of preventive/opportunistic replacement of unit 1
T_{CR1}	time of corrective replacement of unit 1
T_{CG1}	time of corrective general repair of unit 1
T_{M1}	time of minimal repair of unit 1
T_{PR2}	time of preventive/opportunistic replacement of unit 2
T_{CR2}	time of corrective replacement of unit 2
T_{M2}	time of minimal repair of unit 2

When both units are operational, $j\Delta < L_3$, $h(i\Delta, z) < L_1$, and $Max(i, j) < N$, then no PM is performed and the expected sojourn time is:

$$\begin{aligned}\tau_{(z,i,j,0)}(0,0) &= \sum_{k=1}^{m}\left[R_1((k-1)d \mid i\Delta,z) - R_1(kd \mid i\Delta,z)\right] \\ &.R_2(j\Delta+kd \mid j\Delta).kd \\ &+\sum_{k=1}^{m}\left[R_2(j\Delta+(k-1)d \mid j\Delta) - R_2(j\Delta+kd) \mid j\Delta)\right] \\ &.R_1(kd \mid i\Delta,z).kd \\ &+\sum_{z'\geq z} P_{zz'}(i\Delta,(i+1)\Delta).R_1(\Delta \mid i\Delta,z).R_2((j+1)\Delta \mid j\Delta).\Delta\end{aligned} \quad (29)$$

The expected sojourn time of the system when an action is performed equals to the action fulfillment time.

When unit 2 fails, $Max(i, j) < N - 1$, $j\Delta < L_3$, and $h(i\Delta, z) > L_2$, then we have:

$$\tau_{(z,i,j,F_1)}(1,3) = Max(T_{M2}, T_{PR2}) \quad (30)$$

If the age of the system exceeds N, then both units are replaced preventively:

$$\tau_{(z,N,j,0)}(3,3) = Max(T_{PR1}, T_{PR2}) \tag{31}$$

If unit 1 fails and the age of the system exceeds $N-1$, then both units are replaced:

$$\tau_{(z,N-1,j,F_1)}(3,3) = Max(T_{CR1}, T_{PR2}) \tag{32}$$

If unit 2 fails and the age of the system exceeds $N-1$, then both units are replaced:

$$\tau_{(z,N-1,j,F_2)}(3,3) = Max(T_{PR1}, T_{CR2}) \tag{33}$$

C. Expected Cost

The costs of the system are defined in Table III. When no failure occurs, $j\Delta < L_3$, $h(i\Delta, z) < L_1$, and $Max(i,j) < N$, then no action is performed, and the expected cost is:

$$C_{(z,i,j,0)}(0,0) = \sum_{z'\geq z} C_I . P_{zz'}(i\Delta,(i+1)\Delta).R_1(\Delta \mid i\Delta, z).\left(\frac{R_2(j+1)\Delta}{R_2(j\Delta)}\right) \tag{34}$$

If unit 1 fails, $z > 0, j\Delta \geq L_3$, and $Max(i,j) < N-1$, then we have:

$$C_{(z,i,j,F_1)}(2,3) = C_{CG1} + C_{PR2} + Max(T_{CG1}, T_{PR2}).C_L + C_S \tag{35}$$

If the age of the system exceeds N, then both units are replaced preventively:

$$C_{(z,i,j,0)}(3,3) = C_{PR1} + C_{PR2} + Max(T_{PR1}, T_{PR2}).C_L + C_S \tag{36}$$

If unit 1 fails and the age of the system exceeds $N-1$, then both units are replaced:

$$C_{(z,i,j,0)}(3,3) = C_{CR1} + C_{PR2} + Max(T_{CR1}, T_{PR2}).C_L + C_S \tag{37}$$

TABLE III
COST NOTATION

Notation	Definition
C_I	inspection cost
C_{PG1}	cost of preventive/opportunistic general repair of unit 1
C_{PR1}	cost of preventive/opportunistic replacement of unit 1
C_{CR1}	cost of corrective replacement of unit 1
C_{CG1}	cost of corrective general repair of unit 1
C_{M1}	cost of minimal repair of unit 1
C_{PR2}	cost of preventive/opportunistic replacement of unit 2
C_{M2}	cost of minimal repair of unit 2
C_{CR2}	cost of corrective replacement of unit 2
C_S	cost of system setup incurred when the system restarts.
C_L	cost of production loss per unit

If unit 2 fails and the age of the system exceeds $N-1$, then both units are replaced:

$$C_{(z,i,j,0)}(3,3) = C_{PR1} + C_{CR2} + Max(T_{PR1}, T_{CR2}).C_L + C_S \tag{38}$$

IV. APPLICATION

In this section, the results of the modeling of a two-unit series system are presented. The deterioration of unit 1, which is subjected to CBM, follows the gamma process, whose scale and shape parameters are $0.276t$ and 4.86, respectively. The hazard function of unit 1 is Weibull distributed, and the scale parameter, shape parameter, and the covariate coefficient are 85.42 and 4.63 and 0.281, respectively. The data belong to a feed subsystem of a boring machine published in [10]. The deterioration of unit 1 is discretized and takes value on $\Omega = \{0,1,2,\dots,8\}$. The discretization interval w equals to 0.1. The cost and time parameters are presented in Table IV.

TABLE IV
TIME AND COST PARAMETERS

Cost	$	Time	Days
C_I	12	T_{PG1}	1.6
C_{PG1}	357	T_{PR1}	1.8
C_{PR1}	4384	T_{CR1}	2.6
C_{CR1}	6746	T_{CG1}	1.4
C_{CG1}	1130	T_{M1}	0.4
C_{M1}	754	T_{PR2}	0.6
C_{PR2}	1378	T_{CR2}	1
C_{M2}	112	T_{M2}	0.3
C_{CR2}	2119		
C_S	300		
C_L	100		

The length of the inspection interval is $\Delta = 10$ days, which is discretized into 10 intervals with the length of 1 day. The maximum useful lifetime of the system is $N\Delta = 50$.

To achieve the optimal solution, we solved the system of linear equations, according to the first step of the policy iteration algorithm, considering a variety of different control limits for each unit. Table V shows the long-run expected average cost per unit time, where $L_1 = 0.0034$ and different sets of L_2 and L_3 are examined.

TABLE V
CONTROL LIMITS AND AVERAGE COST - POLICY I

L_2 \ L_3	Δ	2Δ	3Δ	4Δ	5Δ
0.0046	$603.10	$357.15	$271.43	$151.76	$122.11
0.0060	$590.15	$357.77	$275.21	$157.83	$127.54
0.0080	$582.85	$358.66	$278.11	$161.74	$131.13
0.0106	$578.67	$359.35	$280.02	$164.20	$133.42

The lowest cost can be achieved when $L_1 = 0.0034$, $L_2 = 0.0046$ and $L_3 = 5\Delta$, which results it the optimal $g^*\left(L_1^*, L_2^*, L_3^*\right) = \$122.11$.

The best solution for policy II, in which replacement is performed instead of general repair, is represented in Table VI where $L_1 = 0.0080$, $L_2 = 0.0106$, and $L_3 = 5\Delta$ and the long-run expected average cost per unit time is $g^*\left(L_1^*, L_2^*, L_3^*\right) = \$166.34$.

TABLE VI
CONTROL LIMITS AND THE AVERAGE COST - POLICY II

L_2 \ L_3	Δ	2Δ	3Δ	4Δ	5Δ
0.0106	$648.32	$434.71	$325.19	$230.33	$166.34

V. CONCLUSION

Although a wide range of maintenance activities are performed on mechanical systems, a large number of proposed policies have incorporated replacement only. In addition, considering the interaction between units is of great significance to formulate an effective maintenance policy, which has been overlooked in many studies. The proposed policy I fills these gaps by allowing minimal and general repair as well as replacement for a two-unit series system subject to random failure.

We examined the results of implementing two maintenance policies and found that policy I outperforms policy II in which repair is not considered. The results show that implementing policy I results in a 26.6% reduction in the long-run expected average cost per unit time compared with policy II. The comparison of two policies confirms that the appropriate scheduling of repair activities results in the effective utilization of equipment lifetime.

The proposed policy is applicable to multi-unit physical assets with a core part whose reliability is of high priority, and units interact with each other. An example of such systems is the engine of haul trucks, which is composed of a gearbox and a clutch, where the gearbox is subject to condition monitoring, and the age information is available for the clutch.

In this research, it is assumed that general repair brings the deterioration of the unit to zero without changing its age. However, this assumption can be relaxed by applying Kijima's models, which incorporate the effect of general repair on the virtual age of the system [4].

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