Using Lagrange Equations to Study the Relative Motion of a Mechanism

R. A. Petre, S. E. Nichifor, A. Craifaleanu, I. Stroe

Abstract**—**The relative motion of a robotic arm formed by homogeneous bars of different lengths and masses, hinged to each other is investigated. The first bar of the mechanism is articulated on a platform, considered initially fixed on the surface of the Earth, while for the second case the platform is considered to be in rotation with respect to the Earth. For both analyzed cases the motion equations are determined using the Lagrangian formalism, applied in its traditional form, valid with respect to an inertial reference system, conventionally considered as fixed. However, in the second case, a generalized form of the formalism valid with respect to a non-inertial reference frame will also be applied. The numerical calculations were performed using a MATLAB program.

Keywords**—**Lagrange equations, relative motion, inertial or non-inertial reference frame.

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I. Introduction

In this paper we investigate the relative motion of a robotic arm comprised of hinged bars of different lengths and masses. Two cases were considered: the first one, when the platform on which the first bar of the mechanism is hinged is fixed on the surface of the Earth, and the second one, when the platform is in rotation with respect to the Earth. The first of these two cases corresponds to the motion of the mechanism with respect to a fixed reference frame, while the other one was analyzed with respect to an inertial, respectively with a non-inertial reference frame. The motion equations were determined using the Lagrangian formalism [1]-[3], applied in its traditional form, valid with respect to an inertial reference frame. However, for the second case, a generalized form of the Lagrangian formalism will also be applied, valid with respect to a non-inertial reference frame [4]-[6]. The numerical simulations were performed using a program developed in MATLAB.

II. Configuration of the Mechanism

A robotic arm consisting of two homogeneous bars is studied. The first bar, *OA*, is hinged in point *O* on the fixed element and the second bar, *AB*, which is articulated in point *A* on the first bar, rotates with respect to *OA* bar about an axis perpendicular to it, located in a horizontal plane, as shown in Fig. 1. The bars have the length l_1 and l_2, respectively, and the masses m_1, m_2, respectively.

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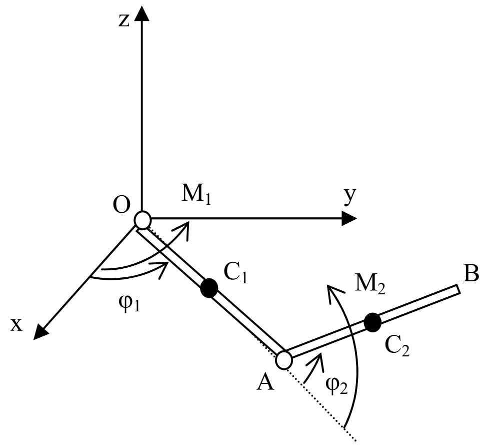

Fig. 1 Mechanism with fixed platform

The bar *OA* is driven by the torque motor with the moment M_1 and the bar *AB* is driven by the bar *OA* via the torque motor with the moment M_2. The variations of moments M_1 and M_2 are determined so that the motion of the mechanism takes place according to the equations of motion:

$$\varphi_1(t) = A_1 \frac{\omega_0}{2\pi}\left(t - \frac{1}{\omega_0}\sin(\omega_0 t)\right), \tag{1}$$

$$\varphi_2(t) = A_2 \frac{\omega_0}{2\pi}\left(t - \frac{1}{\omega_0}\sin(\omega_0 t)\right) . \tag{2}$$

A. Mechanism Hinged on a Fixed Platform

When we considered that the mechanism is hinged on a fixed platform, we applied the Lagrangian formalism in its traditional form [7]:

$$\begin{cases} \frac{d}{dt}\left(\frac{\partial E}{\partial \dot{\varphi}_1}\right) - \frac{\partial E}{\partial \varphi_1} = Q_1 \\ \frac{d}{dt}\left(\frac{\partial E}{\partial \dot{\varphi}_2}\right) - \frac{\partial E}{\partial \varphi_2} = Q_2 \end{cases}, \tag{3}$$

where, E is the kinetic energy of the system and Q_1 and Q_2 are the generalized forces of the system.

Due to the configuration of the mechanism, the kinetic energy of the system has the form:

$$E = \frac{1}{2} J_O \omega_1^2 + \frac{1}{2} m_2 v_{C2}^2 + \frac{1}{2} J_{C2}\left(\omega_2^2 + \omega_1^2 \cos^2 \varphi_2\right) \tag{4}$$

where J_O represents the moment of inertia of the homogeneous bar *OA* with respect to point *O*, J_{C_2} is the moment of inertia of bar *AB* with respect to its center of mass, ω_1^2 is the angular velocity of bar *OA*, ω_2^2 is the angular velocity of bar *AB* and v_{C2} is the linear velocity of the center of mass of bar *AB*,

$$v_{C2}^2 = \frac{l_2^2}{4}\dot{\varphi}_2^2 + \left(l_1 + \frac{l_2}{2}\cos\varphi_2\right)^2 \dot{\varphi}_1^2 \cdot \tag{5}$$

The generalized forces Q_1 and Q_2 from the right side of the Lagrange equations are determined using the virtual work:

$$\delta L = M_1\delta\varphi_1 + M_2\delta\varphi_2, \tag{6}$$

thus,

$$\begin{cases} Q_1 = M_1 \\ Q_2 = M_2 \end{cases}. \tag{7}$$

Replacing (4)-(7) in (3), we obtained the differential equations of motion:

$$\begin{cases} \left(A_{11} + A_{22}\cos^2\varphi_2 + 2A_{12}\cos\varphi_2\right)\ddot{\varphi}_1 - \\ -\left(2A_{22}\cos\varphi_2 + 2A_{12}\right)\dot{\varphi}_1\dot{\varphi}_2\sin\varphi_2 = M_1 \\ A_{22}\ddot{\varphi}_2 + \frac{1}{2}\left(2A_{22}\cos\varphi_2 + 2A_{12}\right)\dot{\varphi}_1^2\sin\varphi_2 = M_2 \end{cases}, \tag{8}$$

where we made use of the following notations:

$$A_{11} = \left(\frac{m_1}{3} + m_2\right) l_1^2, \tag{9}$$

$$A_{22} = \frac{m_2}{3} l_2^2, \tag{10}$$

$$A_{12} = \frac{1}{2} m_2 l_1 l_2. \tag{11}$$

B. The Mechanism Hinged on a Rotating Platform

In this case, we consider that the platform on which the mechanism is hinged is located on the Earth, but in rotation with respect to it, with a constant angular velocity Ω, as shown in Fig. 2.

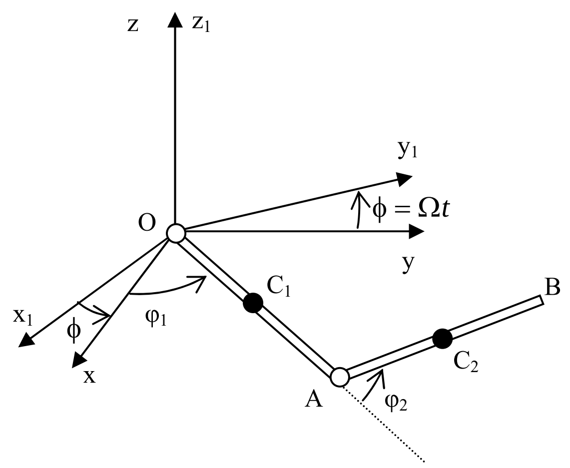

Fig. 2 Mechanism on a mobile platform

For this case, the equations of motion were obtained for two subcases, that is, when we used the Lagrange equations with respect to a fixed reference frame *Oxyz*, and the second subcase when we used the Lagrange equations with respect to the mobile reference frame $O_1x_1y_1z_1$.

1. Lagrange Equations with Respect to a Fixed Reference Frame

In this section we deduce the equations of motion of the robotic arm hinged on the platform located on Earth, that has a rotation motion with respect to it, using Lagrange's equations with respect to the fixed reference frame, *Oxyz*.

We determine the equations of motion using (3), where the kinetic energy of the system, in this case, is

$$E = \frac{1}{2}J_O\left(\omega_1 + \dot{\phi}\right)^2 + \frac{1}{2}m_2 v_{C2}^2 + \frac{1}{2}J_{C2}\left[\omega_2^2 + \left(\omega_1 + \dot{\phi}\right)^2\cos^2\varphi_2\right], \tag{12}$$

where the velocity of the center of mass of the second bar has the form:

$$v_{C2}^2 = \frac{l_2^2}{4}\dot{\varphi}_2^2 + \left(l_1 + \frac{l_2}{2}\cos\varphi_2\right)^2\left(\dot{\varphi}_1 + \dot{\phi}\right)^2 \cdot \tag{13}$$

The generalized forces in the right side of the Lagrange equations will keep their previous form (7), thus, the differential equations of motion will be:

$$\begin{cases} \left(A_{11} + A_{22}\cos^2\varphi_2 + 2A_{12}\cos\varphi_2\right)\cdot\ddot{\varphi}_1 \\ -2\cdot\left(A_{22}\cos\varphi_2 + A_{12}\right)\left(\dot{\varphi}_1 + \Omega\right)\dot{\varphi}_2\sin\varphi_2 = M_1 \\ A_{22}\ddot{\varphi}_2 + \left(A_{22}\cos\varphi_2 + A_{12}\right)\left(\dot{\varphi}_1 + \Omega\right)^2\sin\varphi_2 = M_2 \end{cases}. \tag{14}$$

2. Lagrange Equations with Respect to a Mobile Reference Frame

In this section we deduce the equations of motion of the robotic arm hinged on the platform located on the Earth, that has a rotation motion with respect to it, using Lagrange's equations with respect to a mobile reference frame, $O_1x_1y_1z_1$. This reference system has a constant angular velocity Ω.

Taking into consideration the fact that the motion is studied with respect to a mobile frame, a generalized form of the Lagrangian formalism valid in relation to a non-inertial reference frame [4]-[6] is used.

In order to obtain the generalized transport force, we calculated the kinetic energy for the circular velocities:

$$E_c = E_c^{OA} + E_c^{AB}, \tag{15}$$

where

$$E_c^{OA} = \frac{1}{2}J_o\Omega^2 = \frac{1}{2}\frac{m_1 l_1^2}{3}\Omega^2, \tag{16}$$

$$E_C^{AB} = \frac{1}{2}J_O^{AB}\Omega^2 = \frac{1}{2}\left[\frac{m_2 l_2^2}{12}\cos^2\varphi_2 + m_2\left(x_{1C_2}^2 + y_{1C_2}^2 + z_{1C_2}^2\right)\right]\Omega^2 \tag{17}$$

where J_O^{AB} is the moment of inertia of the bar *AB* with respect

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to point O, and x_{1C_2}, y_{1C_2} and z_{1C_2} are the coordinates of the center of mass of bar AB with respect to the movable reference frame:

$$\begin{cases} x_{1c_2} = l_1 \cos\varphi_1 + \dfrac{l_2}{2}\cos\varphi_2 \cos\varphi_1 \\ y_{1c_2} = l_1 \sin\varphi_1 + \dfrac{l_2}{2}\cos\varphi_2 \sin\varphi_2 \\ z_{1c_2} = \dfrac{l_2}{2}\sin\varphi_2 \end{cases} . \tag{18}$$

For the robotic arm formed by the two homogeneous bars, the expressions found in [4],

$$Q_{kt}^{\omega} = \frac{\partial E_c}{\partial q_k}, (k = 1, 2), \tag{19}$$

depend on the generalized coordinates of the system, φ_1 and φ_2:

$$\begin{cases} Q_{1t}{}^{\omega} = \dfrac{\partial E_c}{\partial \varphi_1} = 0 \\ Q_{2t}{}^{\omega} = \dfrac{\partial E_c}{\partial \varphi_2} = -\dfrac{1}{2}\left(2A_{12} + A_{22}\cos\varphi_2\right)\Omega^2 \sin\varphi_2 \end{cases} \tag{20}$$

From Fig. 2 we can deduce that the bar OA is in rotation, thus the contribution of this element of the mechanism to the generalized Coriolis force is null, but for bar AB, which has an arbitrary relative motion, the generalized Coriolis force [4] has the following expression:

$$Q_{kc}^{AB} = -2\overline{\omega_0} \cdot \left(m_2 \overline{v_{rC_2}} \times \frac{\partial \overline{v_{rC_2}}}{\partial \dot{q}_k} \right) - 2\overline{\omega_0} \cdot \overline{\overline{P_{C_2}}} \cdot \left(\overline{\omega_r} \times \frac{\partial \overline{\omega_r}}{\partial \dot{q}_k} \right), \tag{21}$$

where $\overline{\overline{P_{C_2}}}$ represents the tensor of the planar and centrifugal inertia moments for the second element of the mechanism. This way, the Lagrange equations with respect to a non-inertial reference system will be:

$$\begin{cases} \left(A_{11} + A_{22}\cos^2\varphi_2 + 2A_{12}\cos\varphi_2\right)\ddot{\varphi}_1 - \left(2A_{22}\cos\varphi_2 + 2A_{12}\right)\cdot \\ \cdot\dot{\varphi}_1\dot{\varphi}_2 \sin\varphi_2 = M_1 + \left(2A_{12} + \dfrac{3}{2}A_{22}\cos\varphi_2\right)\dot{\varphi}_2\Omega\sin\varphi_2 + \\ + \dfrac{A_{22}}{2}\dot{\varphi}_1\Omega\sin\varphi_2\cos\varphi_2 \\ A_{22}\ddot{\varphi}_2 + \dfrac{1}{2}\left(2A_{22}\cos\varphi_2 + 2A_{12}\right)\dot{\varphi}_1^2\sin\varphi_2 = M_2 - \\ -\dfrac{1}{2}\left(2A_{12} + A_{22}\cos\varphi_2\right)\Omega^2\sin\varphi_2 - \left(2A_{12} + \dfrac{3}{2}A_{22}\cos\varphi_2\right)\cdot \\ \cdot\dot{\varphi}_1\Omega\sin\varphi_2 - \dfrac{A_{22}}{2}\dot{\varphi}_1\Omega\sin\varphi_2\cos\varphi_2 \end{cases} \tag{22}$$

We can observe that systems (14) and (22) are equivalent, which, again, proves that the two methods of study lead to identical results.

III. Numerical Applications

Several sets of numeric values were considered for the system parameters: A_1, A_2, l_1, l_2, m_1, m_2. For each set, the variation curves of the angles of rotation, angular velocities and angular accelerations of the two bars, as well as the variation curves of the motor moments, for various values of the angular velocity Ω, were determined (Figs. 3-10). The calculations were performed using a program developed in MATLAB.

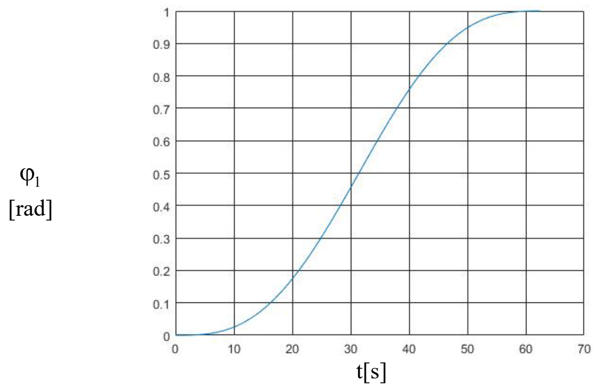

Fig. 3 The Law of Motion for A_1=1*rad*

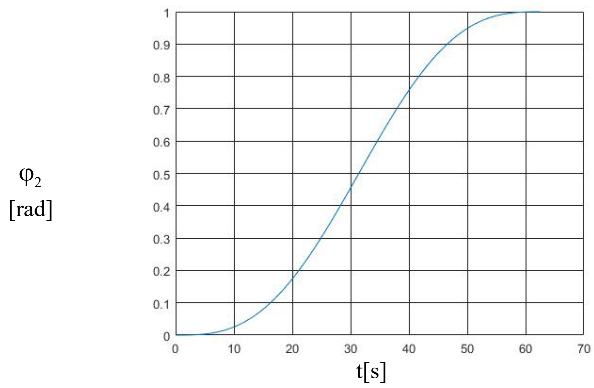

Fig. 4 The Law of Motion for A_2=1*rad*

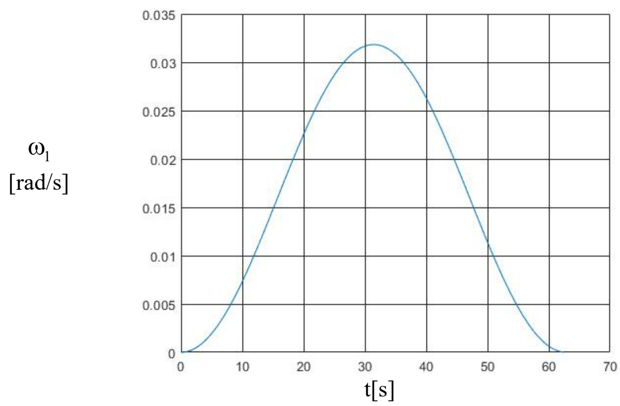

Fig. 5 The angular velocity of the first component of the mechanism

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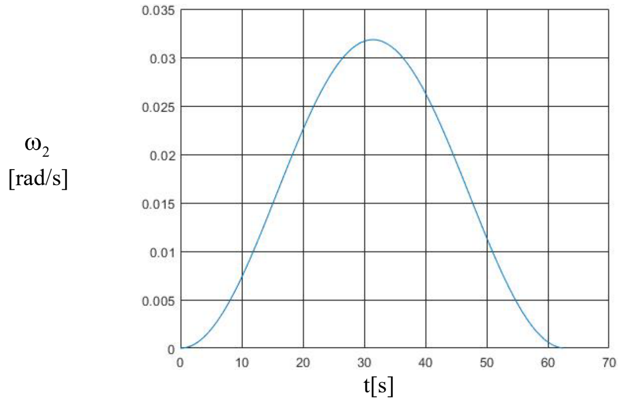

Fig. 6 The angular velocity of the second component of the mechanism

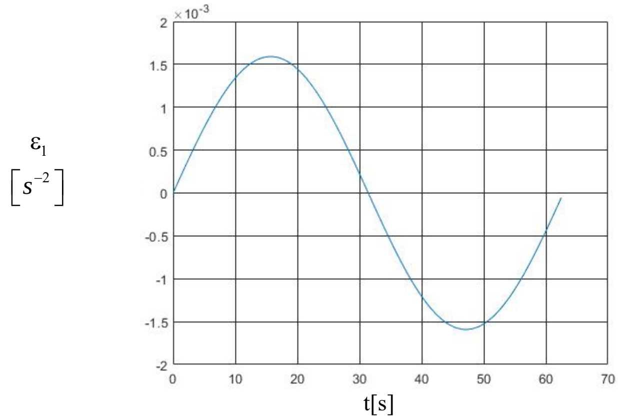

Fig. 7 The angular acceleration of the first component of the mechanism

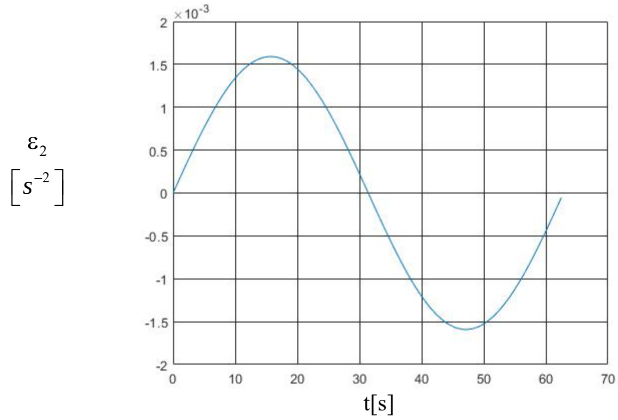

Fig. 8 The angular acceleration of the second component of the mechanism

IV. Conclusion

In both analyzed cases, the motion equations are determined using the Lagrangian formalism, applied in its traditional form, valid with respect to an inertial reference system, conventionally considered as fixed. A generalized form of the formalism valid with respect to a non-inertial reference system has been also applied in the second case.

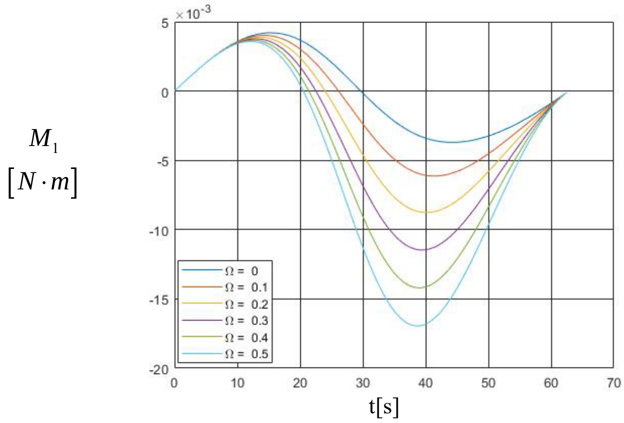

Fig. 9 Motor moments for $A_1 = 1\,rad$, $A_2 = 1\,rad$, $l_1 = l_2 = 1\,m$, $m_1 = m_2 = 1\,kg$

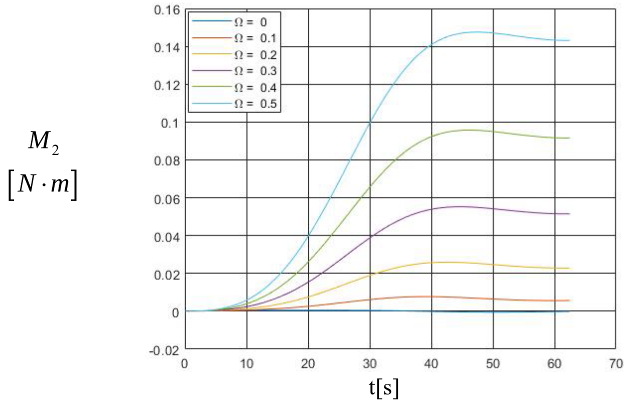

Fig. 10 Motor moments for $A_1 = 1\,rad$, $A_2 = 1\,rad$, $l_1 = l_2 = 1\,m$, $m_1 = m_2 = 1\,kg$

It was noted that the two versions of the Lagrangian formalism have led to the same results.

The numerical studies have shown that the values of the motor moments increase with the values of the amplitudes A_1 and A_2. It also follows that the values of the motor moments generally increase with the platform's angular velocity.

References

[1] I. Stroe, P. Pârvu, “Trajectories in non-inertial reference farmes”, *U.P.B. Sci. Bull. Series D*, Vol. 72, Issue 4, ISSN - 1454-2358, 2010.

[2] T.-V. Nguyen, R.-A. Petre, I. Stroe, Application of Lagrange Equations For Calculus of Internal Forces in a Mechanism, *U.P.B. Sci. Bull., Series* D, Vol. 78, Iss. 4, pg. 15-26, ISSN 1454-2358, 2016.

[3] T.-V. Nguyen, R.-A. Petre, I. Stroe, “Calculus of axial force in a mechanism using Lagrange equations”, INCAS Bulletin, volume 8, issue 2, (online) ISSN 2247–4528, (print) ISSN 2066–8201, ISSN–L 2066–8201, pp. 97-108, DOI: 10.13111/2066-8201.2016.8.2.8, April-June 2016.

[4] I. Stroe, A. Craifaleanu, “Generalization of the Lagrange equations formalism, for motions with respect to non-inertial reference frames”, International Conference on Smart Systems in All Fields of the Life, Aerospace, Robotics, Mechanical Engineering, Manufacturing Systems, Biomechatronics, Neurorehabilitation and Human Motility, ICMERA 2014, Applied Mechanics and Materials Vol. 656, pp. 171-180, Trans Tech Publications, Switzerland, 2014.

[5] I. Stroe, A. Craifaleanu, “Calculus of a compass robotic arm using Lagrange equations in non-inertial reference frames”, Proceedings of the International Conference of Aerospace Sciences "AEROSPATIAL 2012", Bucharest, 11-12 October, pp. 137-141, 2012.

[6] I. Stroe, Ș. Staicu, A. Craifaleanu, "Methods in dynamics of compass robotic arm", Conference Caius Iacob INCAS, 2011.
[7] Roxana-Alexandra, PETRE; Ion, STROE; Andrei, CRAIFALEANU," Rotation-Rotation Mechanism moving with Respect to a Plane", INCAS BULLETIN, vol.11, issue 4, pages 123-131, 2019-12-01, DOI 10.13111/2066-8201.2019.11.4.11

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