Optimal Production and Maintenance Policy for a Partially Observable Production System with Stochastic Demand

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***Abstract*—**In this paper, the joint optimization of the economic manufacturing quantity (EMQ), safety stock level, and condition-based maintenance (CBM) is presented for a partially observable, deteriorating system subject to random failure. The demand is stochastic and it is described by a Poisson process. The stochastic model is developed and the optimization problem is formulated in the semi-Markov decision process framework. A modification of the policy iteration algorithm is developed to find the optimal policy. A numerical example is presented to compare the optimal policy with the policy considering zero safety stock.

***Keywords*—**Condition-based maintenance, economic manufacturing quantity, safety stock, stochastic demand.

I. Introduction

THE traditional economic manufacturing quantity (EMQ) model is rarely applied in the actual manufacturing industries due to simplified and unrealistic assumptions. Therefore, this model has been extended to satisfy the industry requirements and make it more applicable. Many EMQ papers have been published considering imperfect products, stochastic production or demand rates, and production facility deterioration [1]-[6]), but very few papers have been developed considering partial information about the system condition.

One of the main assumptions has been to consider imperfect process or machine deterioration in extended EMQ models. The joint optimization of EMQ and maintenance policy has been investigated in several papers. One of the early works is [7], where the authors studied the effect of random machine breakdowns on the optimal lot size. In the next paper, they included safety stock in their model and obtained the optimal production quantity [8]. To incorporate preventive maintenance (PM) in the model, [9] investigated preventive maintenance and two types of failure in the EMQ model. Machine is replaced upon major failure and it is repaired at a lower cost when minor failure occurs. The optimal lot size and preventive replacement time were found for this model. Later, [10] developed a joint optimization of EMQ and imperfect PM, where the age of the system is reduced proportional to the PM level. The machine replacement is performed when it is out of control or after m inspections, which is a decision variable in the model. Various approaches to modeling and solving the maintenance and lot sizing problem have been proposed in the literature (see e.g. [11]-[13]), however majority of them considered traditional age-based maintenance model. In fact, [14]-[16] are the only references that took condition monitoring (CM) information into account and developed the jointly optimal CBM and EMQ policies considering different assumptions. Continuous monitoring was introduced in [16], and [14], [15] assumed that CM is performed periodically and CM information is available at each sampling epoch. Reference [14] assumed fully observable deterioration of the production facility using proportional hazards model (PHM), whereas [15] considered partially observable machine state where the information is obtained through CM. These papers assumed that demand is deterministic with known constant demand rate, whereas in many real applications, demand can be stochastic. Stochastic demand in lot sizing problems has been considered under different assumptions such as dynamic demand rate [1], [3], [17], Poisson demand arrivals [18]-[20], demand following a particular probability density function [2], [21]. Such a drawback of existing models motivates us to extend the model proposed in [14] by considering stochastic Poisson demand arrival process.

Also, to satisfy the demand when performing maintenance actions and reduce inventory shortages in the system, safety stock is introduced, which is another interesting topic in EMQ models [22]-[24]. In this paper, we assume a safety stock to decrease the lost sales cost in the system during production run. It is assumed that newly produced items are transferred to the inventory system after completion of the production run. We formulate a model to address this gap in the literature. The assumptions and model development are presented in Section II. The SMDP framework considering CM and Bayesian control under the stochastic demand assumption is developed in Section III. Section IV is devoted to the analysis of the proposed model by developing a numerical example and investigating its effectiveness. Finally, conclusions and suggestions for future research are discussed in Section V.

II. Model Formulation

We consider a manufacturing system which produces the lot size Q in each production run, with constant production rate denoted by p, whereas the demand arrives according to a Poisson process with rate λ. The production facility is subject to deterioration and random failure and its condition can be classified to be in one of the three states: a healthy or "as good as new" state (state 0), unhealthy or warning state (state 1), and a failure state (state 2). The healthy and unhealthy states

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are unobservable, only the failure state is observable at any time. The end of each production run provides an opportunity to collect the vector of observations through CM. Once the production facility deterioration exceeds a certain level, then the observations will change considerably, and the machine will be in the warning state. When the machine is identified to be in the warning state, the PM action is initiated. We assume that the products are available to satisfy the demand after completion of a production run. Therefore, the demand during production run will be satisfied from the safety stock, and the unsatisfied demands are considered as lost sales.

We model the machine state process $(X_t : t \in R^+)$ as a hidden Markov process in a partially observable framework with the state space $\Omega = \{0, 1, 2\}$. The system is assumed to start in a healthy state and it can make transitions from state 0 to state 1 with probability p_{01} or from state 0 to state 2 with probability p_{02}, where $p_{01} + p_{02} = 1$. It is assumed that the sojourn times in state 0 and 1 are exponentially distributed, with parameters ν_0 and ν_1, respectively. The instantaneous transition rates for the state process $(X_t : t \in R^+)$ from state z to state z' are given by:

$$q_{zz'} = \begin{cases} \lim_{h\to 0} \frac{P(X_h=z'|X_0=z)}{h} < +\infty, & z \neq z' \\ -\sum_{z\neq z'} q_{zz'}, & z = z' \\ 0, & \text{otherwise}, \end{cases} \tag{1}$$

where $z, z' \in \Omega$ and the state transition rate matrix Q has the form:

$$\boldsymbol{Q} = \begin{bmatrix} -(q_{01} + q_{02}) & q_{01} & q_{02} \\ 0 & -q_{12} & q_{12} \\ 0 & 0 & 0 \end{bmatrix}, \tag{2}$$

where $q_{01}, q_{02}, q_{12} \in (0, \infty)$. We assume that the state process is non-decreasing and the failure state is an absorbing state. Let $\boldsymbol{P} = [P_{z,z'}(t)]_{z,z'\in\Omega}$ represent the transition probability matrix, then the elements of this matrix are obtained by solving the Kolmogorov backward differential equations [25].

We start monitoring the production facility condition at equidistant sampling times $(\Delta, 2\Delta, ...)$, and data vectors $Y_1, Y_2, ... \in R^d$ give partial information about the production facility state, and are assumed to be conditionally independent given the production facility state, and normally distributed (readers are referred to [26] for a discussion and proof of this reasonable assumption). So, the observation Y_n given $X_{n\Delta} = i$ for $i = 0, 1$, has d-dimensional normal distribution $\mathcal{N}_d(\mu_i, \Sigma_i)$, where μ_i, Σ_i are assumed to be known observation process parameters.

III. Optimal Bayesian Control Policy

From the theory of partially observable Markov decision processes, it is well known that the posterior probability statistic that the system is in a warning state is sufficient for decision making [27], [28]. Therefore, upon collecting the samples at the end of each production run, the posterior probability that the production facility is in the unhealthy state is updated. For $n \geq 1$, Π_n denotes the posterior probability at the n^{th} sampling epoch which can be obtained as:

$$\Pi_n = P\Big(X_{n\Delta} = 1 \mid \xi > n\Delta, Y_1, ..., Y_{n\Delta}\Big) = \frac{\theta_1}{\theta_1 + \theta_2} \tag{3}$$

where

$$\theta_1 = f\big(Y_n \mid 1\big)\Big(P_{01}(\Delta)(1 - \Pi_{(n-1)\Delta}) + P_{11}(\Delta)\Pi_{(n-1)\Delta}\Big)$$
$$\theta_2 = f\big(Y_n \mid 0\big)P_{00}(\Delta)\Big(1 - \Pi_{(n-1)\Delta}\Big)$$

$$f(Y_n = y|i) = \frac{1}{\sqrt{(2\pi)^d|\Sigma_i|}} \exp\Big(-\frac{1}{2}(y-\mu_i)^T\Sigma_i^{-1}(y-\mu_i)\Big); \quad i \in \{0, 1\}. \tag{4}$$

At the end of each production run, a sample is collected at a cost C_s and the posterior probability Π_n is updated using Bayes' rule. At sampling epoch $n\Delta$, if Π_n exceeds the control limit, a production facility is inspected to check whether it is in the healthy or warning state and the inspection cost is C_I. If the production facility is found to be in the healthy state, it will be left operational without further actions. Otherwise, PM is performed at a cost C_P. If a failure occurs, failure replacement is carried out immediately with the corresponding cost C_F. We also assume that the machine failure can occur only in the production phase and the failure is observable at any time. The products are stored at the end of a production run at the holding cost rate of C_H, and the demands are satisfied from the inventory stock. If there are unsatisfied demands, then the lost sales cost occurs denoted by C_L. Also, the set-up cost of C_u is charged upon initiating the new production run. The objective is to find the optimal values of the production lot size Q^*, the control limit $\bar{\Pi}^* \in [0, 1]$, and safety stock level s^* that minimize the long-run expected average cost per unit time.

We develop an efficient computational algorithm in the semi-Markov decision process (SMDP) framework to determine the optimal production and maintenance policy. Suppose that at the end of the n^{th} production run, the production facility has not failed, i.e. $\xi > n\Delta$. The state space of the posterior probability [0,1] is discretized into K sub-intervals. The intervals should be selected small enough to provide accurate result in a reasonable time. For a fixed K, the SMDP is defined to be in state (z, i), where z is the coded value of the posterior probability, and i represents the inventory level. The coded value of the posterior probability is z, if the current value of Π_n lies in the interval $[\frac{z-1}{K}, \frac{z}{K})$.

Therefore, the state space of the SMDP is defined as $\Psi = \{(0, s)\} \cup \{(z, i) : z \in (0, 1]\} \cup \{(PM, l)\} \cup \{(F, j)\}$, where $\{(PM, l)\}$ and $\{(F, j)\}$ are the preventive maintenance and failure states with the corresponding inventory levels l, j, respectively. Now, for a set of SMDP states Ψ, the long-run expected average cost is determined by the transition probabilities, expected costs, and expected sojourn times, which are defined as:

$P_{h,l}$= the probability that at the next production run the production facility will be in state $l \in \Psi$ given the current state is $h \in \Psi$.

C_h= the expected cost incurred until the next production run given the current state is $h \in \Psi$.

τ_h= the expected sojourn time until the next production run given the current state is $h \in \Psi$.

For given production lot size, control limit, and safety stock, and the long-run expected average cost per unit time, $g(Q, \bar{\Pi}, s)$, can be obtained by iteratively solving the system of linear (5):

$$\begin{aligned} Z_h &= C_h - g(Q, \bar{\Pi}, s)\tau_h + \sum_{l \in \Psi} P_{h,l} Z_l, \text{ for each } h, l \in \Psi, \\ Z_j &= 0, \quad \text{for some } j \in \Psi. \end{aligned} \tag{5}$$

Now, we need to determine the SMDP quantities, i.e. $P_{h,l}, \tau_h, C_h$ for $h, l \in \Psi$, to obtain the optimal production and maintenance policy minimizing the long-run expected average cost [25].

A. Transition Probabilities

The SMDP transition probabilities for all states are derived in this section. Since, the system may start in state $(0, s)$, then we have:

1) The SMDP transition probability $P_{(0,s),(z',j)}$ from state $(0, s)$ to state (z', j) where $z' < \bar{\Pi}$ is given by:

$$\begin{aligned} P_{(0,s),(z',j)} &= P\Big(\frac{z'-1}{K} \le \Pi_1 < \frac{z'}{K} | \xi > \Delta, \Pi_0 = 0\Big) \\ &\cdot R(\Delta|\Pi_0 = 0) \cdot \frac{(\lambda\Delta)^m e^{-\lambda\Delta}}{m!}; \ j = Q + s - m, \\ & m = 0, 1, ..., s-1. \end{aligned} \tag{6}$$

If the demand during production run exceeds the safety stock level, then the next state will be (z', Q), and we have:

$$\begin{aligned} P_{(0,s),(z',Q)} &= P\Big(\frac{z'-1}{K} \le \Pi_1 < \frac{z'}{K} | \xi > \Delta, \Pi_0 = 0\Big) \\ &\cdot R(\Delta|\Pi_0 = 0) \cdot \sum_{m=s}^{\infty} \frac{(\lambda\Delta)^m e^{-\lambda\Delta}}{m!}. \end{aligned} \tag{7}$$

where $R(\Delta|\Pi_0 = 0)$ is the conditional reliability function at the first production run which is given by:

$$R(t|\Pi_0 = 0) = 1 - P_{02}(t). \tag{8}$$

The general formulas for the conditional reliability function and for the transition probabilities will be developed in (15) and (17), respectively.

2) If the posterior probability exceeds the control limit, then inspection is performed and the result can be true or false alarm. If it is a false alarm, then the system will be in the state $(0, j)$, and the transition probability can be derived as:

$$\begin{aligned} P_{(0,s),(0,j)} &= \sum_{z' \in M} P\Big(\frac{z'-1}{K} \le \Pi_1 < \frac{z'}{K} | \xi > \Delta, \Pi_0 = 0\Big) \\ &\cdot R(\Delta|\Pi_0 = 0) \cdot \frac{(\lambda\Delta)^m e^{-\lambda\Delta}}{m!} \cdot (1 - z'); \ j = Q + s - m, \\ & m = 0, 1, ..., s-1, \end{aligned} \tag{9}$$

where $M = \{z' : z' \ge \bar{\Pi}\}$.

When the demand during production run is greater than the safety stock level, then the system makes transition to the state $(0, Q)$:

$$\begin{aligned} P_{(0,s),(0,Q)} &= \sum_{z' \in M} P\Big(\frac{z'-1}{K} \le \Pi < \frac{z'}{K} | \xi > \Delta, \Pi_0 = 0\Big) \\ &\cdot R(\Delta|\Pi_0 = 0) \cdot \sum_{m=s}^{\infty} \frac{(\lambda\Delta)^m e^{-\lambda\Delta}}{m!} \cdot (1 - z'). \end{aligned} \tag{10}$$

However, if the inspection reveals that it is true alarm, then the system will be in the state (PM, j), and the transition probability is given by:

$$\begin{aligned} P_{(0,s),(PM,j)} &= \sum_{z' \in M} P\Big(\frac{z'-1}{K} \le \Pi < \frac{z'}{K} | \xi > \Delta, \Pi_0 = 0\Big) \\ &\cdot R(\Delta|\Pi_0 = 0) \cdot \frac{(\lambda\Delta)^m e^{-\lambda\Delta}}{m!} \cdot z'; \ j = Q + s - m, \\ & m = 0, 1, ..., s-1, \end{aligned} \tag{11}$$

where the demand is less than safety stock level, otherwise, the system makes transition to the state (PM, Q), and we have:

$$\begin{aligned} P_{(0,i),(PM,Q)} &= \sum_{z' \in M} P\Big(\frac{z'-1}{K} \le \Pi < \frac{z'}{K} | \xi > \Delta, \Pi_0 = 0\Big) \\ &\cdot R(\Delta|\Pi_0 = 0) \cdot \sum_{m=i}^{\infty} \frac{(\lambda\Delta)^m e^{-\lambda\Delta}}{m!} \cdot z'. \end{aligned} \tag{12}$$

3) Upon failure, the system will be in the state (F, j), and we have:

$$P_{(0,s),(F,j)} = \begin{cases} \int_0^{\Delta} \frac{-d}{dt} R(t|\Pi_0 = 0) \cdot \frac{(\lambda t)^m e^{-\lambda t}}{m!} \, dt; \\ j = q + s - m, \ m = 0, 1, ..., s-1 \\ \int_0^{\Delta} \frac{-d}{dt} R(t|\Pi_0 = 0) \cdot \sum_{m=s}^{\infty} \frac{(\lambda t)^m e^{-\lambda t}}{m!} dt; \\ j = q, \end{cases} \tag{13}$$

where $q = \lfloor pt \rfloor$.

4) The SMDP transition probability $P_{(z,i),(z',j)}$ from state (z, i) to state (z', j) where $z, z' < \bar{\Pi}$ is given by:

$$\begin{aligned} P_{(z,i),(z',j)} &= P\Big(\frac{z'-1}{K} \le \Pi_n < \frac{z'}{K} | \xi > n\Delta, \Pi_{n-1} = z\Big) \\ &\cdot R(\Delta|\Pi_{n-1} = z) \cdot \frac{(\lambda\Delta)^m e^{-\lambda\Delta}}{m!}; \ j = Q + s - m, \\ & m = 0, 1, ..., s-1 \end{aligned} \tag{14}$$

where $R(\Delta|\Pi_{n-1} = z)$ is the conditional reliability function at the n^{th} production run which is given by:

$$\begin{aligned} & R(t|\Pi_{n-1}) = \\ & (1 - \Pi_{n-1}) \cdot (1 - P_{02}(t)) + \Pi_{n-1} \cdot (1 - P_{12}(t)). \end{aligned} \tag{15}$$

If the demand during production run exceeds the safety stock level, then the system makes transition to the state (z', Q), and we have:

$$\begin{aligned} P_{(z,i),(z',Q)} &= P\Big(\frac{z'-1}{K} \le \Pi_n < \frac{z'}{K} | \xi > n\Delta, \Pi_{n-1} = z\Big) \\ &\cdot R(\Delta|\Pi_{n-1} = z) \cdot \sum_{m=s}^{\infty} \frac{(\lambda\Delta)^m e^{-\lambda\Delta}}{m!}. \end{aligned} \tag{16}$$

The first part of (14) is given by:

$$P\Big(\frac{z'-1}{K} \le \Pi_n < \frac{z'}{K} \mid \xi > n\Delta, \Pi_{n-1} = z\Big) = \quad (17)$$
$$P\Big[2 \cdot \ln\Big[\frac{(1-\frac{z'}{K})D^1_{\Pi_{n-1}}}{\frac{z'}{K}D^0_{\Pi_{n-1}}} \cdot \delta\Big] - C < V_n \le 2$$
$$\cdot \ln\Big[\frac{(1-\frac{z'-1}{K})D^1_{\Pi_{n-1}}}{\frac{z'-1}{K}D^0_{\Pi_{n-1}}} \cdot \delta\Big] - C | X_{n\Delta}=0\Big] \cdot \Big[\frac{D^0_{\Pi_{n-1}}}{D^1_{\Pi_{n-1}} + D^0_{\Pi_{n-1}}}\Big]$$
$$+P\Big[2 \cdot \ln\Big[\frac{(1-\frac{z'}{K})D^1_{\Pi_{n-1}}}{\frac{z'}{K}D^0_{\Pi_{n-1}}} \cdot \delta\Big] - C < V_n \le 2$$
$$\cdot \ln\Big[\frac{(1-\frac{z'-1}{K})D^1_{\Pi_{n-1}}}{\frac{z'-1}{K}D^0_{\Pi_{n-1}}} \cdot \delta\Big] - C | X_{n\Delta}=1\Big] \cdot \Big[\frac{D^1_{\Pi_{(n-1)}}}{D^1_{\Pi_{n-1}} + D^0_{\Pi_{n-1}}}\Big].$$

where

$$D^0_{\Pi_{n-1}} = P_{00}(\Delta)(1-\Pi_{n-1})$$
$$D^1_{\Pi_{n-1}} = P_{01}(\Delta)(1-\Pi_{n-1}) + P_{11}(\Delta)\Pi_{n-1}, \quad (18)$$

and

$$V_n = (Y_n - B)^T A (Y_n - B)$$
$$A = \Sigma_1^{-1} - \Sigma_0^{-1}$$
$$B = (\Sigma_1^{-1} - \Sigma_0^{-1})^{-1}(\Sigma_1^{-1}\mu_1 - \Sigma_0^{-1}\mu_0)$$
$$C = (\mu_1{}^T\Sigma_1^{-1}\mu_1 - \mu_0^T\Sigma_0^{-1}\mu_0) - B^T(\Sigma_1^{-1}\mu_1 - \Sigma_0^{-1}\mu_0)$$
$$\delta = \big(|\Sigma_1| \cdot |\Sigma_0|^{-1}\big)^{1/2}. \quad (19)$$

We can simplify the posterior probability calculation under the assumption $\Sigma_0 \ne \Sigma_1$, and we have:

$$\Pi_n = \frac{D^1_{\Pi_{n-1}}}{\delta \cdot \exp\Big[\frac{1}{2}\big(V_n + C\big)\Big] D^0_{\Pi_{n-1}} + D^1_{\Pi_{n-1}}}, \quad (20)$$

The probabilities in (17) can be derived by using the closed formula for the cumulative distribution function of $(V_n|X_{n\Delta})$, where $(Y_n - B|X_{n\Delta} = i)$ follows normal distribution $N(\mu_i - B, \Sigma_i)$ [29].

5) When the posterior probability exceeds the control limit $\bar{\Pi}$, full inspection is performed and it can be false alarm or true alarm. If a false alarm occurred, then the production facility is in as good as new condition, so the system makes transition to state $(0, j)$.

$$P_{(z,i),(0,j)} = \sum_{z' \in M} P\Big(\frac{z'-1}{K} \le \Pi_n < \frac{z'}{K} | \xi > n\Delta, \Pi_{n-1} = z\Big)$$
$$\cdot R(\Delta|\Pi_{n-1} = z) \cdot \frac{(\lambda\Delta)^m e^{-\lambda\Delta}}{m!} \cdot (1-z');$$
$$j = Q + s - m, \; m = 0, 1, ..., s-1 \quad (21)$$

After being in the state $(0, j)$, $j > s$, then the inventory is depleted to reach the state $(0, s)$, and we have:

$$P_{(0,j),(0,s)} = 1. \quad (22)$$

However, when the demand during production run is greater than the safety stock level, then the system makes transition to the state $(0, Q)$:

$$P_{(z,i),(0,Q)} = \sum_{z' \in M} P\Big(\frac{z'-1}{K} \le \Pi_n < \frac{z'}{K} | \xi > n\Delta, \Pi_{n-1} = z\Big)$$
$$\cdot R(\Delta|\Pi_{n-1} = z) \cdot \sum_{m=s}^{\infty} \frac{(\lambda\Delta)^m e^{-\lambda\Delta}}{m!} \cdot (1-z'); \quad (23)$$

6) If the result of inspection reveals that it was true alarm, then the system goes to the (PM, j) state, and the corresponding transition probabilities are given by:

$$P_{(z,i),(PM,j)} = \sum_{z' \in M} P\Big(\frac{z'-1}{K} \le \Pi_n < \frac{z'}{K} | \xi > n\Delta, \Pi_{n-1} = z\Big)$$
$$\cdot R(\Delta|\Pi_{n-1} = z) \cdot \frac{(\lambda\Delta)^m e^{-\lambda\Delta}}{m!} \cdot z'; \; j = Q + s - m,$$
$$m = 0, 1, ..., s-1 \quad (24)$$

if the demand is less than safety stock level, otherwise, the system makes transition to the state (PM, Q), and the corresponding transition probability is derived as:

$$P_{(z,i),(PM,Q)} = \sum_{z' \in M} P\Big(\frac{z'-1}{K} \le \Pi_n < \frac{z'}{K} | \xi > n\Delta, \Pi_{n-1} = z\Big)$$
$$\cdot R(\Delta|\Pi_{n-1} = z) \cdot \sum_{m=s}^{\infty} \frac{(\lambda\Delta)^m e^{-\lambda\Delta}}{m!} \cdot z'; \quad (25)$$

7) When observable failure occurs, then the system goes to the state (F, j).

$$P_{(z,i),(F,j)} = \begin{cases} \int_0^\Delta \frac{-d}{dt} R(t|\Pi_{n-1} = z) \cdot \frac{(\lambda t)^m e^{-\lambda t}}{m!} \, dt; \\ \quad j = q + s - m, \; m = 0, 1, ..., s-1 \\ \int_0^\Delta \frac{-d}{dt} R(t|\Pi_{n-1} = z) \cdot \sum_{m=s}^{\infty} \frac{(\lambda t)^m e^{-\lambda t}}{m!} dt; \\ \quad j = q. \end{cases} \quad (26)$$

8) When the machine is in the (PM, j) mandatory replacement is performed, and we have:

$$P_{(PM,j),(0,s)} = 1. \quad (27)$$

9) Finally, when the machine is in the failure state (F, j), the inventory level may be less than safety stock level or not. Therefore, the system makes transition to the state $(0, s)$, and we have:

$$P_{(F,j),(0,s)} = 1, \quad j \ge s \quad (28)$$

However, when the inventory is less than safety stock level upon failure, then the new production run is initiated and the system goes to the state (z', l):

$$P_{(F,j),(z',l)} = P\Big(\frac{z'-1}{K} \le \Pi_1 < \frac{z'}{K} | \xi > \Delta, \Pi_0 = 0\Big)$$
$$\cdot R(\Delta|\Pi_0 = 0) \cdot \frac{(\lambda\Delta)^m e^{-\lambda\Delta}}{m!}; \; l = Q + j - m,$$
$$m = 0, 1, ..., j-1, \; j < s. \quad (29)$$

If the demand during inspection interval exceeds the current inventory level, then the next state will be

(z', Q), and we have:

$$P_{(F,j),(z',Q)} = P\Big(\frac{z'-1}{K} \le \Pi_1 < \frac{z'}{K} | \xi > \Delta, \Pi_0 = 0\Big) \cdot R(\Delta|\Pi_0 = 0) \cdot \sum_{m=j}^{\infty} \frac{(\lambda\Delta)^m e^{-\lambda\Delta}}{m!},\ j < s. \quad (30)$$

Also, when the posterior probability crosses the control limit, then the system will be in the state $(0, l)$ or (PM, l), depending on whether false or true alarm occurs, respectively. The transition probabilities are given by:

$$P_{(F,j),(0,l)} = \sum_{z' \in M} P\Big(\frac{z'-1}{K} \le \Pi_1 < \frac{z'}{K} | \xi > \Delta, \Pi_0 = 0\Big) \cdot R(\Delta|\Pi_0 = 0) \cdot \frac{(\lambda\Delta)^m e^{-\lambda\Delta}}{m!} \cdot (1-z');\ l = Q+j-m,\ m = 0, 1, ..., j-1,\ j < s. \quad (31)$$

When the demand during production run is greater than the inventory on hand upon failure, then the system makes transition to the state $(0, Q)$:

$$P_{(F,j),(0,Q)} = \sum_{z' \in M} P\Big(\frac{z'-1}{K} \le \Pi < \frac{z'}{K} | \xi > \Delta, \Pi_0 = 0\Big) \cdot R(\Delta|\Pi_0 = 0) \cdot \sum_{m=j}^{\infty} \frac{(\lambda\Delta)^m e^{-\lambda\Delta}}{m!} \cdot (1-z'),\ j < s. \quad (32)$$

However, upon detection of true alarm, the system will be in the state (PM, l):

$$P_{(F,j),(PM,l)} = \sum_{z' \in M} P\Big(\frac{z'-1}{K} \le \Pi < \frac{z'}{K} | \xi > \Delta, \Pi_0 = 0\Big) \cdot R(\Delta|\Pi_0 = 0) \cdot \frac{(\lambda\Delta)^m e^{-\lambda\Delta}}{m!} \cdot z';\ l = Q + j - m,\ m = 0, 1, ..., j-1,\ j < s, \quad (33)$$

if the demand is less than safety stock level, otherwise, the system makes transition to the state (PM, Q), and we have:

$$P_{(F,j),(PM,Q)} = \sum_{z' \in M} P\Big(\frac{z'-1}{K} \le \Pi < \frac{z'}{K} | \xi > \Delta, \Pi_0 = 0\Big) \cdot R(\Delta|\Pi_0 = 0) \cdot \sum_{m=j}^{\infty} \frac{(\lambda\Delta)^m e^{-\lambda\Delta}}{m!} \cdot z',\ j < s. \quad (34)$$

When failure occurs before the production run completion, the system will go to the state (F, l), and we have:

$$P_{(F,j),(F,l)} = \begin{cases} \int_0^{\Delta} \frac{-d}{dt} R(t|\Pi_0 = 0) \cdot \frac{(\lambda t)^m e^{-\lambda t}}{m!}\, dt; \\ l = q + j - m,\ m = 0, 1, ..., j-1 \\ \\ \int_0^{\Delta} \frac{-d}{dt} R(t|\Pi_0 = 0) \cdot \sum_{m=j}^{\infty} \frac{(\lambda t)^m e^{-\lambda t}}{m!}\, dt; \\ j = q. \end{cases} \quad (35)$$

In the next two sections, the expected sojourn times and the expected costs in each state are derived.

B. Expected Sojourn Times

The expected sojourn times should be calculated for all SMDP states. For the initial state, the system whether survives till next inspection epoch or not. If the production facility works properly, then the expected sojourn time will be equal to the sampling interval, however if the production facility has failed before the next inspection epoch ($\Delta = Q/p$), then the mean sojourn time for the initial state depends on the failure time.

So, the sojourn time for the initial state $(0, s)$ can be written as:

$$\begin{aligned}\tau_{(0,s)} &= E(\text{sojourn time} \mid \Pi_0 = 0, \xi > 0) \\ &= E(\text{sojourn time}|\Pi_0 = 0, \xi > \Delta) \cdot R(\Delta|\Pi_0 = 0) + \\ &\int_0^{\Delta} E(\text{sojourn time}|\Pi_0 = 0, \xi > 0, \xi = t) \cdot \Big(\frac{-d}{dt} R(t|\Pi_0 = 0)\Big) dt \\ &= \frac{Q}{p} \cdot R(\Delta \mid \Pi_0 = 0) + \int_0^{\Delta} t \cdot \frac{-d}{dt} R(t \mid \Pi_0 = 0) dt, \end{aligned} \quad (36)$$

The expected sojourn time in the state (z, i), for which the posterior probability does not cross the control limit is given by:

$$\begin{aligned}\tau_{(z,i)} &= E(\text{sojourn time} \mid \Pi_n = z, \xi > n\Delta) \\ &= E(\text{sojourn time}|\Pi_n = z, \xi > (n+1)\Delta) \cdot R(\Delta|\Pi_n = z) + \\ &\int_0^{\Delta} E(\text{sojourn time}|\Pi_n = z, \xi > n\Delta, \xi = n\Delta + t) \cdot \Big(\frac{-d}{dt} R(t|\Pi_n = z)\Big) dt \\ &= \frac{Q}{p} \cdot R(\Delta|\Pi_n = z) + \int_0^{\Delta} t \cdot \frac{-d}{dt} R(t|\Pi_n = z) dt + E(W_{i-s}) \end{aligned} \quad (37)$$

where W_{i-s} is the time to deplete the inventory to reach the safety stock level, which follows Erlang distribution, so (37) can be written as:

$$\tau_{(z,i)} = \frac{Q}{p} \cdot R(\Delta|\Pi_n = z) + \int_0^{\Delta} t \cdot \frac{-d}{dt} R(t|\Pi_n = z) dt + \frac{i-s}{\lambda} \quad (38)$$

If the posterior probability crosses the control limit and a false alarm occurs, then the system will be in the state $(0, j),\ j > s$ and the expected sojourn time at this state depends on the time when inventory is depleted to reach the state $(0, s)$.

$$\tau_{(0,j)} = E(W_{j-s}),\ j > s. \quad (39)$$

If the inspection reveals the true alarm, then the system goes to (PM, j) state and the expected sojourn time is given by:

$$\tau_{(PM,j)} = E(W_{j-s}) \quad (40)$$

When failure occurs, then the system state will be (F, j), and the expected sojourn time in this state when $j \ge s$ will be:

$$\tau_{(F,j)} = E(W_{j-s}) \quad (41)$$

However, when the inventory upon failure is less than safety stock level $(F, j),\ j < s$, then the expected sojourn time is given by:

$$\tau_{(F,j)} = \frac{Q}{p} \cdot R(\Delta \mid \Pi_0 = 0) + \int_0^{\Delta} t \cdot \frac{-d}{dt} R(t|\Pi_0 = 0) dt, \quad (42)$$

C. Expected Costs

The expected cost incurred in each SMDP state is computed in this section.

For the initial state $(0, s)$, the expected cost can be obtained as:

$$\begin{aligned} C_{(0,s)} &= E(\text{cost} \mid \Pi_0 = 0, \xi > 0) \quad (43)\\ &= E(\text{cost} \mid \Pi_0 = 0, \xi > \Delta) \cdot R(\Delta \mid \Pi_0 = 0) \\ &+ \int_0^\Delta E(\text{cost} \mid \Pi_0 = 0, \xi > 0, \xi = t) \cdot (\frac{-d}{dt} R(t|\Pi_0{=}0))dt \\ &= C_u + \Big(C_s + H_p(\Delta) + L_p(\Delta)\Big) \cdot R(\Delta|\Pi_0 = 0) + C_I \\ &\cdot \sum_{z' \in M} P\Big(\frac{z'-1}{K} \le \Pi < \frac{z'}{K} |\xi > \Delta, \Pi_0{=}0\Big) \cdot R(\Delta|\Pi_0{=}0) \\ &+ \int_0^\Delta \Big(C_F + H_p(t) + L_p(t)\Big) \cdot \frac{-d}{dt} R(t|\Pi_0 = 0))dt \end{aligned}$$

where $H_p(t)$ and $L_p(t)$ are expected holding and lost sales costs during production run in t time.

$$\begin{aligned} C_{(0,s)} &= C_u + \Big(C_s + C_H \cdot \Big[s \cdot \Delta \cdot \sum_{j=0}^{s} \frac{(\lambda\Delta)^j \cdot e^{-\lambda\Delta}}{j!} - \frac{\lambda\Delta^2}{2} \\ &\cdot \sum_{j=0}^{s-1} \frac{(\lambda\Delta)^j \cdot e^{-\lambda\Delta}}{j!} + \frac{s(s+1)}{2\lambda} \cdot (1 - \sum_{j=0}^{s+1} \frac{(\lambda\Delta)^j \cdot e^{-\lambda\Delta}}{j!})\Big] \\ &+ C_L \cdot \sum_{m=s+1}^{\infty} (m-s) \cdot \frac{(\lambda\Delta)^m \cdot e^{-\lambda\Delta}}{m!}\Big) \cdot R(\Delta|\Pi_0{=}0) + C_I \\ &\cdot \sum_{z' \in M} P\Big(\frac{z'-1}{K} \le \Pi < \frac{z'}{K} |\xi > \Delta, \Pi_0{=}0\Big) \cdot R(\Delta|\Pi_0{=}0) + \\ &\int_0^\Delta \Big(C_F + C_H \cdot \Big[s \cdot t \cdot \sum_{j=0}^{s} \frac{(\lambda t)^j \cdot e^{-\lambda t}}{j!} - \frac{\lambda t^2}{2} \cdot \sum_{j=0}^{s-1} \frac{(\lambda t)^j \cdot e^{-\lambda t}}{j!} \\ &+ \frac{s(s+1)}{2\lambda} \cdot (1 - \sum_{j=0}^{s+1} \frac{(\lambda t)^j \cdot e^{-\lambda t}}{j!})\Big] + C_L \cdot \sum_{m=s+1}^{\infty} (m-s) \\ &\cdot \frac{(\lambda t)^m \cdot e^{-\lambda t}}{m!}\Big) \cdot \frac{-d}{dt} R(t|\Pi_0{=}0))dt \quad (44) \end{aligned}$$

The average cost incurred until the next decision epoch for state (z, i) such that the posterior probability does not cross the control limit, is given by:

$$\begin{aligned} C_{(z,i)} &= E(\text{cost}|\Pi_n{=}z, \xi{>}n\Delta) = E(\text{cost}|\Pi_n{=}z, \xi{>}(n+1)\Delta) \\ &\cdot R(\Delta|\Pi_n{=}z) + \int_0^\Delta E(\text{cost} \mid \Pi_n{=}z, \xi{>}n\Delta, \xi = n\Delta{+}t) \\ &\cdot (\frac{-d}{dt} R(t \mid \Pi_n = z))dt = C_u + \Big[C_s + H_p(\Delta) + L_p(\Delta)\Big] \\ &\cdot R(\Delta|\Pi_n = z) + C_I \cdot \sum_{z' \in M} P\Big(\frac{z'-1}{K} \le \Pi < \frac{z'}{K} |\xi > \Delta, \Pi_n{=}z\Big) \\ &\cdot R(\Delta|\Pi_n{=}z) + \int_0^\Delta \Big[C_F + H_p(t) + L_p(t)\Big] \cdot (\frac{-d}{dt} R(t|\Pi_n{=}z))dt \\ &+ H_d(i-s) \quad (45) \end{aligned}$$

where $H_d(q)$ is the holding cost of depleting q products during depletion period.

$$\begin{aligned} C_{(z,i)} &= C_u + \Big(C_s + C_H \cdot \Big[s \cdot \Delta \cdot \sum_{j=0}^{s} \frac{(\lambda\Delta)^j \cdot e^{-\lambda\Delta}}{j!} - \frac{\lambda\Delta^2}{2} \\ &\cdot \sum_{j=0}^{s-1} \frac{(\lambda\Delta)^j \cdot e^{-\lambda\Delta}}{j!} + \frac{s(s+1)}{2\lambda} \cdot (1 - \sum_{j=0}^{s+1} \frac{(\lambda\Delta)^j \cdot e^{-\lambda\Delta}}{j!})\Big] \\ &+ C_L \cdot \sum_{m=s+1}^{\infty} (m-s) \cdot \frac{(\lambda\Delta)^m \cdot e^{-\lambda\Delta}}{m!}\Big) \cdot R(\Delta|\Pi_n{=}z) + C_I \cdot \\ &\sum_{z' \in M} P\Big(\frac{z'-1}{K} \le \Pi < \frac{z'}{K} |\xi > \Delta, \Pi_n{=}z\Big) \cdot R(\Delta|\Pi_n{=}z) + \\ &\int_0^\Delta \Big(C_F + C_H \cdot \Big[s \cdot t \cdot \sum_{j=0}^{s} \frac{(\lambda t)^j \cdot e^{-\lambda t}}{j!} - \frac{\lambda t^2}{2} \cdot \sum_{j=0}^{s-1} \frac{(\lambda t)^j \cdot e^{-\lambda t}}{j!} \\ &+ \frac{s(s+1)}{2\lambda} \cdot (1 - \sum_{j=0}^{s+1} \frac{(\lambda t)^j \cdot e^{-\lambda t}}{j!})\Big] + C_L \cdot \sum_{m=s+1}^{\infty} (m-s) \cdot \\ &\frac{(\lambda t)^m \cdot e^{-\lambda t}}{m!}\Big) \cdot \frac{-d}{dt} R(t|\Pi_n{=}z)dt + C_H \cdot \frac{(i-s)(i-s+1)}{2\lambda} \quad (46) \end{aligned}$$

When the posterior probability exceeds the control limit, then the system will go to the state $(0, j)$, $j > s$ and the average cost in this state is given by:

$$C_{(0,j)} = H_d(j-s) = C_H \cdot \frac{(j-s)(j-s+1)}{2\lambda}. \quad (47)$$

If the inspection reveals that it was true alarm, then the average cost incurred in (PM, j) state is:

$$C_{(PM,j)} = C_P + H_d(j-s) = C_P + C_H \cdot \frac{(j-s)(j-s+1)}{2\lambda}. \quad (48)$$

Also, upon failure the system state will be (F, j), and the expected cost is given by:

$$\begin{aligned} C_{(F,j)} &= C_F + H_d(j-s) \\ &= C_F + C_H \cdot \frac{(j-s)(j-s+1)}{2\lambda}, \; j > s, \quad (49) \end{aligned}$$

where the inventory on hand upon failure is greater than the safety stock level. For $j \le s$, we have:

$$\begin{aligned} C_{(F,j)} &= C_F + C_u + \Big(C_s + C_H \cdot \Big[j \cdot \Delta \cdot \sum_{k=0}^{j} \frac{(\lambda\Delta)^k \cdot e^{-\lambda\Delta}}{k!} - \\ &\frac{\lambda\Delta^2}{2} \cdot \sum_{k=0}^{j-1} \frac{(\lambda\Delta)^k \cdot e^{-\lambda\Delta}}{k!} + \frac{j(j+1)}{2\lambda} \cdot (1 - \sum_{k=0}^{j+1} \frac{(\lambda\Delta)^k \cdot e^{-\lambda\Delta}}{k!})\Big] \\ &+ C_L \cdot \sum_{m=j+1}^{\infty} (m-j) \cdot \frac{(\lambda\Delta)^m \cdot e^{-\lambda\Delta}}{m!}\Big) \cdot R(\Delta|\Pi_0{=}0) + C_I \\ &\cdot \sum_{z' \in M} P\Big(\frac{z'-1}{K} \le \Pi < \frac{z'}{K} |\xi > \Delta, \Pi_0{=}0\Big) \cdot R(\Delta|\Pi_0{=}0) + \\ &\int_0^\Delta \Big(C_F + C_H \cdot \Big[j \cdot t \cdot \sum_{k=0}^{j} \frac{(\lambda t)^k \cdot e^{-\lambda t}}{k!} - \frac{\lambda t^2}{2} \cdot \sum_{k=0}^{j-1} \frac{(\lambda t)^k \cdot e^{-\lambda t}}{k!} \\ &+ \frac{j(j+1)}{2\lambda} \cdot (1 - \sum_{k=0}^{j+1} \frac{(\lambda t)^k \cdot e^{-\lambda t}}{k!})\Big] + C_L \cdot \sum_{m=j+1}^{\infty} (m-j) \end{aligned}$$

$$\cdot \frac{(\lambda t)^m \cdot e^{-\lambda t}}{m!}\Big) \cdot \frac{-d}{dt} R(t|\Pi_0 = 0))dt, \ j \leq s. \tag{50}$$

Now, all the SMDP quantities have been determined and the optimal policy is found by iteratively solving (5).

IV. Experimental Results

We illustrate the proposed computational procedure by extending the numerical example introduced in [14]. We assume that the production facility deterioration follows a hidden Markov process with the set of possible states $\Omega = \{0, 1, 2\}$. States 0 and 1 are unobservable, representing the healthy and warning states, respectively, and state 2 refers to the observable failure state. The sojourn times in healthy and warning states are exponentially distributed with parameters $\nu_0 = q_{01} + q_{02}$, and $\nu_1 = q_{12}$. The transition rates of the state process are given by:

$$q_{01} = 0.024, \quad q_{02} = 0.004 \quad \text{and} \quad q_{12} = 0.3.$$

The observation process $Y_1, Y_2, ... \in R^2$ represents the information collected through CM at equidistant sampling epochs and it is assumed to follow normal distribution $\mathcal{N}_2(\mu_0, \Sigma_0)$ and $\mathcal{N}_2(\mu_1, \Sigma_1)$ depending on whether the system is in the healthy or unhealthy state, where:

$$\mu_0 = \begin{pmatrix} 0.21 \\ -0.01 \end{pmatrix} \qquad \Sigma_0 = \begin{pmatrix} 1.50 & 0.61 \\ 0.60 & 1.90 \end{pmatrix}$$

$$\mu_1 = \begin{pmatrix} 0.75 \\ 0.54 \end{pmatrix} \qquad \Sigma_1 = \begin{pmatrix} 1.81 & 1.97 \\ 1.97 & 2.22 \end{pmatrix}.$$

Then, the corresponding transition probability matrix is obtained by solving Kolmogorov backward differential equations and it is given by:

$$\boldsymbol{P} = [P_{i,j}(t)] =$$

$$\begin{bmatrix} e^{-\nu_0 t} & \frac{q_{01}(e^{-\nu_1 t} - e^{-\nu_0 t})}{\nu_0 - \nu_1} & 1 - e^{-\nu_0 t} - \frac{q_{01}(e^{-\nu_1 t} - e^{-\nu_0 t})}{\nu_0 - \nu_1} \\ 0 & e^{-\nu_1 t} & 1 - e^{-\nu_1 t} \\ 0 & 0 & 1 \end{bmatrix},$$

which is used to compute the SMDP transition probabilities. The production facility produces 20 items per time unit. Also, the demand arrival process follows Poisson distribution with parameter $\lambda = 8$. The cost parameters $C_s = 1, C_I = 10, C_L = 2, C_H = 0.5, C_P = 100, C_F = 500, C_u = 20$ are considered in this experiment. The computational algorithm for the optimal decision variables $(Q^*, \bar{\Pi}^*, s^*)$ requires an appropriate discretization level K. The results obtained by running the algorithm on an Intel Core (TM) i5 CPU with 2.27 GHz reveals that $K = 40$ is sufficient to get the precise results in a reasonable computational time.

TABLE I
The Optimal Parameters and the Long-Run Expected Average Cost per Unit Time

Optimal Policy	Lot size (Q^*)	Control limit ($\bar{\Pi}^*$)	s^*	Average cost per unit time
Proposed model	30	0.250	11	17.5842
Zero safety stock model	20	0.275	0	20.0315

We applied the proposed model and obtained the results shown in Table I. Table I shows the long-run expected average cost per unit time and the optimal decision variables for the proposed model and also for the model with zero safety stock. The results in Table I confirm the superiority of the proposed model when compare with the model considering zero safety stock.

V. Conclusions and Future Research

In this paper, we have developed a joint optimization model combining both EMQ and CBM with stochastic demand. The production facility deterioration has been described by a hidden Markov process with the set of three possible states. The first two operational states representing the healthy and unhealthy states are unobservable and only failure state is observable at any time. We have assumed that the demand arrival process can be modeled as a Poisson process and the safety stock has been introduced to reduce the effect of lost sales cost in the system. The computational algorithm has been developed in the SMDP framework minimizing the long-run expected average cost per unit time. We have considered the production and maintenance costs including set-up cost, inventory holding, sampling, inspection, preventive, corrective maintenance, and shortage costs. Numerical example has been provided to illustrate the effectiveness of the proposed model which was compared with the zero safety stock model. The results indicate that substantial cost savings can be achieved by introducing safety stock in the system. Further improvements can be expected by relaxing the assumption of negligible maintenance times, which is a suitable topic for future research.

Acknowledgment

The authors would like to thank for the financial support provided by the Ontario Centers of Excellence (OCE) under Grant No. 201461, and by Natural Sciences and Engineering Research Council of Canada under Grant No. RGPIN 121384-11.

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