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C Vibration Analysis of a Beam on Elastic Foundation with Elastically Restrained Ends Using Spectral Element Method

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Abstract**—**In this study, a spectral element method (SEM) is employed to predict the free vibration of a Euler-Bernoulli beam resting on a Winkler foundation with elastically restrained ends. The formulation of the dynamic stiffness matrix has been established by solving the differential equation of motion which was transformed to frequency domain. Non-dimensional natural frequencies and shape modes are obtained by solving the partial differential equations, numerically. Numerical comparisons and examples are performed to show the effectiveness of the SEM and to investigate the effects of various parameters, such as the springs at the boundaries and the elastic foundation parameter on the vibration frequencies. The obtained results demonstrate that the present method can also be applied to solve the more general problem of the dynamic analysis of structures with higher order precision.

Keywords**—**Elastically supported Euler-Bernoulli beam, free-vibration, spectral element method, Winkler foundation.

I. Introduction

THE analysis of dynamic interaction between a structure, its foundation and the underlying soil is a classical discipline in engineering. Therefore, the concept of beam resting on elastic foundation proves to be a significant tool for modeling and analysis of the highway, railroad, structural, and geotechnical engineering problems such as highway pavement, railroad tracks, continuously supported pipelines, and strip footings. Different types of foundation models such as Winkler, Pasternak, Hetenyi, Kerr, Vlasov and Viscoelastic are developed and applied in the analysis of structures on elastic foundations. Winkler model is the most well-known and widely used mechanical model. According to this model, the soil medium is modeled by a set of mutually independent elastic vertical spring elements that introduces a linear algebraic relationship between the normal displacement of the structure and the contact pressure.

There are a large number of papers on vibration of beams resting on Winkler elastic foundation have been published. Ozgur et al. [1] used an efficient method for the analysis of the free vibration behavior of Euler-Bernoulli beams on an elastic foundation with elastic restraints. Kacar et al. [2] studied the vibration of a Euler-Bernoulli beam resting on a variable Winkler elastic foundation using the differential transform method. Zhou [3] studied a general solution to vibrations of beams on a variable Winkler elastic foundation. The exact solution to free vibration of elastically restrained Timoshenko beam on an arbitrary variable elastic foundation using Green Function is presented by Ghannadiasl et al [4]. Rao and Naidu [5] studied the free vibration and stability behavior of a simply supported uniform beam or column with nonlinear elastic end restraints against rotation by using the finite element method. Kim and Kim [6] studied the vibration of Euler-Bernoulli beam with generally restrained boundary conditions using Fourier series. Balkaya et al. [7] employed a simulation method called the differential transform method to predict the vibration of a Euler–Bernoulli and Timoshenko beam (pipeline) resting on an elastic soil. Tazabekova et al. [8] calculated the free vibration characteristics for a Euler-Bernoulli beam on a Winkler linear elastic foundation using He's Variational Iteration Method.

In this paper, the free vibration of a Euler-Bernoulli beam resting on an elastic foundation with elastic restrains is studied by using the SEM. First, the efficiency of the proposed method is demonstrated via different examples. Then, some numerical examples are performed to investigate the effects of various parameters, such as the springs at the boundaries, the elastic foundation parameter to examine how these parameters affect the vibration frequencies.

II. Problem Formulation

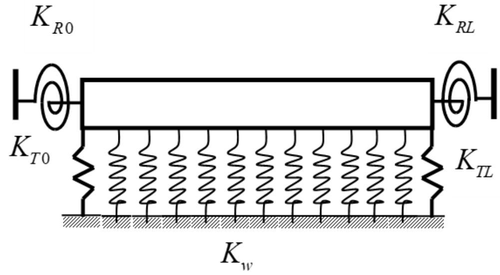

Fig. 1 The beam on elastic foundation with elastically restrained ends

Consider the problem of a Euler–Bernoulli beam of length ℓ and with constant flexural stiffness EI, resting on a Winkler-type foundation with elastically restrained ends is depicted in Fig. 1. The governing differential equation of this problem can be expressed as

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$$\rho A \frac{\partial^2 v(x,t)}{\partial t^2} + EI \frac{\partial^4 v(x,t)}{\partial x^4} + K_w v(x,t) = 0 \tag{1}$$

where ρ is the mass density, A is the cross-sectional area of the beam, K_w is the elastic coefficient of Winkler foundation, K_{T0}, K_{TL} are translational spring constants and K_{R0}, K_{RL} are rotational springs constants and $v(x,t)$ is the transverse deflection at the axial location x and time t.

The solution of (1) can be obtained by the separation method of variables.

$$v(x,t) = V(x).e^{i\omega t} \tag{2}$$

Equation (1) can be expressed as:

$$\frac{d^4 V(x)}{dx^4} - \alpha^4 V(x) = 0 \tag{3}$$

defining $\alpha^4 = \frac{\rho A \omega^2 - K_w}{EI}$ where ω is the natural circular frequency.

The solution of (3) can be expressed as

$$V(x,\omega) = \langle \cos\alpha x \quad \sin\alpha x \quad \cosh\alpha x \quad \sinh\alpha x \rangle \{C_i\} \tag{4}$$

where $\{C_i\} = \begin{bmatrix} C_1 & C_2 & C_3 & C_4 \end{bmatrix}^T$ is the constant vector.

The nodal displacement vector $\{q_e\}$ can be deduced by using the kinematic conditions at the ends of the beam.

$$\{q_e\} = [D] e^{i\omega t}.\{C_i\} \tag{5}$$

where $\{q_e\} = \begin{bmatrix} v_1 & \vartheta_1 & v_2 & \vartheta_2 \end{bmatrix}^T$ is the vector of nodal displacements and the matrix $[D]$ has the form

$$[D] = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & \alpha & 0 & \alpha \\ \cos\alpha\ell & \sin\alpha\ell & \cosh\alpha\ell & \sinh\alpha\ell \\ -\alpha.\sin\alpha\ell & \alpha.\cos\alpha\ell & \alpha.\sinh\alpha\ell & \alpha.\cosh\alpha\ell \end{bmatrix}$$

Corresponding to the general problem (Fig. 1), the nodal force vector can be deduced by using boundary conditions of the system, which are:

$$T_1 = EI \frac{d^3 v(x,t)}{dx^3} + K_{T0} v(x,t), \quad x = 0$$

$$T_2 = EI \frac{\partial^3 v(x,t)}{\partial x^3} - K_{TL} v(x,t), \quad x = \ell$$

$$M_1 = EI \frac{\partial^2 v(x,t)}{\partial x^2} - K_{R0} \frac{\partial v(x,t)}{\partial x}, \quad x = 0$$

$$M_2 = EI \frac{\partial^2 v(x,t)}{\partial x^2} + K_{RL} \frac{\partial v(x,t)}{\partial x}, \quad x = \ell$$

In matrix form, the forces at the ends of beam are

$$\{F_e\} = \begin{bmatrix} T_1 & T_2 & M_1 & M_2 \end{bmatrix} = [F] e^{i\omega t} \{C_i\} \tag{6}$$

with

$$[F] = \begin{bmatrix} F_{11} & -F_{12} & F_{11} & F_{12} \\ -F_{21} & -F_{22} & F_{21} & -F_{22} \\ F_{31} & F_{32} & F_{33} & F_{34} \\ F_{41} & F_{42} & F_{43} & F_{44} \end{bmatrix} \tag{7}$$

where

$$F_{11} = K_{T0} \quad , \quad F_{12} = EI\alpha^3$$

$$F_{21} = EI\alpha^2 \quad , \quad F_{22} = K_{R0}.\alpha$$

$$F_{31} = EI\alpha^3.\sin\alpha\ell - K_{TL}.\cos\alpha\ell \quad F_{32} = -EI\alpha^3.\cos\alpha\ell - K_{TL}.\sin\alpha\ell$$

$$F_{33} = EI\alpha^3.\sinh\alpha\ell - K_{TL}.\cosh\alpha\ell$$

$$F_{34} = EI\alpha^3.\cosh\alpha\ell - K_{TL}.\sinh\alpha\ell$$

$$F_{41} = -EI\alpha^2.\cos\alpha\ell - K_{RL}\alpha.\sin\alpha\ell$$

$$F_{42} = -EI\alpha^2.\sin\alpha\ell + K_{RL}\alpha.\cos\alpha\ell$$

$$F_{43} = EI\alpha^2.\cosh\alpha\ell + K_{RL}\alpha.\sinh\alpha\ell$$

$$F_{44} = EI\alpha^2.\sinh\alpha\ell + K_{RL}\alpha.\cosh\alpha\ell$$

Using (5) and (6), the expression between the nodal force and the degree of freedom vectors can be deduced

$$\{F_e\} = [F][D]^{-1}\{q_e\} \tag{8}$$

The spectral stiffness matrix for a finite spectral element of non-classical boundary conditions can be evaluated by

$$\left[K_e^B\right] = [F][D]^{-1} \tag{9}$$

III. Numerical Results

In this section, illustrative examples were presented to examine the present problem. Firstly, to validate the proposed method, the results of a beam on elastic foundation with classical boundary conditions are presented and compared with available results. Then, examples were exhibited to show the effects of Winkler elastic foundation and spring stiffness parameters on the dimensionless frequency parameters of the beam. In this study, the material properties of the beam set all values to unity, such as $L=I=E=A=\rho=1$.

A. Validation of the Proposed Method

In order to show the efficiency and convergence of the proposed method, free vibration of a Euler–Bernoulli beam resting on a Winkler soil with classical boundary conditions are investigated and compared to [7], [2].

It should be noted that, by letting $K_{T0}=\infty$, $K_{TL}=0$, $K_{R0}=\infty$, and $K_{RL}=0$, will automatically degenerate into a cantilever beam. Frequencies ω_i(rad/s) of a simply supported beam can be achieved by using the values $K_{T0}=\infty$, $K_{TL}=\infty$, $K_{R0}=0$, and $K_{RL}=0$.

TABLE I
NATURAL FREQUENCIES Ω_i(RAD/S) OF A SIMPLY SUPPORTED BEAM RESTING ON A WINKLER FOUNDATION

Method	ω_1	ω_2	ω_3
SEM	9.92014	39.49108	88.83207
DTM	9.92014	39.4911	88. 8321
DQEM	9.92014	39.4913	89.4002
Exact solution	9.92014	39.4911	88.8321

Firstly, the Winkler elastic parameter is $K_w=1$ and the fact that the results shown in Tables I and II are in good agreements with study given by Balkaya [7] for beams resting on the Winkler elastic foundation under different boundary conditions confirms the present formulation.

TABLE II
NATURAL FREQUENCIES Ω_i(RAD/S) OF A CANTILEVER BEAM RESTING ON A WINKLER FOUNDATION

Method	ω_1	ω_2	ω_3
SEM	3.65546	22.05717	61.70532
DTM	3.65546	22.0572	61.7053
DQEM	3.65544	22.0572	61.7057

In the second example, dimensionless parameters ($\lambda^4=\dfrac{\rho A\ell^4}{EI}\omega^2$) of the simply supported-simply supported case at different K_w values are given extensively. The present results are compared with various results in [2]. The results are given in Table III.

B. Beam on Elastic Foundation with Elastically Restrained Ends

In this example, the analysis is performed to investigate the effects of elastic foundation parameters and spring stiffness parameters on the dimensionless frequency parameters of the beam.

The results of analysis for both cases are depicted in Figs. 2-5 fixing the spring stiffness parameters (K_{T0}, K_{TL}, K_{R0}, K_{RL}) and varying the foundation stiffness parameter (K_w).

TABLE III
FREQUENCY PARAMETERS for S-S BEAM on ELASTIC FOUNDATION

K_w		λ_1	λ_2	λ_3
10	SEM	3.21929	6.29324	9.42776
	exact	3.219	6.293	9.427
	[2]	3,219291184	6,293239752	9,427762796
50	SEM	3.48442	6.33298	9.43967
	exact	3,484	6.333	9.439
	[2]	3,484424567	6,33298318	9,439673875
100	SEM	3.74836	6.38163	9.45450
	exact	3.748	6.382	9.454
	[2]	3,748364250	6,381633293	9,454499603
500	SEM	4.94388	6.73581	9.57067
	exact	4.944	6.736	9.571
	[2]	4,943880409	6,735814452	9,570668085
1000	SEM	5.75562	7.11211	9.71018
	exact	5.756	7.112	9.710
	[2]	5,755620336	7,112107040	9,710176091
2000	SEM	6.76738	7.72357	9.97242
	exact	6.767	7.724	9.972
	[2]	6,767383474	7,723570755	9,972420206

TABLE IV
THE FIRST THREE FREQUENCY PARAMETERS FOR $K_{T0}=K_{TL}=10$, $K_{R0}=K_{RL}=10^5$ AND $K_w=10,\ldots,2000$

K_w	$K_{T0}=K_{TL}=10$ $K_{R0}=K_{RL}=10^5$		
	λ_1	λ_2	λ_3
10	2.32961	3.48297	6.33333
50	2.88684	3.69875	6.37234
100	3.30597	3.92429	6.42010
500	4.77405	5.02415	6.76859
1000	5.65056	5.80705	7.13999
2000	6.70361	6.79922	7.74538

TABLE V
THE FIRST THREE FREQUENCY PARAMETERS FOR $K_{T0}=K_{TL}=10^5$, $K_{R0}=K_{RL}=10$ AND $K_w=10,\ldots,2000$

K_w	$K_{T0}=K_{TL}=10^5$ $K_{R0}=K_{RL}=10$		
	λ_1	λ_2	λ_3
10	4.18893	7.06963	10.05244
50	4.31881	7.09777	10.06227
100	4.46626	7.13247	10.07452
500	5.31480	7.39340	10.17092
1000	6.00220	7.68499	10.28770
2000	6.92361	8.18488	10.51000

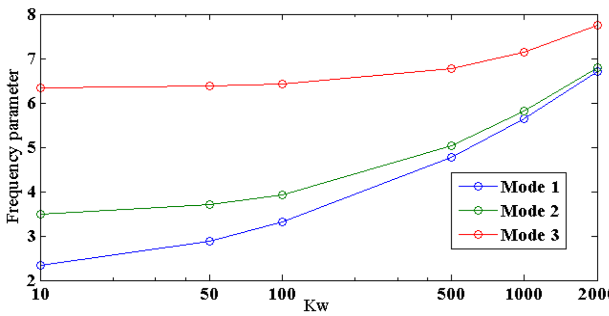

Fig. 2 The effects of different spring parameters on the vibration frequencies. $K_{T0}=K_{TL}=10$ and $K_{R0}=K_{RL}=10^5$

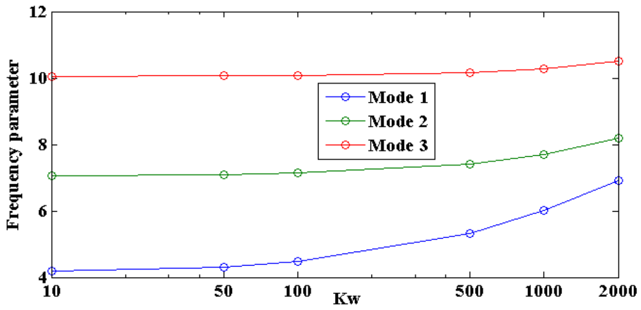

Fig. 3 The effects of different spring parameters on the vibration frequencies. $K_{T0}=K_{TL}=10^5$ an $K_{R0}=K_{RL}=10$

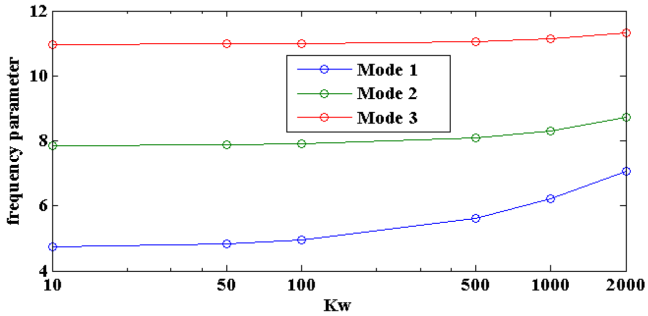

Fig. 4 The effects of different spring parameters on the vibration frequencies ($K_{T0}=K_{TL}=10^5$ an $K_{R0}=K_{RL}=10^5$)

TABLE VI

THE FIRST THREE FREQUENCY PARAMETERS FOR $K_{T0}=K_{TL}=10^5$, $K_{R0}=K_{RL}=10^5$ AND K_w=10,…, 2000

K_w	$K_{T0}=K_{TL}=10^5$	$K_{R0}=K_{RL}=10^5$	
	λ_1	λ_2	λ_3
10	4.75138	7.84851	10.97054
50	4.84199	7.86911	10.97811
100	4.94853	7.89464	10.98754
500	5.62294	8.09047	11.06217
1000	6.22298	8.31684	11.15337
2000	7.07083	8.72099	11.32935

TABLE VII

THE FIRST THREE FREQUENCY PARAMETERS FOR $K_{T0}=10^5$, $K_{TL}=10$, $K_{R0}=10^5$,$K_{RL}=10$ AND K_w=10,…,2000

K_w	$K_{T0}=10^5$ $K_{R0}=10^5$	$K_{TL}=10$	$K_{RL}=10$
	λ_1	λ_2	λ_3
10	2.83173	5.34978	8.36455
50	3.19573	5.41393	8.38158
100	3.52445	5.49104	8.40273
500	4.85217	6.01512	8.56644
1000	5.69824	6.52178	8.75871
2000	6.73234	7.28018	9.10918

Fig. 2: $K_{T0}=K_{TL}=10$, $K_{R0}=K_{RL}=10^5$ and Kw varying. Changing the foundation stiffness parameter (K_w) results in an increase in the first, second and third mode.

Fig. 3: $K_{T0}=K_{TL}=10^5$, $K_{R0}=K_{RL}=$ 10 and Kw varying. Changing the foundation stiffness parameter (K_w) results in an increase in the first mode and second mode whereas third mode is not much affected from the existence of the foundation stiffness parameter.

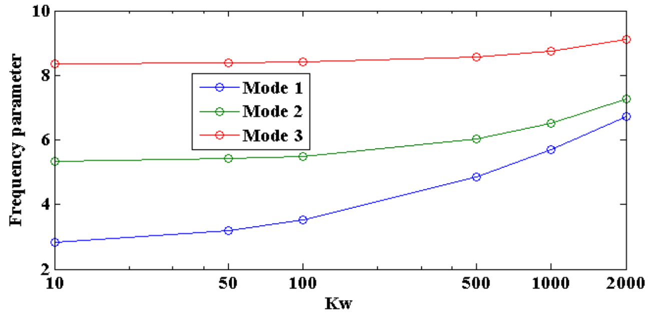

Fig. 5 The effects of different spring parameters on the vibration frequencies ($K_{T0}=10^5$, $K_{R0}=10^5$, $K_{TL}=10$, $K_{RL}=10$).

Figs. 4 and 5: fixing the K_{T0}, K_{TL}, K_{R0}, K_{RL} and Kw varying. Changing the foundation stiffness parameter (K_w) results in an increase in the first mode and second mode, whereas third mode is not much affected from the existence of the foundation stiffness parameter.

IV. Conclusion

This study presents the free vibration of elastically restrained Euler-Bernoulli beam on a Winkler-type elastic foundation using SEM. Based on the exact solution of the differential equation of vibration, the dynamic stiffness matrix of the system is formulated in the frequency domain. At first, the accuracy and efficiency of the proposed method have been evaluated by comparing the obtained results with those obtained by other methods in literature for special cases boundary conditions simply supported and cantilever beams. The results obtained are found to be in close agreement with the exact solution. In addition, some numerical examples have been presented to show the effect of elastic supports stiffness parameters at the beam ends, foundation stiffness parameters on the natural frequencies parameters. Also, the natural frequency parameter increases as the stiffness of the system increases. It should be noted that the increase in the stiffness of the translational springs is more effective on the frequency parameters of the system than the increase in the stiffness of the rotational springs.

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