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Einstein's General Equation of the Gravitational Field

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***Abstract*—The generalization of relativistic theory of gravity based essentially on the principle of equivalence stipulates that for all bodies, the grave mass is equal to the inert mass which leads us to believe that gravitation is not a property of the bodies themselves, but of space, and the conclusion that the gravitational field must curved space-time what allows the abandonment of Minkowski space (because Minkowski space-time being nonetheless null curvature) to adopt Riemannian geometry as a mathematical framework in order to determine the curvature. Therefore the work presented in this paper begins with the evolution of the concept of gravity then tensor field which manifests by Riemannian geometry to formulate the general equation of the gravitational field.

***Keywords*—Inertia, principle of equivalence, tensors, Riemannian geometry.

I. Introduction

GENERAL Relativity is the theory of Gravitation determined by Einstein equations, General Relativity is geometric theory which is based essentially on Riemannian and Lorentzian geometry. It is the theory which describes matter and energy near massive objects like stars, galaxies, black holes, and it describes the universe as a whole (curvature). Since the Special Relativity is valid when gravitational effects are negligible, Einstein desired to generalize his theory to describe the gravity when the gravitational field is considered; this imposed the replacement of Minkowski metric η, by the space-time Lorentzian metric, denoted by g. η metric is defined by the signature (+, -, -, -) that determined the flatness of Minkowski space (this signature is used specifically in special relativity because the speed of light is the limit speed) [4]; in this case the g metric is equivalent to the η metric, but in general relativity when the gravitational field is considered, the equivalence principle imposes that the light follows a geodesic line which is expressed as the curvature of space-time produced from the effects of the matter (energy) when the signature of the metric changes to (-, +, +, +) which is represented by the space-time Lorentzian metric [2], [1, section 1.11].

The physical facts that inspired Einstein's genius are the principle of general covariance [1, section 1.5] and the Galileo Newton equivalence principle.

The principle of equivalence goes back to Copernic's idea that said that the laws of physics must be similar everywhere in the universe, and then this idea was developed by Einstein, who emphasized that *the force of gravity by which objects affect each other is indistinguishable from forces of inertia*. In physics, there are two terms: the mass of inertia and the gravitational mass. The principle of equivalence says that these two masses are the same for systems whose speed is much lower than the speed of light. This can be understood from Newtonian mechanics equations and Einstein mechanics equations. The fundamental law of Newtonian dynamics $F = m_I \gamma$ relates the acceleration γ of a test particle in Galileo Newton absolute space-time $E^3 \times R$ to the force F acting on it and to its *inertial mass* m_I. In Newtonian mechanics, this inertial mass is a constant, depending only on the nature of the particle. One can defined *gravitational mass* m_G through the gravitational potential U generated by a particle P, and satisfies the Poisson equation

$$\Delta U = -4\pi G\rho$$

where ρ is the gravitational mass density. This particle generates a potential with value $U = m_G/r$at a point $x$at distance $r$from the origin O.

Up to here, we can identify the gravitational mass from the Newtonian principle of equality of action and reaction, where it is a parameter associated with material bodies submissive or creating a gravitational force, what leads to expressed the Galileo-Newton equivalence principle by the equivalence of gravitational mass and the inertial mass; this is known also as the weak equivalence principle.

We can also speak about the weak Einstein equivalence principle, which stipulates that equations of motion of massive point-like objects in free fall are independent of their mass because and from General relativity theory they follow time-like geodesics of the metric when these equations contain terms which expressed inertial forces that appear as a Christoffel symbols and described by another side their relative acceleration, so we can say that this principle stated that there is no intrinsic splitting between gravity and inertial-type forces (see Chapter I [2]). The Newtonian equations in space are replaced in relativistic dynamics by a spatio-temporal equation given by $m_0(\ddot{u} + \Gamma^k_{ij}\dot{u}^i\dot{u}^j) = F^\alpha$ at local coordinates when the connection coefficients Γ disappear, the last equation is reduced to an analogue of Newton's equation in a Galilean frame. This remark leads to interpret the term implying Γ as a kind of inertial force, therefore we can conclude that the gravity and relative acceleration, at that point, exactly balanced (One of them reflects the other) [2, section III]. Newtonian gravitational potential does not affect the Newtonian absolute space-time structure. Then, Einstein's new idea is that space and time are united in a four-dimensional curved Lorentzian manifold whose metric g is linked with the energy content of space-time. This Lorentzian metric governs the space-time *causalit y*structure [2, section I], it is defined as a pseudo-Riemannian metric g and is called

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a Lorentzian metric if the signature of the quadratic form defined by g is (- ++....+). In the case of a manifold with a Lorentzian metric, we denote its dimension by $n + 1$ and use Greek indices $\alpha, \beta, ... = 0, 1, 2, ..., n$ for local coordinates and local frames. A Lorentzian metric is a quadratic form

$$g \equiv g^{\alpha\beta} dx^{\alpha} dx^{\beta} \quad (1)$$

A metric of fundamental physical importance in the case $n = 3$ is the Minkowski metric. For a general n, it is the flat metric on R^{n+1}thatreads

$$g \equiv -(dx^0)^2 + \sum_1^3 (dx^i)^2 \quad (2)$$

Einstein's equivalence principle considered as a unification of gravitation and inertia that is the road of Einstein to General Relativity theory. In General Relativity Einstein said that the matter (energy) causes space-time to curve and he presented an evidence from new description of the equation of motion of an objects in free falling which is translated by geodesics lines that by definition are the minimum path between two markedly different points, and their equations of motion are therefore independent of their mass, reading, in arbitrary coordinates [3], [4],

$$\ddot{u}^k + \Gamma_{ij}^k \dot{u}^i \dot{u}^j = 0 \quad (3)$$

where Γ^k is Christoffel's Symbols, u^k is the world line defined as $u^k = {}_{dx}k$ and s is the proper time. The connection Γ of the metric represents in the coordinates x^{α} both gravitation and inertia.

II. Geometric Properties

A. Poisson Equation

Since the material is described as a continuous medium of mass density ρ, the potential U becomes

$$U(p) = -G \int \frac{\rho(t, P\prime) dv\prime}{r_{pp\prime}} \quad (4)$$

where $r_{pp\prime} = |PP'| = |R - R'|$ is the distance between the point P where$U$ is evaluated and the point $P'$of the extended body creating the field, and dV is the volume element in the chosen coordinates. If the mass distribution P' is symmetric spherical, the potential U at P is equal to that created by a point of the same total mass located at the center of the distribution. Indeed, we place the body at the origin of the frame; by symmetry we can place the point P in $(0,0,r)$, components Cartesian of the position vector of a point P' are in spherical coordinates: $(r'\sin\theta\cos\varphi, r'\sin\theta\sin\varphi, r'\cos\theta)$; the potential in P is therefore

$$U(r) = -G \int \frac{\rho(r', t)}{\sqrt{r'^2 + r^2 - 2rr'\cos\theta}} r'^2 \sin\theta dr' d\theta d\varphi \quad (5)$$

which integrates with a view to give,

$$U(r) = -\frac{GM}{r}, \text{if } r > r_0 \quad (6)$$

where $M = 4\pi \int_0^{r_0} \rho r'^2$ the total mass of the body and r is its radius.

Using the similarity between the classical laws of gravitational field and electromagnetics field (note that this similarity ends in the borders of pecial relativity when the mass of bodies is not absolute but the electric charge remains unchanged) we can introduce the Poison equation as a deferencial equation between $U$and ρ, which transforms the Newtonian theory of gravitation into a field theory [6]:

$$\Delta U = 4\pi G\rho \quad (7)$$

Since the study of the gravitational potential was using spherical coordinate it has to define *Gaussian curvature K* of a surface at a point as the product of the principal curvatures $k_1$andk_2 of the surface at the given point: $K = k_1 k_2$. For example, a sphere of radius r has Gaussian curvature $1/r^2$. A positive Gaussian curvature means that the principal curvatures are of the same sign. The surface is locally similar to a distorted sphere, and the geometry is locally elliptic. If the Gaussian curvature vanishes, one of the principal curvatures have to vanish and the geometry is locally planar and the geometry is locally Euclidean. Finally, if the Gaussian curvature is negative the surface is locally saddle-shaped, i.e. like a hyperbolic surface. In this case, the geometry is said to be locally hyperbolic [7].

B. Christoffel's Symbol

Christoffel symbol is important element in mathematics of General Relativity and it has several properties; we show here some of them [2], [5]. Therefore, the Quantities du^k are contravariant components of the vector dM on the natural basis e_k. On the other hand, the vectors e'^i will be able to be determined by calculating the elementary variations of the vectors e_i, respect to the natural frame (M, e_i), when we passed from the point M to the point M'; then $e'_i = e_i + de_i$. The calculation of the vectors de_i then remains the essential problem to be solved. We are going at first to study an example of this type of calculation in spherical coordinates. The de_i differentials are thus decomposed on the natural basis e_i. If we denote by ω^k the contravariant components of the vector de_i, this one is written

$$de_i = \omega^k e_k$$

The components ω^k of the vectors de_i are differential forms (linear combinations of differentials). For example,

$$\omega^2 = d\theta/r, \ \omega^3 = dr/r + \cot\theta d\theta$$

If we generally write u_i the spherical coordinates, we have:

$$u^1 = r, \ u^2 = \theta, \ u^3 = \phi \quad (8)$$

The differentials of the coordinates are then noted:

$$du^1=dr,\ du^2=d\theta,\ du^3=d\varphi \quad (9)$$

and the components ω^j are written in a general way:

$$\omega^k=\Gamma^j du^k \quad (10)$$

where the quantities Γ^{ki} are functions of $r,\ \theta,\ \varphi$ which will be explicitly obtained by identifying each component ω^k. For example, the component ω^3 is written with the notation of the (10)

$$\omega^3= (dr/r) + \cot\theta d\theta= \Gamma^3 du^1+ \omega^3 du^2+ \Gamma^3 du^3 \quad (11)$$

By doing the same with the nine components ω^j, we obtain the 27 terms Γ^j. For any curvilinear coordinate system, these quantities Γ^j are called *Christoffel Symbols.*

C. Christoffel Symbols of the Second Kind

For a punctual space ε_n and a system of curvilinear coordinates u_i, any differential vectors $de_i= \omega^k e_k$ of the natural base are written on this basis:

$$de_i=\omega^j e_j=\Gamma^j du^k e_j \quad (12)$$

The calculation of the n^3 symbols of Christoffel is done from $n(n+1)/2$ quantities g_{ij}. Starting from the definition of these quantities:

$$g_{ij}= e_i\cdot e_j \quad (13)$$

We obtain by differentiation of this last relation where ($de_j=\omega_j^l e_l$):

$$dg_{ij}= e_i\cdot de_j+ e_j\cdot de_i= g_{il}\omega^l+ g_{jl}\omega^l \quad (14)$$

The expression $g_{il}\omega^l$ represents the covariant component ω_{ij} of the vector de_j, taking into account the contravariant components as a function of the Christoffel symbols:

$$\omega_{ij} = g_{il}\omega_j^l = g_{il}\Gamma_{kj}^l du^k \quad (15)$$

Then, we obtain the expression linking the Christoffel symbols of the first and second kind:

$$\Gamma_{kij}=g_{il}\Gamma^l$$

Substituting this last relation in (15), we obtain:

$$\omega_{ij}=\Gamma_{kij}du^k \quad (16)$$

The expression linking the Christoffel symbols of each species is obtained:

$$\Gamma_{kij}= g_{jl}\Gamma^l \text{ or } \quad \Gamma^j= g^{jl}\Gamma_{kli} \quad (17)$$

On the other hand, the differential of the function g_{ij} is also written:

$$dg_{ij}=(\partial_k g_{ij})du^k \quad (18)$$

from where by identifying the coefficients of the differential du^k in these last two expressions:

$$\Gamma_{jik}+ \Gamma_{kji}= \partial_k g_{ij} \quad (19)$$

$$\Gamma_{kji}+ \Gamma_{ikj}= \partial_i g_{jk} \quad (20)$$

$$\Gamma_{ikj}+ \Gamma_{jik}= \partial_j g_{ki} \quad (21)$$

Let's sum (19) and (20) and subtract (21) it be- comes:

$$\Gamma_{kji}= 1/2(\partial_k g_{ij}+ \partial_i g_{jk}-\partial_j g_{ki}) \quad (22)$$

This is the expression of the Christoffel symbols of the first kind according to the partial derivatives of the g_{ij} components of the fundamental tensor. We get those from second species from (17) and (22), namely:

$$^i=g^{il}\Gamma_{klj}=1/2g^{il}(\partial_k g_{jl}+\partial_j g_{lk}-\partial_l g_{kj}) \quad (23)$$

Expressions (22) and (23) allow the effective calculation of symbols of Christoffel for a given metric. When the quantities g_{ij} area priori data, we can study the properties of the point space defined by the data of this metric, which is the case of the spaces of Riemann.

D. Absolute Differential of a Vector

We write a vector V with components v^i on basis e_i as [4]: $V = v^i e_i$ and its differential $dV= dv^i e_i+v^i de_i$ but we know that $de_i = \omega^j e_j$ and $de_h = \omega^i e_i$, which allows us to get $v^h de_h= v^h\omega^i e_i$

$$dV=dv^i e_i+v^h\omega^i e_i \quad (24)$$

The quantities $dV= dv^i e_i+ v^h\omega^i e_i$ constitute the covariant components, with respect to the basis e_i, of the absolute differential of the vector V. Contravariant components of vector dv can be written as:

$$\nabla v^i= dv+\omega^i v^h \quad (25)$$

By a similar computation: $v_i=dv_i\,\omega^h v_h$ can develop the differential dv^k and introducing the symbols of Christoffel instead of ω^i,

$$dv_k= (\dot{\partial}_j v_k- v_i\Gamma^i)dy^j \quad (26)$$

We deduce the absolute differential of the vector V and its expression on a natural frame $(M,\ e_i)$:

$$\nabla v^i = \frac{d}{dy^k} v^i dy^k + \Gamma_{kh}^i v^h dy^k = \left(\partial_k v^i + \Gamma_{kh}^i v^h\right)dy^k$$

where $\omega^j=\Gamma^i dy^k$ so we find that,

$$\frac{\nabla v^i}{dy^i} = \nabla_k v^i = \partial_k v^i + \Gamma_{kh}^i v^h$$

$$Then\ dV = dv^k e_k = (\partial_j v^k + v^i \Gamma_{ij}^k) dy^j e_k \quad (27)$$

Case of a T tensor of the second order defined by mixed components t^j:

$$\nabla t_i^j = dt_i^j + \omega_i^h t_h^j + \omega_h^j t_i^h \quad (28)$$

and as a value of the covariant derivative of the tensor T:

$$\nabla_k t_i = \partial_k t_i - \Gamma_{ki} t_h + \Gamma_{kh} t_i \quad (29)$$

E. Ricci Theorem

The absolute difference of the fundamental tensor g_{ij} is zero. Indeed, let g_{ij} be the fundamental content [4],

$$\nabla g_{ij} = dg_{ij} - \omega_i^h g_{hj} - \omega_j^h g_{ih} \quad (30)$$

Indeed: $e_i \cdot e_j = g_{ij}, e_i de_j + e_j de_i = dg_{ij} de_i = \omega_i^j e_j = \omega_i^h e_h, de_j = \omega_j^h e_j$ so we find that:

$$dg_{ij} = \omega_j^h g_{ih} + \omega_i^h g_{ih} \quad (31)$$

by substitution we find:

$$\nabla g_{ij} = 0 \quad (32)$$

. Change of Basis

Consider two curvilinear coordinate systems, u^i and u'^j, corresponding to natural basis e_i and e'_j, let two bases $(e_1, e_2, e_3,)$ and (e'_1, e'_2, e'_3) be of a vector space E_n. Each vector of a basis can be decomposed on the other basis in the following form:

$$e_i = A'^k_i e'_k \quad (33)$$

$$e'_k = A^i_k e_i \quad (34)$$

where the coefficients A'^k_i and A^i_k written depend the partial derivatives as the following forms:

$$A'^k_i = \partial_i u'^k A^i_k = \partial_k u^i \quad (35)$$

On the other hand, we have the following expressions of the differentials:

$$dA'^k_i = \partial_l A'^k_i du^l \,;\, du'^h = A'^h_l du^l \quad (36)$$

By identifying the coefficients of the same vector e_j we obtain also:

$$\omega_i^j = A'^k_i A^j_m \omega'^m_k + A^j_k dA'^k_i \quad (37)$$

By expressing the quantities that appear in (37) according to Christoffel's symbols, it becomes:

$$\Gamma_{li}^j du^l = A'^k_i A^j_m \Gamma'^m_{hk} du'^h + A^j_k dA'^k_i \quad (38)$$

On the other hand, we have the following expressions of the differentials:

$$dA'^k_i = \partial_l A'^k_i du^l du'^h = A'^h_l du^l \quad (39)$$

So, as we see it can write directly:

$$\Gamma_{li}^j = A'^k_i A^j_m A'^h_l \Gamma'^m_{hk} + A^j_k \partial_l A'^k_i \quad (40)$$

F. Definition of Riemann Spaces:

A Riemann space is a variety to which a metric has been attached. This means that in each part of the manifold, represented analytically by means of a coordinate system u^i, we have given a metric defined by the quadratic form: $ds^2 = g_{ij} du^i du^j$

The coefficients g_{ij} are not entirely arbitrary and must verify the following conditions:

- The components g_{ij} are symmetrical: $g_{ij} = g_{ji}$.
- The determinant of the matrix $[g_{ij}]$ is not zero.
- The differential form ds^2 and therefore the concept of distance defined by g_{ij} is invariant to any coordinate system change.
- All partial derivatives of order two of g_{ij} exist and are continuous (we say that g_{ij} are of class C^2).

A Riemannian space is therefore a space of points, each being spotted by n coordinates u^i, with any metric of the form ds^2 verifying the above statements. This metric is called Riemannian. If the metric is positive definite, that is, when $g_{ij} v^i v^j$, for any vector v not null, we say that space is properly Riemannian. In this case, the determinant of the matrix [gij] is strictly positive and all the eigenvalues of this matrix are strictly positive.

G. Riemannian Curvature

The displacement associated with an elementary cycle is reduced to a rotation of an angle $d\varphi$ in the case of a two-dimensional Riemann space. This rotation is expressed using the rotation tensor Ω^l for a space of any dimension. If the Riemannian curvature at a point M does not change with the orientation of the direction in M, then M is called an isotropic point. We prove that the Riemannian curvature, at an isotropic point, is given by [5]:

$$K = \frac{R_{ijkl}}{g_{ik} g_{jl} - g_{il} g_{jk}} = \frac{R_{1212}}{g}$$

where R_{ijkl} is the form of covariant components of the Riemann-Christoffel tensor R^k and

$$R_{ijkl} = g_{jk} R^k \quad (42)$$

Riemann-Christoffel tensor is written as:

$$R^l_{ijk} = \partial_j \Gamma^l_{ik} - \partial_k \Gamma^l_{ij} + \Gamma^r_{ik} \Gamma^l_{jr} - \Gamma^r_{ij} \Gamma^l_{kr} \quad (43)$$

In normal coordinates z^i, the expression of the curvature tensor simplifies since the symbols of Christoffel are all null; it becomes:

$$R_{ijrs} = \frac{1}{2}\left(\partial_{ri}\, g_{sj} + \partial_{sj} g_{ir} - \partial_{rj} g_{is} - \partial_{si} g_{rj}\right) \quad (44)$$

The expression of the curve given by (41) is reduced for a space of two dimensions to:

$$K = \frac{R_{1212}}{g_{11}g_{22} - g_{12}^2} = \frac{R_{1212}}{g} \quad (45)$$

H. Ricci Tensor and Scalar Curvature

The contraction of the Riemann-Christoffel R^k_{irs}tensor with respect to the indices *k* and *r* leads to the tensor:

$$R_{is} = R^k_{iks} = \partial_k \Gamma^k_{is} - \partial_s \Gamma^k_{ik} + \Gamma^l_{is}\Gamma^k_{kl} - \Gamma^l_{ik}\Gamma^k_{sl} \quad (46)$$

The tensor R_{is} is called Ricci tensor and we will see that it enters in the equations of general relativity. Its mixed components are given by:

$$R^i = g^{ik} R_{kj} \quad (47)$$

The contraction of the Ricci tensor allows to obtain the scalar curvature noted *R*:

$$R = g^{ij} R_{ij} = R^i_i \quad (48)$$

A Riemann space is flat if and only if its curvature is zero in all point.

Now we may to introduce the term expressed the geometry of curved space time, this term is ***Einstien tensor*** given as

$$S_{ij} = g_{ik} S^k_j = R_{ij} - \frac{1}{2} g_{ik} R$$

It verifies the identities:

$$\nabla_k S_r = 0 \quad (50)$$

And since it verifies these identities, it is called a conservative tensor. For Subsections II *B-F* see also [4].

III. Einstein's Field Equations

Having decided upon a description of gravity and its action on matter that is based on the idea of a Riemann manifold (i.e. fitted with a metric connection), general relativity chooses the Riemann tensor, which characterizes curved space-times i.e. the metric of space-time. The principle of equivalence in General Relativity (GR) translates gravitation by the curvature associated with the metric g of space-time. In next steps we determine the tensors which describe the matter and field with assure that for weak field these tensors reduced to classical theory expressed by Poison equation, [3], [1], Poison equation for weak (Newtonian) field is given by:

$$\nabla \varphi = 4\pi G \rho \quad (51)$$

where ρ is the density of mass and φ is the gravitational potential. Its solution for a point particle of mass m is

$$\varphi = -\frac{G\, m}{r}$$

In classical theory, any distribution of matter is associated with a gravitational potential via the Poisson equation. In the theory of general relativity, a distribution of matter no longer creates a gravitational field, but it curves the space-time.

Thanks to the equivalence of mass and energy that Einstein had discovered before, we might be tempted to use the mass density *ρ,* appears in (51), as the source of the relativistic gravitational field, and consider it as the density of total energy [1, Section1.14]. It took a considerable time for Einstein to settle on the choice of equations connecting the metric g with sources. From a physical point of view, he was inspired on the one hand by the equivalence principle and from the energy and momentum correlation (due to the equivalence mass-energy) [2] and on the other hand by the conservation laws for the various stress energy momentum tensors *T* found in Special Relativity. From a mathematical point of view Einstein's theory of gravity, space-time is modeled by a Lorentzian manifold of dimension four (M; g). The distribution of matter and energy in the universe is described by a field of symmetrical bilinear forms of zero divergence, denoted T and called energy-momentum tensor as we mentioned. The geometry of space-time is encoded by a second tensor, called Einstein tensor, noted $G_{\mu\nu}$. The geometric equations found by Einstein in 1915, with the help of the mathematician Marcel Grossmann [2] are:

$$Einstein(g) := Ricci(g) - gR(g) = G_E T \quad (53)$$

So,

$$S_{\alpha\beta} =: R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = G_E T_{\alpha\beta}$$

The factor G_E is a phenomenological dimensional constant.

$$G_E = 8\pi\, G_N \quad (55)$$

where the mass density is considering source which creates the gravitational field in classical theory. It may consider the stress energy momentum symmetric $2 -$ tensor $T_{\alpha\beta}$ as a source of gravitational field in GR theory (gravitational field that equivalent to curved space-time) which represent all the energies, momenta and stresses present in the space-time.

Where G_N is the Newtonian gravitational constant, the contracted Bianchi identities show that the Einstein equations are compatible only if the tensor T satisfies the equations, called conservation laws,

$$\nabla_\alpha T = 0 \quad (56)$$

So we shall reject ρ as source and instead insist that the generaliza6tion of Newton's mass density should be T^{00}.

For the particles, we can say that the stress energy tensor will not be conserved if the particles are subject to forces that act at a distance, which expressed by: $\nabla_\alpha T^{\alpha\beta} = G^\beta$ where G^β is the density of the external force f^βacting on the system (for an isolated system, $G^{\beta}=0$).

Let us recall that in the Newton limit when the field is so weak, the time-time component of the metric tensor is approximately given by [1, section 1.20]

$$g_{00} \approx -(1+2\varphi)$$

Here φ is the Newtonian potential, determined by Poisson's equation. Furthermore, the energy density T_{00} for nonrelativistic matter is just equal to its mass density

$$T_{00}=\rho \tag{59}$$

Combining the above, we have then

$$\Delta g_{00}=-8\pi G T_{00} \tag{60}$$

Equation (60) leads us to guess that the equation which governs gravitational fields of arbitrary strength must take the form [3]

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{61}$$

$G_{\mu\nu}$ is a tensor determined by Ricci tensor and curvature scalar [3].

In 1917 Einstein introduced the concept the cosmological constant Λ to counter-balance the effects of gravity and achieve a static universe, then, the cosmological constant is the energy density of space, or vacuum energy, that arises in Albert Einstein's field equations of general relativity. Einstein abandoned the concept in 1931 after *Hubble's* discovery of the expanding universe. From the 1930 until the late 1990, most physicists assumed the cosmological constant to be equal to zero. Einstein wrote May 23, 1923 a postcard addressed to the mathematician *Weyl*: "If the universe is not quasi-static, then to the devil the cosmological constant!" [8]

Another approach is to try to determine what may be the vacuum fluctuation energy at the Planck length scale. *Quantum field theory possesses vacuum fluctuations* that can be interpreted as a term of cosmological constant, which leads to a value of Λ of the order of the inverse of Planck's surface. Finally we give here the full form of Einstein's field equations as [3, section 8]:

$$R_{\mu\nu}- g_{\mu\nu}R- g_{\mu\nu}\Lambda =8\pi G \; {}_{c}4 \; T_{\mu\nu} \tag{62}$$

And since the cosmological constant is zero, they reduce to the form,

$$G_{\mu\nu}= {}_{c}4 \; T_{\mu\nu} \tag{63}$$

where $G_{\mu\nu}$ is Einstein tensor given by,

$$G_{\mu\nu}= R_{\mu\nu} - {}_{2} \, g_{\mu\nu} R \tag{64}$$

$R_{\mu\nu}$: Ricci curvature tensor, $g_{\mu\nu}$: metric tensor, R: scalar curvature, $T_{\mu\nu}$: stress energy tensor.

We can introduce the Einstein equations in vacuum as: in the vacuum where a domains empty of matter or field, the source of energy becomes zero, so

$$T_{\mu\nu}=0 \tag{65}$$

This leads to the consequence that Ricci tensor equal to zero.

$$R_{\mu\nu}=0 \tag{66}$$

An empty space whose Ricci tensor disappears is called a "Ricci-plat" space.

"This does not mean that space-time is flat in the absence of any matter or energy: the curvature of space is represented by the Riemann tensor, not by the Ricci tensor." (see [7, p.181]) The idea is that the Riemann tensor is decomposed into two components, the Weyl tensor, the Ricci tensor and the curvature scalar:

$$WEY\ L = RIEMANN-RICCI \tag{67}$$

It is the cancellation of this tensor which is the condition for the consistent flatness of space-time.

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