World Academy of Science, Engineering and Technology
International Journal of Physical and Mathematical Sciences
Vol:14, No:9, 2020

Weyl Type Theorem and the Fuglede Property

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Abstract—Given $\mathcal{H}$ a Hilbert space and $\mathcal{B}(\mathcal{H})$ the algebra of bounded linear operator in $\mathcal{H}$, let δ_{AB} denote the generalized derivation defined by A and B. The main objective of this article is to study Weyl type theorems for generalized derivation for (A, B) satisfying a couple of Fuglede.

Keywords—Fuglede Property, Weyl's theorem, generalized derivation, Aluthge Transformation.

I. Introduction

THe algebra of all bounded operators on a complex infinite dimensional Hilbert space $\mathcal{H}$ will be denoted by $\mathcal{B}(\mathcal{H})$. For $A, B \in \mathcal{B}(\mathcal{H})$, let $\delta_{AB} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ and $\Delta_{AB} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ denote the itgeneralized derivation $\delta_{AB} = AX - XB$ and the itelementary operator $\Delta_{AB} = AXB - X$. Let $d_{AB} = \delta_{AB}$ or Δ_{AB}. If T is a bounded linear operator on a normed linear space $\mathcal{X}$, we have

$$\begin{aligned} \ker(d_{AB}) \perp \Re(d_{AB}) \Rightarrow & \\ \ker(d_{AB}) \cap cl(\Re(d_{AB})) &= \{0\} \\ \Rightarrow \ker(d_{AB}) \cap \Re(d_{AB}) &= \{0\} \\ \Leftrightarrow asc(d_{AB}) &\leq 1 \end{aligned}$$

[1, Page 25]. Here $asc(d_{AB})$ denotes the itascent of d_{AB}, $cl(\Re(d_{AB}))$ denote the closure of the range of d_{AB} and $\ker(d_{AB}) \perp \Re(d_{AB})$ denotes that the kernel of d_{AB} is orthogonal to the range of d_{AB} in the sense of G. Birkhoff. A normed linear subspace $\mathcal{M}$ is said to be orthogonal to a normed linear subspace $\mathcal{M}$ in the sense of Birkhoff, written as, $\mathcal{M} \perp \mathcal{N}$, if $\|m\| \leq \|m+n\|$ for all $m \in \mathcal{M}$ and $n \in \mathcal{N}$. This concept of orthogonality is not symmetric, i.e., $\mathcal{M} \perp \mathcal{N}$ does not imply $\mathcal{N} \perp \mathcal{M}$, but the concept does agree with the usual concept of orthogonality in the case in which $\mathcal{X} = \mathcal{H}$. The range-kernel orthogonality of d_{AB} has been considered by a number of authors, see ([1], [2], [3], [4], [5] and [6]). A sufficient condition guaranteeing $\ker(d_{AB}) \perp \Re(d_{AB})$ is that $\ker(d_{AB}) \subseteq \ker(d_{A^*B^*})$ [3]. The inclusion $\ker(d_{AB}) \subseteq \ker(d_{A^*B^*})$, known in the literature as the Putnam-Fuglede commutativity theorem.

Definition I. Let $A, B \in \mathcal{B}(\mathcal{H})$. We say that (A, B) is a couple of Fuglede (Shortly, $(A, B) \in FP$) if $AX = XB$ implies that $A^*X = XB^*$, for all $X \in \mathcal{B}(\mathcal{H})$.

Lemma II. ([7]) Let $A, B \in \mathcal{B}(\mathcal{H})$. Then the following assertions equivalent.

(i) (A, B) is a couple of Fuglede.

(ii) If $AX = XB$, then $cl(\Re(X))$ reduces A, $\ker(X)^\perp$ reduces B, and $A|_{cl(\Re(X))}$, $B|_{\ker(X)^\perp}$ are unitarily equivalent normal operators.

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Lemma III. *([8])* Let $A, B \in \mathcal{B}(\mathcal{H})$. If (A, B) is a couple of Fuglede, then $\ker(d_{AB}) \subseteq \ker(d_{A^*B^*})$.

II. Main Results and Their Proofs

Let $A \in \mathcal{B}(\mathcal{H})$ have the polar decomposition $A = U|A|$. Then the first Aluthge transform $\widetilde{A} = |A|^{\frac{1}{2}}U|A|^{\frac{1}{2}}$, and if $\widetilde{A}$ has the polar decomposition $\widetilde{A} = V|\widetilde{A}|$, then the second Aluthge $\widehat{A} = |\widetilde{A}|^{\frac{1}{2}}V|\widetilde{A}|^{\frac{1}{2}}$. It is known that $A, \widetilde{A}$ and $\widehat{A}$ have the same point spectrum, the same approximate point spectrum and the same spectrum. Furthermore, $\widehat{A}$ has a normal part if and only if A has a normal part.

Lemma IV. Let $A, B \in \mathcal{B}(\mathcal{H})$. If $(A, B) \in FP$, then $(A - \lambda, B) \in FP$.

Proof. Suppose that $(A, B) \in FP$. Then

$$(A - \lambda)X = AX - \lambda X = XB - X\lambda = X(B - \lambda).$$

Now

$$\begin{aligned} (A-\lambda)^*X &= (A^* - \overline{\lambda})X = A^*X - \overline{\lambda}X \\ &= XB^* - X\overline{\lambda} \quad \text{since } (A, B) \in FP \\ &= X(B - \lambda)^*. \end{aligned}$$

Hence $(A - \lambda, B) \in FP$.

Lemma V. ([9]) Let $A, B \in \mathcal{B}(\mathcal{H})$ be normal. If there is a quasiaffinity $X \in \Delta_{AB}^{-1}(0)$, then B is invertible and $X \in \delta_{AB}^{-1}(0)$.

Lemma VI.([9]) If $A, B \in \mathcal{B}(\mathcal{H})$ are normal, then $d_{AB}^{-2}(0) = d_{AB}^{-1}(0)$.

Lemma VII. Let $A, B \in \mathcal{B}(\mathcal{H})$. If (A, B) is a couple of Fuglede, then $\ker(\delta_{AB} - \lambda) \subseteq \ker(\delta_{A^*B^*} - \overline{\lambda})$ for all $\lambda \in \mathbf{C}$, where $\overline{\lambda}$ denote the complex conjugate of λ.

Proof. Since (A, B) is a couple of Fuglede, it follows from Lemma I that $cl(\Re(X))$ reduces A, $\ker(X)^\perp$ reduces B, and $A|_{cl(\Re(X))}$, $B|_{\ker(X)^\perp}$ are unitarily equivalent normal operators. Then with respect to the orthogonal decomposition $\mathcal{H} = cl(\Re(X)) \oplus \overline{\Re(X)}^\perp$ and $\mathcal{H} = \ker(X) \oplus \ker(X)^\perp$, A and B can be respectively represented as $A = A_n \oplus A_p$ and $B = B_n \oplus B_p$, where A_n, B_n are normal operators and A_p, B_p are pure parts; now assume $X \in \ker(\delta_{AB} - \lambda)$,

$X : \ker(X)^\perp \oplus \ker(X) \to cl(\Re(X)) \oplus cl(\Re(X))^\perp$, have the corresponding matrix representation $X = [X_{ij}]_{i,j=1}^2$. Then

$$\ker(\delta_{AB}-\lambda) = \begin{pmatrix} (\delta_{A_nB_n}-\lambda)X_{11} & (\delta_{A_nB_p}-\lambda)X_{12} \\ (\delta_{A_pB_n}-\lambda)X_{21} & (\delta_{A_pB_p}-\lambda)X_{22} \end{pmatrix} = 0.$$

Since the operator $A_n - \lambda$ (resp., $B_n - \lambda$) is normal. Since $cl(\Re(X))$ reduces A, $\ker(X)^\perp$ reduces B, we have the pure parts of A and B are injective. It follows from an application of the Fuglede-Putnam property to $(\delta_{A_nB_p}-\lambda)X_{12} = (\delta_{A_pB_n}-\lambda)X_{21} = 0$ that $X_{12} = X_{21} = 0$. Define the second Aluthge transforms as above. Then

$$(\delta_{A_pB_p}-\lambda)X_{22} = 0 \Longleftrightarrow (\delta_{A_pT_p}-\lambda)Y = 0,$$

where we have set $\widehat{B^*}^* = T_p$ and $Y = |\widetilde{A_p}|^{\frac{1}{2}}|A_p|^{\frac{1}{2}}X_{22}|B_p^*|^{\frac{1}{2}}|\widetilde{B_p^*}|^{\frac{1}{2}}$. Since $|\widetilde{A_p}|^{\frac{1}{2}}, |A_p|^{\frac{1}{2}}, |B_p^*|^{\frac{1}{2}}$ and $|\widetilde{B_p^*}|^{\frac{1}{2}}$ are quasiaffinities, which implies that $X_{22} = 0$ and $X = X_{11} \oplus 0$. Since $\ker(\delta_{A_nB_n}-\lambda) \subseteq \ker(\delta_{A_n^*B_n^*}-\bar{\lambda})$ we have $\ker(\delta_{AB}-\lambda) \subseteq \ker(\delta_{A^*B^*}-\bar{\lambda})$ for all $\lambda \in \mathbf{C}$.

Lemma VIII. If $(A, B) \in FP$ and $\lambda \in \mathbf{C}$, then $asc(\delta_{AB}-\lambda) \le 1$.

Proof. Let $X \in (\delta_{AB}-\lambda)^{-1}(0)$. Then $AX - X(B+\lambda) = 0 = A^*X - X(B^*+\bar{\lambda})$ implies that $\overline{\Re(X)}$ reduces A and $X^{-1}(0)^\perp$ reduces $B+\lambda$. Since $X \in (d_{AB}-\lambda)^{-1}(0)$, we have AX and $X(B+\lambda) \in (\delta_{AB}-\lambda)^{-1}(0)$, $A^*AX = AX(B+\lambda)^* = AA^*X$ and $X(B+\lambda)^*(B+\lambda) = A^*X(B+\lambda) = X(B+\lambda)(B+\lambda)^*$. Hence $A_1 = A|_{\overline{\Re(X)}}$ and $B_1 = (B+\lambda)|_{X^{-1}(0)^\perp}$ are normal operators.

Suppose now that $Y \in (\delta_{AB}-\lambda)^{-2}(0)$. Set $(\delta_{AB}-\lambda)Y = X$, let $X_1 : X^{-1}(0)^\perp \longrightarrow \overline{\Re(X)}$ be the quasiaffinity defined by setting $X_1h = Xh$ for each $h \in \mathcal{H}$ and let $Y : X^{-1}(0)^\perp \oplus X^{-1}(0) \longrightarrow \overline{\Re(X)} \oplus \overline{\Re(X)}^\perp$ have the matrix representation $Y = [Y_{ij}]_{i,j=1}^2$. Then $0 = \delta_{AB}(X) = \delta_{A_1B_1}(X_1) \oplus 0 = \delta_{A_1B_1}^2(Y_{11}) \oplus 0$. The operators A_1 and B_1 being normal, it follows from Lemma III that $\delta_{A_1B_1}(Y_{11}) = 0$. Hence $X = \delta_{A_1B_1}(Y_{11}) \oplus 0 = 0$ implies $(\delta_{AB}-\lambda)(Y) = 0$ and so $asc(\delta_{AB}-\lambda) \le 1$.

Let $\mathcal{X}$ be a complex Banach space. A Banach space operator $T \in \mathcal{B}(\mathcal{X})$ has the single-valued extension property, or SVEP, at a point $\lambda \in \sigma(T)$ if for every open disc $\mathcal{D}$ centered at λ the only analytic function $f : \mathcal{D} \longrightarrow \mathcal{X}$ satisfying $(T-\mu)f(\mu) = 0$ is the function $f \equiv 0$; T has SVEP if it has SVEP at every $\lambda \in \sigma(T)$.

Corollary IX. If $(A, B) \in FP$, then δ_{AB} has SVEP.

Proof. The finite ascent property of $(\delta_{AB}-\lambda)$ implies SVEP [10].

Remark. Recall from [11] that $\sigma(\delta_{AB}) = \{\lambda \in \sigma(A)-\sigma(B) : \lambda = \alpha - \beta, \alpha \in \sigma(A) \text{ and } \beta \in \sigma(B)\}$.

Theorem X. If $(A, B) \in FP$, then $\Re(d_{AB}-\lambda)$ is closed for each $\lambda \in iso\,\sigma(d_{AB})$.

Proof. Let $\lambda \in iso\,\sigma(\delta_{AB})$. Then $0 \in iso\,\sigma(\delta_{AB}-\lambda)$, where $\sigma(\delta_{AB}-\lambda) = \sigma(A) - \sigma(B+\lambda)$. Hence $\sigma(A) \cap \sigma(B+\lambda)$ consists of points which are isolated in both $\sigma(A)$ and $\sigma(B+\lambda)$. In particular, $\sigma(A) \cap \sigma(B+\lambda)$ does not contain any limit points of $\sigma(A) \cap \sigma(B+\lambda)$. There exists a finite set $S = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ of distinct values α_i such that $S = \sigma(A) \cap \sigma(B+\lambda)$ and each $\alpha_i, 1 \le i \le n$, is an isolated point of both $\sigma(A)$ and $\sigma(B+\lambda$. Let

$$H_1 = \bigvee_{i=1}^n (B-\alpha_i)^{*^{-1}}(0), \quad H_1' = \bigvee_{i=1}^n (A-\alpha_i)^{-1}(0),$$
$$H_2 = \mathcal{H} \ominus H_1 \text{ and } H_2' = \mathcal{H} \ominus H_1'.$$

Then A and B have the direct sum decompositions $A = A_1 \oplus A_2$ and $B = B_1 \oplus B_2$, where $A_1 = A|_{H_1'}$ and $B_1 = B|_{H_1}$ are normal operators. Let $X = [X_{ij}]_{i,j=1}^2$, it is seen that

$$(\delta_{AB}-\lambda)X = \begin{pmatrix} (\delta_{A_1B_1}-\lambda)X_{11} & (\delta_{A_1B_2}-\lambda)X_{12} \\ (\delta_{A_2B_1}-\lambda)X_{21} & (\delta_{A_2B_2}-\lambda)X_{22} \end{pmatrix},$$

where $A_2 = A|_{H_2'}$, $B_2 = B|_{H_2}$ and $\sigma(A_i) \cap \sigma(B_i+\lambda) = \emptyset$ for all $1 \le i, j \le 2$ such that $i, j \ne 1$ and so that $0 \notin \sigma(\delta_{AB}-\lambda)$ for all $1 \le i, j \le 2$ such that $i, j \ne 1$. Therefore, $\Re(\delta_{AB}-\lambda)$ is closed.

III. The Operator d_{AB} and Weyl's Theorem

Let us denote by $\alpha(T)$ the dimension of the kernel and by $\beta(T)$ the codimension of the range. Recall that the operator $T \in \mathcal{B}(\mathcal{X})$ is said to be itupper semi-Fredholm, $T \in SF_+(\mathcal{X})$, if the range of $T \in \mathcal{B}(\mathcal{X})$ is closed and $\alpha(T) < \infty$, while $T \in \mathcal{B}(\mathcal{X})$ is said to be lower semi-Fredholm, $T \in SF_-(\mathcal{X})$, if $\beta(T) < \infty$. An operator $T \in B(\mathcal{X})$ is said to be itsemi-Fredholm if $T \in SF_+(\mathcal{X}) \cup SF_-(\mathcal{X})$ and Fredholm if $T \in SF_+(\mathcal{X}) \cap SF_-(\mathcal{X})$. If T is semi-Fredholm then the itindex of T is defined by $ind(T) = \alpha(T) - \beta(T)$. A bounded linear operator T acting on a Banach space $\mathcal{X}$ is itWeyl if it is Fredholm of index zero and Browder if T is Fredholm of finite ascent and descent. The Weyl spectrum $\sigma_w(T)$ and Browder spectrum $\sigma_b(T)$ of T are defined by $\sigma_w(T) = \{\lambda \in \mathbf{C} : T - \lambda I \text{ is not Weyl}\}$ and $\sigma_b(T) = \{\lambda \in \mathbf{C} : T - \lambda I \text{ is not Browder}\}$. Let $E^0(T) = \{\lambda \in \text{iso}\,\sigma(T) : 0 < \alpha(T-\lambda) < \infty\}$ and let $\pi_0(T) := \sigma(T) \setminus \sigma_b(T)$ all *Riesz points* of T. According to Coburn [12], Weyl's theorem holds for T if $\sigma(T) \setminus \sigma_w(T) = E^0(T)$, and that Browder's theorem holds for T if $\sigma(T) \setminus \sigma_w(T) = \pi^0(T)$. Let $SF_+^-(\mathcal{X}) = \{T \in SF_+ : \text{ind}\,(T) \le 0\}$. The *upper semi Weyl spectrum* is defined by $\sigma_{SF_+^-}(T) = \{\lambda \in \mathbf{C} : T - \lambda \notin SF_+^-(\mathcal{X})\}$. According to Rakočević [13], an operator $T \in B(\mathcal{X})$ is said to satisfy a-Weyl's theorem if $\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = E_a^0(T)$, where $E_a^0(T) = \{\lambda \in \text{iso}\,\sigma_a(T) : 0 < \alpha(T-\lambda I) < \infty\}$. It is known [13] that an operator satisfying a-Weyl's theorem satisfies Weyl's theorem, but the converse does not hold in general.

In the following we prove that if $(A, B) \in FP$, then δ_{AB} satisfies the property that its quasinilpotent part $H_0(d_{AB}-\lambda)$,

$$H_0(\delta_{AB}-\lambda) = \{X \in \mathcal{B}(\mathcal{H}) : \lim_{n\to\infty} \|(\delta_{AB}-\lambda)^n X\| = 0\}$$

equals $(\delta_{AB}-\lambda)^{-1}(0)$ for all $\lambda \in \operatorname{iso}\sigma(\delta_{AB})$. This implies that δ_{AB} satisfies Weyl's theorem, δ^*_{AB} satisfies a-Weyl's theorem.

Lemma XI. If $(A,B) \in FP$, then $H_0(\delta_{AB}-\lambda) = (\delta_{AB}-\lambda)^{-1}(0)$ for all $\lambda \in \operatorname{iso}\sigma(\delta_{AB})$.

Proof. Evidently, the non-zero points α_i (resp.,$\overline{\beta}$), $1 \le i \le m$, are normal eigenvalues of A (resp., B^*). Let $M_{1i} = (A-\alpha_i)^{-1}(0)$, $N_{1i} = (B-\mathcal{B}_i)^{-1}(0)$ $(= (B-\beta_i)^{*^{-1}})$, $M_1 = \bigoplus_{i=1}^m M_{1i}, N_1 = \bigoplus_{i=1}^m, M_2 = M_1^\perp$ and $N_2 = N_1^\perp$; let $A = A_1 \oplus A_2 \in \mathcal{L}(M_1 \oplus M_2)$ and $B = B_1 \oplus B_2 \in \mathcal{L}(N_1 \oplus N_2)$. Then $\sigma(A_2) = \sigma(A) \setminus \{\alpha_1, \cdots, \alpha_m\}$ and $\sigma(B_2) = \sigma(B) \setminus \{\beta_1, \cdots, \beta_m\}$ $\lambda \notin \sigma(\delta_{A_kB_t})$ for all $1 \le k,t \le 2$ other than $k=t=1$.

Let $X \in H_0(\delta_{AB}-\lambda)$, and let $X \in \mathcal{L}(N_1 \oplus N_2, M_1 \oplus M_2)$ have the representation $X = [X_{ij}]_{i,j=1}^2$. Then

$$(\delta_{AB}-\lambda)^n X = \begin{pmatrix} * & * \\ & (\delta_{A_2B_2}-\lambda)^n X_{22} \end{pmatrix}$$

(for some, as yet, non specified entries $*$). Since $\lim\limits_{n\to\infty} \|(\delta_{AB}-\lambda)^n X\|^{\frac{1}{n}} = 0$ implies $\lim\limits_{n\to\infty} \|(\delta_{A_2B_2}-\lambda)^n X_{22}\|^{\frac{1}{n}} = 0$, and since $\delta_{A_2B_2}-\lambda$ is invertible, we have $X_{22}=0$, and then

$$(\delta_{AB}-\lambda)^n X = \begin{pmatrix} * & (\delta_{A_1B_2}-\lambda)^n X_{12} \\ (\delta_{A_2B_1}-\lambda)^n X_{21} & 0 \end{pmatrix}$$

(for some, as yet, non specified entry $*$). Again, $\lim\limits_{n\to\infty} \|(\delta_{AB}-\lambda)^n X\|^{\frac{1}{n}} = 0$ implies $\lim\limits_{n\to\infty} \|(\delta_{A_1B_2}-\lambda)^n X_{12}\|^{\frac{1}{n}} = \lim\limits_{n\to\infty} \|(\delta_{A_2B_1}-\lambda)^n X_{21}\|^{\frac{1}{n}} = 0$, and since $\delta_{A_1B_2}-\lambda$ and $\delta_{A_2B_1}-\lambda$ are invertible, we have $X_{12} = 0 = X_{21}$. Hence, $(\delta_{AB}-\lambda)^n X = (\delta_{A_1B_1}-\lambda)^n X_{11}$. Let $X_{11} = [Y_{ij}]_{1\le i,j\le m} \in \mathcal{L}(\bigoplus_{i=1}^m N_{1i}, \bigoplus_{i=1}^m M_{1i})$. Then, for $1 \le i,j \le m$,

$$\begin{aligned}(d_{A_1B_1}-\lambda)^n(X_{11}) &= ((L_{A_1-\alpha_i} - R_{B_1-\beta_j}) + \\ &\quad (\alpha_i - \beta_j - \lambda))^n [Y_{ij}]_{1\le i,j\le m} \\ &= \left(\sum_{k=0}^n \binom{n}{k} (L_{A_1-\alpha_i} - R_{B_1-\beta_j})^k \right. \\ &\quad \left. \times (\alpha_i - \beta_j - \lambda)^{n-k}\right) [Y_{ij}]_{1\le i,j\le m}\end{aligned}$$

where we have set $L_{A_1-\alpha_i}R_{B_1} + \alpha_i R_{B_1-\beta_j} = T$. Since $(A-\alpha_i)|_{M_{1i}} = 0 = (B_1-\beta_i)|_{N_{1i}}$, it follows that

$$(\delta_{A_1B_1}-\lambda)^n(X_{11}) = (\alpha_i - \beta_j - \lambda))^n [Y_{ij}]_{1\le i,j\le m}$$

Recall, $\lim\limits_{n\to\infty} \|(\delta_{A_1B_1}-\lambda)^n X_{11}\|^{\frac{1}{n}} = 0$; hence $\lim\limits_{n\to\infty} |\alpha_i - \beta_j - \lambda| \, \|Y_{ij}\|^{\frac{1}{n}} = 0$. Thus $Y_{ij} = 0$ for all i,j such that $i \neq j$. This implies that $X = X_{11} = \bigoplus_{i=1}^m Y_{ij} \in (\delta_{AB}-\lambda)^{-1}(0)$. Hence $H_0(\delta_{AB}-\lambda) \subset (\delta_{AB}-\lambda)^{-1}(0)$. Since the reverse inclusion holds for every operator, we must have $H_0(\delta_{AB}-\lambda) = (\delta_{AB}-\lambda)^{-1}(0)$.

For an operator $T \in \mathcal{B}(\mathcal{X})$, the *analytic core* $\mathcal{K}(T-\lambda)$ of $T-\lambda$ is defined by

$$\begin{aligned}\mathcal{K}(T-\lambda) &= \{x \in \mathcal{X} : \text{there exists a sequence} \\ &\quad \{x_n\} \subset \mathcal{X} \text{ and } \delta > 0 \text{ for which} \\ &\quad x = x_0, (T-\lambda)x_{n+1} = x_n \text{ and} \\ &\quad \|x_n\| \le \delta^n \|x\| \text{ for all } n = 1,2,\cdots\}.\end{aligned}$$

We note that $H_0(T-\lambda)$ and $\mathcal{K}(T-\lambda)$ are generally non-closed hyperinvariant subspaces of $T-\lambda$ such that $(T-\lambda)^{-q}(0) \subseteq H_0(T-\lambda)$ for all $q = 0,1,2,\cdots$ and $(T-\lambda)\mathcal{K}(T-\lambda) = \mathcal{K}(T-\lambda)$ [14]. Recall from [14] that if $0 \in \operatorname{iso}\sigma(T)$, then $H_0(T)$ and $\mathcal{K}(T)$ are closed and $\mathcal{X} = H_0(T) \oplus \mathcal{K}(T)$.

Theorem XII. If $(A,B) \in FP$, then δ_{AB} satisfies Weyl's theorem and δ^*_{AB} satisfies a-Weyl's theorem.

Proof. Let $\lambda \in \operatorname{iso}\sigma(d_{AB})$. Then by Lemma III, $H_0(\delta_{AB}-\lambda) = (\delta_{AB}-\lambda)^{-1}(0)$ implies

$$\begin{aligned}\mathcal{B}(\mathcal{H}) &= H_0(\delta_{AB}-\lambda) \quad \oplus\mathcal{K}(\delta_{AB}-\lambda) \\ &= (\delta_{AB}-\lambda)^{-1}(0) \quad \oplus\mathcal{K}(\delta_{AB}-\lambda).\end{aligned}$$

Hence

$$\begin{aligned}\Re(\delta_{AB}-\lambda) = &\quad 0 \oplus (\delta_{AB}-\lambda)(\mathcal{K}(\delta_{AB}-\lambda)) \\ &= \mathcal{K}(\delta_{AB}-\lambda)\end{aligned}$$

and

$$\mathcal{B}(\mathcal{H}) = (\delta_{AB}-\lambda)^{-1}(0) \oplus \Re(\delta_{AB}-\lambda).$$

Thus, isolated points of $\sigma(\delta_{AB})$ are simple poles of the resolvent of δ_{AB} (i.e., δ_{AB} is simply polaroid). Observe from the argument above that if we let $\pi^0(\delta_{AB})$ denote the set of finite rank poles of the resolvent of δ_{AB}, then $\pi^0(\delta_{AB}) = E^0(\delta_{AB})$. Since δ_{AB} has SVEP by Lemma II , it follows that δ_{AB} satisfies Weyl's theorem [15, Theorem 3.85], i.e., $\sigma(\delta_{AB}) \setminus \sigma_w(\delta_{AB}) = E^0(\delta_{AB})$.
Since δ^*_{AB} has SVEP, $\sigma(\delta_{AB}) = \sigma(\delta^*_{AB}) = \sigma_a(\delta^*_{AB})$ (see [10, Proposition 1.3.2]), which implies $E^0(\delta^*_{AB}) = E^0_a(\delta^*_{AB})$. Again, since $\lambda \notin \sigma_{SF_+^-}(\delta^*_{AB})$ if and only if $\delta^*_{AB}-\lambda$ is upper semi-Fredholm and $\operatorname{ind}(\delta^*_{AB}-\lambda) \le 0$, and since δ_{AB} has SVEP and $\delta^*_{AB}-\lambda$ is upper semi-Fredholm implies $\operatorname{ind}(\delta^*_{AB}-\lambda) \ge 0$, $\lambda \notin \sigma_{SF_+^-}(\delta^*_{AB})$ implies $\lambda \notin \sigma_w(\delta^*_{AB})$, and this, since $\sigma_{SF_+^-}(T) \subseteq \sigma_w(T)$ for every operator $T \in \mathcal{B}(\mathcal{X})$, implies $\sigma_{SF_+^-}(\delta^*_{AB}) = \sigma_w(\delta^*_{AB}) = \sigma_w(\delta_{AB})$. As seen above, δ_{AB} is simply polaroid; hence δ^*_{AB} is simply polaroid, with $\pi^0(\delta_{AB}) = \pi^0(\delta^*_{AB})$. This implies that $E^0(\delta_{AB}) = \pi^0(\delta_{AB}) = \pi^0(\delta^*_{AB}) \subseteq E^0(\delta^*_{AB}) = E^0(\delta^*_{AB})$. Since $\lambda \in E^0(\delta^*_{AB})$ implies $\lambda \in \operatorname{iso}\sigma(\delta_{AB})$ and so $\lambda \in \pi^0(\delta_{AB})$, it follows that $E^0(\delta_{AB}) = E^0_a(\delta^*_{AB})$. Hence,

$$\begin{aligned}E^0(\delta_{AB}) = &\quad \sigma(\delta_{AB}) \setminus \sigma_w(\delta_{AB}) \\ &= \sigma_a(\delta^*_{AB}) \setminus \sigma_{SF_+^-}(\delta^*_{AB}) \\ &= E^0_a(\delta^*_{AB}),\end{aligned}$$

i.e., δ^*_{AB} satisfies a-Weyl's theorem.

Recall from [13] that an operator $T \in \mathcal{B}(\mathcal{X})$ is said to be obeys property (w) if $\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = E^0(T)$. From the proof of Theorem III, we have

Corollary XIII. If $(A, B) \in FP$, then δ^*_{AB} satisfies property (w).

For a Banach space operator T, let $H(\sigma(T))$ denote the set of functions which are holomorphic on an open neighborhood of $\sigma(T)$.

Corollary XIV. If $(A, B) \in FP$, then $f(\delta_{AB})$ satisfies Weyl's theorem and $f(\delta^*_{AB})$ satisfies a-Weyl's theorem for every $f \in H(\sigma(\delta_{AB}))$.

Proof. It follows from Theorem III that δ_{AB} satisfies Weyl's theorem. That is,

$$\sigma(\delta_{AB}) \setminus E^0(\delta_{AB}) = \sigma_w(\delta_{AB}).$$

Now the polaroid property of δ_{AB} implies, [15, Lemma 3.89], that

$$f(\sigma(\delta_{AB}) \setminus E^0(\delta_{AB})) = \sigma(f(\delta_{AB})) \setminus E^0(f(\delta_{AB}))$$

for every $f \in H(\sigma(\delta_{AB}))$. Also, since δ_{AB} has SVEP,

$$f(\sigma_w(\delta_{AB})) = \sigma_w(f(\delta_{AB}))$$

for every $f \in H(\sigma(\delta_{AB}))$. Hence,

$$\begin{aligned}\sigma(f(\delta_{AB})) \setminus E^0(f(\delta_{AB})) &= f(\sigma(\delta_{AB})) \setminus f(E^0(\delta_{AB})) \\ &= f(\sigma(\delta_{AB}) \setminus E^0(\delta_{AB})) \\ &= f(\sigma_w(\delta_{AB})) = \sigma_w(f(\delta_{AB})).\end{aligned}$$

That is, $f(\delta_{AB})$ satisfies Weyl's theorem.

To prove that $f(\delta^*_{AB})$ satisfies a-Weyl's theorem, we recall that δ_{AB} is polaroid if and only δ^*_{AB} is polaroid, δ^*_{AB} is isoloid. Applying [15, Lemma 3.89], it thus follows that

$$f(\sigma(\delta^*_{AB})) \setminus f(E^0(\delta^*_{AB})) = \sigma(f(\delta^*_{AB})) \setminus E^0(f(\delta^*_{AB})).$$

Again, since δ_{AB} has SVEP implies $f(\delta_{AB})$ has SVEP [15, Theorem 2.39],

$$\begin{aligned}\sigma(f(\delta_{AB})) &= \sigma(f(\delta^*_{AB})) = \sigma_a(f(\delta^*_{AB})), \\ & E^0(f(\delta^*_{AB})) = E^0_a(\delta^*_{AB}) \\ \sigma_w(f(\delta_{AB})) &= \sigma_w(f(\delta^*_{AB})) = \sigma_{SF_+^-}(f(\delta^*_{AB})) \\ & and\ \sigma_{SF_+^-}(f(\delta^*_{AB})) = f(\sigma_{SF_+^-}(\delta^*_{AB}))\end{aligned}$$

[15, Corollary 3.72]. Thus δ^*_{AB} satisfies a-Weyl's theorem, we have that

$$\begin{aligned}\sigma_a(f(\delta^*_{AB})) \setminus E^0_a(f(\delta^*_{AB})) &= f(\sigma(\delta^*_{AB}) \setminus E^0(\delta^*_{AB})) \\ &= f(\sigma_w(\delta^*_{AB})) \\ &= \sigma_{SF_+^-}(f(\delta^*_{AB})).\end{aligned}$$

That is, $f(\delta^*_{AB})$ satisfies a-Weyl's theorem.

IV. Conclusion

In this paper we investigated Weyl's type theorems for generalized derivation for (A, B) satisfying a couple of Fuglede. Many questions are raised by this work. First one such, the examination of the conditions which enable one to easily apply the Weyl types theorems and Fuglede-Putnam theorem, we have discussed, which are mostly stated as purely mathematical results. The second question of which our theorems can give constructive proofs. Other questions can be posed and indeed all are under investigation and will be considered elsewhere.

Acknowledgment

The author thank the referee for useful comments and suggestions that will improve the quality of this paper.

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